DETERMINISTIC STICK-SLIP DYNAMICS IN A ONE-DIMENSIONAL RANDOM POTENTIAL *

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In physics, engineering and other settings, it is important to understand the macroscopic behaviour of systems whose evolution is determined by microscale effects. It seems natural to consider these microscale effects to be random in nature (and, therefore, beyond the scope of classical averaging and homogenization theory) and to constitute some perturbation of a well-understood smooth structure. In the case of rate-*dependent* viscous systems, analysis of how the random microstructure determines the macroscopic behaviour can be found in the Green-Kubo relations and many further developments since their introduction in the 1950s [2] [4]. In the realm of rate-*independent* plasticity theory, there is a large literature (see, for example, [7]) surrounding ordinary differential inclusions such as

$$-\nabla V\big(X_t\big) + f(t) \in \partial\psi^\gamma\big(\dot{X}_t\big); \quad X_0 = x_0; \tag{1}$$

which have been very successful in modelling plastic effects with their associated structures of hysteresis loops, yield surfaces and stick-slip dynamics. Stick-slip evolutions such as the movement of a dislocation line in a crystalline structure or the Barkhausen effect in a magnetic domain can be seen in this way, cf. [3]. Our interest lies in rigorously justifying such differential inclusions as scaling limits of evolutions in perturbations of the potential V. Here we present such a derivation for a one-dimensional example.

This derivation has been known since the 1990s in the case of a periodic perturbation: it follows easily from, for example, [1] [6] that any forced one-dimensional gradient flow $X^\varepsilon : \mathbb{R}_{\geq 0} \to \mathbb{R}$ satisfying

$$\dot{X}^\varepsilon_t = -\kappa X^\varepsilon_t - g\left(\frac{X^\varepsilon_t}{\varepsilon}\right) + f(\varepsilon t); \quad X^\varepsilon_0 = x_0;$$

for $\varepsilon, \kappa > 0$, $g \in C^0(\mathbb{R}; [\gamma^-, \gamma^+])$ periodic and surjective and $f \in \mathrm{BC}^0(\mathbb{R}_{\geq 0}; \mathbb{R})$ exhibits rate-independent stick-slip behaviour in the limit as $\varepsilon \downarrow 0$:

$$X^0_t := \lim_{\varepsilon \downarrow 0} X^\varepsilon_{t/\varepsilon}$$

is the unique solution of the ordinary differential inclusion (1), where $\psi^\gamma : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is the convex, 1-homogeneous dissipation functional induced by g and given by

$$\psi^\gamma(\dot{x}) := \begin{cases} \gamma^- \dot{x}; & \dot{x} \leq 0; \\ \gamma^+ \dot{x}; & \dot{x} \geq 0. \end{cases}$$

Our result is that the same conclusion holds $\mathbb{P}$-almost surely for perturbations g belonging to a wide class of stochastic processes $g : \Omega \times \mathbb{R} \to [\gamma^-, \gamma^+]$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is some probability space. This result is noteworthy since X^0 is in principle a stochastic process dependent on the choice of "landscape parameter" $\omega \in \Omega$.

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Theorem 1 (TJS-FT, 2006–07)**.** *Let $f \in \mathrm{BC}^0(\mathbb{R}_{\geq 0}; \mathbb{R})$ and let $g : \Omega \times \mathbb{R} \to [\gamma^-, \gamma^+]$ be a stochastic process. Then X^0 $\mathbb{P}$-almost surely satisfies* (1) *if, and only if, g satisfies the following "continuity and frequent attainment of bounds" conditions, collectively denoted* (✠)*:*

1. *The sample realizations of g must be continuous $\mathbb{P}$-almost surely.*
2. *Define a sequence of stopping distances (random variables) $D_n^\pm : \Omega \to [0, +\infty]$ as follows: let $D_0^+(\omega)$ be the least $x \geq 0$ such that $g(\omega, x) = \gamma^+$; let $D_{n+1}^+(\omega)$ be the least increase upon $\sum_{i=0}^n D_i^+(\omega)$ such that the process $g(\omega, \cdot)$ attains both values $\gamma^\pm$ in the interval*

$$\left(\sum_{i=0}^{n} D_i^+(\omega), \sum_{i=0}^{n+1} D_i^+(\omega) \right].$$

Define D_n^- for $x \leq 0$ similarly. Intuitively, each $D_n^\pm$ is the first return distance of g from γ^+ back to γ^+ via the opposite extreme γ^-. It is then required that:

   (a) *For each $n \geq 0$, $D_n^\pm < +\infty$ $\mathbb{P}$-almost surely.*
   (b) *The series $\sum_{n \geq 0} D_n^\pm = +\infty$ $\mathbb{P}$-almost surely.*
   (c) *The ratio $D_{n+1}^\pm / \sum_{i=0}^n D_i^\pm \to 0$ as $n \to \infty$ $\mathbb{P}$-almost surely.*

A fortiori*, the law $(X^0)_*(\mathbb{P})$ induced on $C^0(\mathbb{R}_{\geq 0}; \mathbb{R})$ is a Dirac measure centred on the unique deterministic solution to* (1).

Note that a continuous, periodic, surjective function $g : \mathbb{R} \to [\gamma^-, \gamma^+]$ certainly satisfies (✠) (and so Theorem 1 generalizes [1]), and that a simple prototype for a stochastic process satisfying (✠) is given by a two-sided, doubly reflected Wiener process $g : \Omega \times \mathbb{R} \to [\gamma^-, \gamma^+]$.

The key step in the proof of Theorem 1 is the identification of the interval

$$\mathcal{A}^\gamma(f(t)) := \left[\frac{f(t) - \gamma^+}{\kappa}, \frac{f(t) - \gamma^-}{\kappa} \right] \subset \mathbb{R}$$

as a suitable limit of sets in the sense of [5]. In a general metric space $(\mathbb{M}, d)$, the *Kuratowski limit inferior* of a family of subsets $\{A_\varepsilon \subseteq \mathbb{M}\}_{\varepsilon > 0}$ is defined to be

$$\operatorname*{Li}_{\varepsilon \downarrow 0} A_\varepsilon := \left\{ x \in \mathbb{M} \,\middle|\, \limsup_{\varepsilon \downarrow 0} d_{\mathrm{H}}(x, A_\varepsilon) = 0 \right\},$$

where $d_{\mathrm{H}}(x, A_\varepsilon) := \inf_{y \in A_\varepsilon} d(x, y)$ is the usual Hausdorff semi-distance.

Lemma 2 (TJS-FT, 2006–07)**.** *Let*

$$A_\varepsilon^{g(\omega)}(F) := \left\{ x \in \mathbb{R} \,\middle|\, -\kappa x - g\left(\omega, \frac{x}{\varepsilon}\right) + F = 0 \right\}$$

be the fixed-point set for the dynamics with a given realization $\omega \in \Omega$ at scale $\varepsilon > 0$ and constant loading $f(t) \equiv F$. Then g satisfies (✠) *if, and only if,*

$$\operatorname*{Li}_{\varepsilon \downarrow 0} A_\varepsilon^{g(\omega)}(F) = \mathcal{A}^\gamma(F) \text{ for } \mathbb{P}\text{-almost all } \omega \in \Omega.$$

One interpretation of Theorem 1 is that in modelling deterministic dynamics in a wiggly potential $x \mapsto \frac{\kappa}{2} x^2 + \varepsilon G\left(\omega, \frac{x}{\varepsilon}\right)$, with

$$G(\omega, x) := \int_0^x g(\omega, x')\, \mathrm{d}x',$$

it does not matter a great deal exactly which perturbation $G(\omega, \cdot)$ one chooses, since the same rate-independent limit is obtained in $\mathbb{P}$-almost all cases. In some sense, random perturbations satisfying (✠) are no worse than periodic ones.

It would be of interest to extend the above one-dimensional argument to higher- or even infinite-dimensional cases, and/or to consider the addition of a noise term representing the effect of a heat bath.

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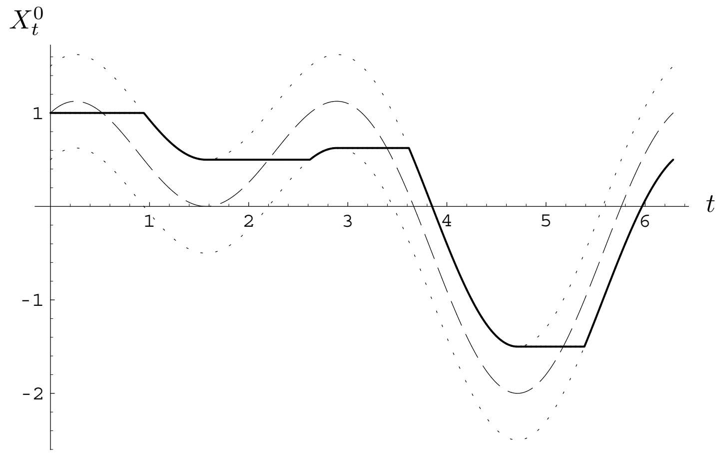

Figure 1: The $\mathbb{P}$-almost sure trajectory of X_t^0 (solid) against t for some continuous forcing $f(t)$ (dashed), showing the relationship with $\max \mathcal{A}^\gamma(f(t))$ and $\min \mathcal{A}^\gamma(f(t))$ (dotted).

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