CHAPTER 2: THE INTEGRAL THEOREMS OF VECTOR CALCULUS

2.1: GREEN'S THEOREM

Definition: Given a planar vector field $\mathbf{v} : \mathbb{R}^2 \to \mathbb{R}^2$ we define

$$\operatorname{curl} \mathbf{v} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}$$

where $\mathbf{v}(x, y) = (v_1(x, y), v_2(x, y))$.

Theorem: (Green's Theorem for a rectangle) *Let* $R = [\alpha, \beta] \times [\xi, \eta] \subset \mathbb{R}^2$ *be a rectangle in the plane and* $\mathbf{v} : R \to \mathbb{R}^2$ *a planar vector field. Then*

$$\iint_R \operatorname{curl} \mathbf{v} \, dxdy = \int_{\partial R} \mathbf{v} \cdot d\mathbf{r} \tag{G}$$

where ∂R *denotes the boundary of the rectangle* R.

Proof:

$$\int_\xi^\eta \int_\alpha^\beta \frac{\partial v_2}{\partial x} dxdy = \int_\xi^\eta (v_2(\beta, y) - v_2(\alpha, y)) dy \tag{1}$$

$$\int_\alpha^\beta \int_\xi^\eta \frac{\partial v_1}{\partial x} dydx = \int_\alpha^\beta (v_1(x, \eta) - v_1(x, \xi)) dx \tag{2}$$

The RHS of the equation (G) is

$$\begin{aligned}
\int_{\partial R} \mathbf{v} \cdot d\mathbf{r} &= \int_\alpha^\beta v_1(t, \xi) dt + \int_\xi^\eta v_2(\beta, t) dt + \int_\beta^\alpha v_1(t, \eta) dt + \int_\eta^\xi v_1(\alpha, t) dt \\
&= \int_\alpha^\beta v_1(x, \xi) dx + \int_\xi^\eta v_2(\beta, y) dy - \int_\alpha^\beta v_1(x, \eta) dx - \int_\xi^\eta v_1(\alpha, y) dy \\
&= \int_\alpha^\beta (v_1(x, \xi) - v_1(x, \eta)) dx + \int_\xi^\eta (v_2(\beta, y) - v_2(\alpha, y)) dy \\
&= -(2) + (1)
\end{aligned}$$

■

Definition: A *region* in $\mathbb{R}^n$ is a subset $\Omega \subset \mathbb{R}^n$ for which there exists a function $\omega : \mathbb{R}^n \to \mathbb{R}$ satisfying

(i) all partial derivatives of ω exist and are continuous;
(ii) $\Omega = \{\mathbf{x} \in \mathbb{R}^n \mid \omega(\mathbf{x}) < 0\}$;
(iii) $\nabla\omega(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in \partial\Omega = \{\mathbf{x} \in \mathbb{R}^n \mid \omega(\mathbf{x}) = 0\}$.

Definition: If $\Omega \subset \mathbb{R}^n$ is a region defined by ω then for all $\mathbf{x} \in \partial\Omega$ $\nabla\omega(\mathbf{x})$ is an *outward normal* to Ω and the *unit outward normal* is

$$\mathbf{n}(\mathbf{x}) = \frac{\nabla\omega(\mathbf{x})}{\|\nabla\omega(\mathbf{x})\|}.$$

Example: If $\omega(x,y) = x^2 + y^2 - \rho^2$ then $\Omega = \{(x,y) \mid x^2 + y^2 < \rho^2\}$ = the disc of radius ρ centred on 0. An outward normal is $\nabla\omega(x,y) = (2x, 2y)$ and $\mathbf{n}(x,y) = (x,y)/\rho$.

Definition: If $\Omega \subset \mathbb{R}^n$ is a region then the *closure* of Ω is $\overline{\Omega} = \Omega \cup \partial\Omega$.

Theorem: (Green's Theorem) *Let* $\Omega \subset \mathbb{R}^2$ *be a region and let* $\mathbf{v} : \overline{\Omega} \to \mathbb{R}^2$ *be a continuously differentiable vector field. Let* $\boldsymbol{\tau}$ *be the unit tangent to* $\partial\Omega$ *obtained by rotating* $\mathbf{n}$ *through* $\pi/2$ *anticlockwise. Then*

$$\iint_\Omega \operatorname{curl}\mathbf{v}\, dA = \int_{\partial\Omega} \mathbf{v}\cdot\boldsymbol{\tau}\, ds = \oint_{\partial\Omega} \mathbf{v}\cdot d\mathbf{r}.$$

Example: Verify Green's Theorem for $\mathbf{v}(x,y) = (-x^2 y, y^2 x)$ on the disc $\Omega = \{(x,y) \mid x^2 + y^2 < 4\}$.

$$\begin{aligned}
\operatorname{curl}\mathbf{v} &= \frac{\partial}{\partial x}(y^2 x) - \frac{\partial}{\partial y}(-x^2 y) = x^2 + y^2 \\
\iint_\Omega \operatorname{curl}\mathbf{v}\, dA &= \iint_\Omega (x^2 + y^2)\, dA \\
&= \int_0^{2\pi}\int_0^2 r^2 r\, dr d\theta \\
&= \int_0^{2\pi} r^4/4\Big|_0^2\, d\theta \\
&= 8\pi
\end{aligned}$$

Parameterize $\partial\Omega$ by $\mathbf{r}(t) = (2\cos t, 2\sin t)$ for $0 \le t < 2\pi$. Then

$$\begin{aligned}
\oint_{\partial\Omega} \mathbf{v}\cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{v}(\mathbf{r}(t))\cdot\frac{d\mathbf{r}}{dt}\, dt \\
&= \int_0^{2\pi} (-8\cos^2 t\sin t, 8\sin^2 t\cos t)\cdot(-2\sin t, 2\cos t)\, dt \\
&= \int_0^{2\pi} 16\cos^2 t\sin^2 t + 16\sin^2 t\cos^2 t\, dt \\
&= \int_0^{2\pi} 8\sin^2 2t\, dt \\
&= \int_0^{2\pi} (4 + 4\cos 4t)\, dt \\
&= 8\pi
\end{aligned}$$

2.2: MORE GENERAL REGIONS

The condition that $\nabla\omega(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in \partial\Omega$ makes sure that $\partial\Omega$ has no sharp corners. However, this rules out polygons and polyhedra, which we do want to be able to consider. A careful discussion of 'piecewise regions' is too cumbersome, and so we shall have to use our intuition when considering such regions.

Example: If $\Omega = [0,2]\times[0,1]\subset\mathbb{R}^2$ then $\partial\Omega$ is parameterized by

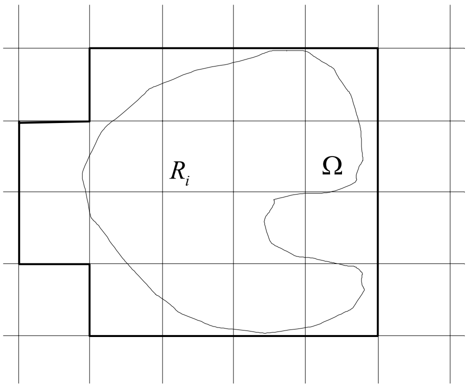
$$\mathbf{r}(t) = \begin{cases} (t,0) & 0\le t<2 \\ (2,t-2) & 2\le t<3 \\ (5-t,1) & 3\le t<5 \\ (0,6-t) & 5\le t<6 \end{cases}$$

For the purposes of calculating line integrals around $\partial\Omega$ we can use

$\mathbf{r}_L(t) = (t,0)$ for $0\le t<2$ for the lower segment L,
$\mathbf{r}_\rho(t) = (2,t)$ for $0\le t<1$ for the right segment ρ,
$\mathbf{r}_U(t) = (2-t,1)$ for $0\le t<2$ for the upper segment U,
$\mathbf{r}_\lambda(t) = (0,1-t)$ for $0\le t<1$ for the left segment λ.

Sketch Proof of Green's Theorem: Enclose Ω in a rectangular grid and apply Green's Theorem for a rectangle to each R_i and sum the results. Then

$$\sum_i \iint_{R_i} \operatorname{curl}\mathbf{v}\,dxdy = \sum_i \int_{\partial R_i} \mathbf{v}\cdot d\mathbf{r}.$$



Observe that we get cancellations from rectangles that share a common edge. Thus, since $i\ne j \Rightarrow R_i\cap R_j=\varnothing$,

$$\sum_i \int_{\partial R_i} \mathbf{v}\cdot d\mathbf{r} = \int_{\partial\left(\bigcup_i R_i\right)} \mathbf{v}\cdot d\mathbf{r}$$
$$\sum_i \iint_{R_i} \operatorname{curl}\mathbf{v}\,dxdy = \iint_{\bigcup_i R_i} \operatorname{curl}\mathbf{v}\,dxdy$$

As the grid gets finer '$\bigcup_i R_i \to \Omega$' and '$\partial\left(\bigcup_i R_i\right) \to \partial\Omega$' so

$$\lim_{\text{grid size}\to 0} \iint_{\bigcup_i R_i} \operatorname{curl}\mathbf{v}\,dxdy = \lim_{\text{grid size}\to 0} \int_{\partial\left(\bigcup_i R_i\right)} \mathbf{v}\cdot d\mathbf{r}$$
$$\iint_\Omega \operatorname{curl}\mathbf{v}\,dA = \oint_{\partial\Omega} \mathbf{v}\cdot d\mathbf{r}$$

■

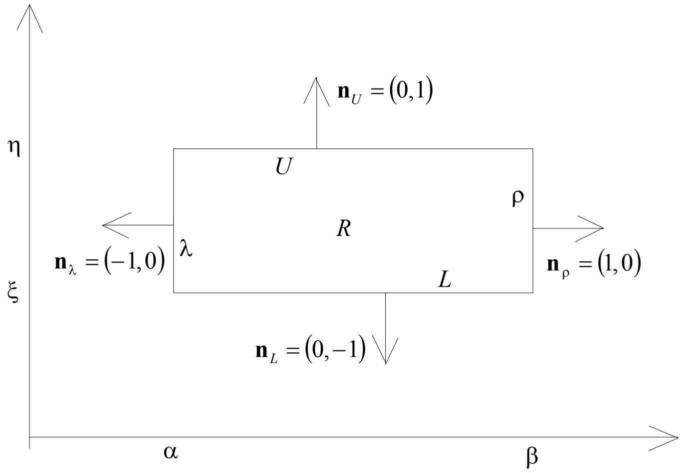
2.3: THE GAUSS DIVERGENCE THEOREM IN THE PLANE

Definition: Let $\Omega \subset \mathbb{R}^2$ be a region and $\mathbf{v} : \overline{\Omega} \to \mathbb{R}^2$ a planar vector field. The *flux* of $\mathbf{v}$ across $\partial\Omega$ (out of Ω) is defined by

$$\text{flux}\,\mathbf{v} = \Phi(\mathbf{v}) = \int_{\partial\Omega} \mathbf{v}\cdot\mathbf{n}\,ds\,,$$

where $\mathbf{n}$ is the outward unit normal to Ω.

Example: Calculate the flux of a general vector field out of the rectangle $R = [\alpha, \beta] \times [\xi, \eta]$.



The flux is

$$\begin{aligned} &\int_\alpha^\beta \mathbf{v}(t,\xi)\cdot(0,-1)\,dt \\ &+\int_\xi^\eta \mathbf{v}(\beta,t)\cdot(1,0)\,dt \\ &+\int_\alpha^\beta \mathbf{v}(t,\eta)\cdot(0,1)\,dt \\ &+\int_\xi^\eta \mathbf{v}(\alpha,t)\cdot(-1,0)\,dt \\ =&\int_\alpha^\beta (v_2(t,\eta)-v_2(t,\xi)) + \int_\xi^\eta (v_1(\beta,t)-v_1(\alpha,t))\,dt \\ =&\int_\alpha^\beta\int_\xi^\eta \frac{\partial v_2}{\partial y}(t,y)\,dy\,dt + \int_\xi^\eta\int_\alpha^\beta \frac{\partial v_1}{\partial x}(x,t)\,dx\,dt \end{aligned}$$

Thus we have Gauss' Divergence Theorem for a rectangle:

$$\iint_R \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}\right) dx\,dy = \int_{\partial R} \mathbf{v}\cdot\mathbf{n}\,ds$$

Definition: The *divergence* of a vector field $\mathbf{v}:\mathbb{R}^n \to \mathbb{R}^n$ is denoted by $\nabla\cdot\mathbf{v}$ and defined by

$$\nabla\cdot\mathbf{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \ldots + \frac{\partial v_n}{\partial x_n}.$$

Theorem: (Gauss' Divergence Theorem in the Plane) *Let* $\Omega \subset \mathbb{R}^2$ *be a (piecewise) region and* $\mathbf{v}:\overline{\Omega} \to \mathbb{R}^2$ *a planar vector field. Then*

$$\iint_\Omega \nabla\cdot\mathbf{v}\,dA = \int_{\partial R} \mathbf{v}\cdot\mathbf{n}\,ds$$

where $\mathbf{n}$ *is the outward unit normal to* Ω.

Example: Let

$$\mathbf{v}(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right).$$

Calculate $\nabla\cdot\mathbf{v}$ and the flux of $\mathbf{v}$ across the circle of radius 2 centred on the origin.

$$\begin{aligned}\nabla\cdot\mathbf{v} &= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\\ &= \frac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)-y(2y)}{(x^2+y^2)^2}\\ &= \frac{x^2+y^2-2x^2+x^2+y^2-2y^2}{(x^2+y^2)^2}\\ &= 0\end{aligned}$$

Parameterize the circle by $\mathbf{r}(t) = (2\cos t, 2\sin t)$ for $0 \le t < 2\pi$.

$$\begin{aligned}\Phi(\mathbf{v}) &= \int_0^{2\pi} \mathbf{v}(\mathbf{r}(t))\cdot\mathbf{n}(\mathbf{r}(t))\left\|\tfrac{\partial\mathbf{r}}{\partial t}\right\| dt\\ &= \int_0^{2\pi} \left(\tfrac{\cos t}{2}, \tfrac{\sin t}{2}\right)\cdot(\cos t, \sin t)2\,dt\\ &= \int_0^{2\pi} 1\,dt\\ &= 2\pi\end{aligned}$$

However, $\mathbf{v}$ is not defined at 0 and so we cannot apply Gauss' Theorem on the disc $\{(x,y) \mid x^2+y^2<4\}$. Hence, Gauss' Theorem is not violated.

Application: Vector fields for which $\nabla\cdot\mathbf{v}=1$ everywhere allow us to use Gauss' Theorem to calculate areas. For example, $\mathbf{v}(x,y) = (x/2, y/2)$, $\nabla\cdot\mathbf{v} = \tfrac{1}{2}+\tfrac{1}{2} = 1$.

$$\iint_\Omega 1\,dxdy = \iint_\Omega \nabla\cdot\mathbf{v}\,dxdy = \int_{\partial\Omega} \mathbf{v}\cdot\mathbf{n}\,ds$$

Now parameterize $\partial\Omega$ by $(x(t), y(t))$ for $a \le t \le b$.

$$\begin{aligned}\text{Area}(\Omega) &= \int_{\partial\Omega} \mathbf{v}\cdot\mathbf{n}\, ds \\ &= \frac{1}{2}\int_a^b (x(t), y(t))\cdot\left(\tfrac{dy}{dt}, -\tfrac{dx}{dt}\right)dt \\ &= \tfrac{1}{2}\int_{\partial\Omega} x\,dy - y\,dx\end{aligned}$$

2.4: AREA, SURFACE AND VOLUME INTEGRALS

Definitions: Let $E \subset \mathbb{R}^2$ be a planar region. We say that $\mathbf{r}: E \to \mathbb{R}^3$ is *regular* if $\frac{d\mathbf{r}}{dx}\times\frac{d\mathbf{r}}{dy} \ne 0$ everywhere in E. The image $\mathbf{r}(E)$ is then a *surface* $S \subset \mathbb{R}^3$ and $\mathbf{r}$ is called a *parameterization* of S.

As is the case with curves, different parameterizations may give the same surface. For example, consider the upper unit hemisphere. In polar coordinates:

$$\mathbf{r}(\theta,\phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \text{ for } 0 \le \theta < \pi/2,\ 0 \le \phi < 2\pi .$$

Alternatively, as the graph of $\sqrt{x^2+y^2}$ over the unit disc:

$$\mathbf{r}(x,y) = \left(x, y, \sqrt{x^2+y^2}\right) \text{ for } x^2+y^2 < 1.$$

Definition: The *unit normal* to a surface S parameterized by $\mathbf{r}$ is given by

$$\mathbf{n}(x,y) = \left(\frac{d\mathbf{r}}{dx}\times\frac{d\mathbf{r}}{dy}\right)\Bigg/\left\|\frac{d\mathbf{r}}{dx}\times\frac{d\mathbf{r}}{dy}\right\|.$$

Definition: The *area* of a surface S parameterized by $\mathbf{r}: E \to \mathbb{R}^3$ is

$$\text{Area}(S) = \iint_E \left\|\frac{d\mathbf{r}}{dx}\times\frac{d\mathbf{r}}{dy}\right\| dxdy.$$

Exercise: Check that this definition is independent of the parameterization $\mathbf{r}$. (Hint: use the change of variables formula.)

Definition: Given a surface S parameterized by $\mathbf{r}: E \to \mathbb{R}^3$ the *surface integral* of $f: \mathbb{R}^3 \to \mathbb{R}$ over S is

$$\iint_S f\, dA = \iint_E f(\mathbf{r}(x,y))\left\|\frac{d\mathbf{r}}{dx}\times\frac{d\mathbf{r}}{dy}\right\| dxdy.$$

In particular, $\text{Area}(S) = \iint_S 1\, dA$.

Example: Consider a torus parameterized by

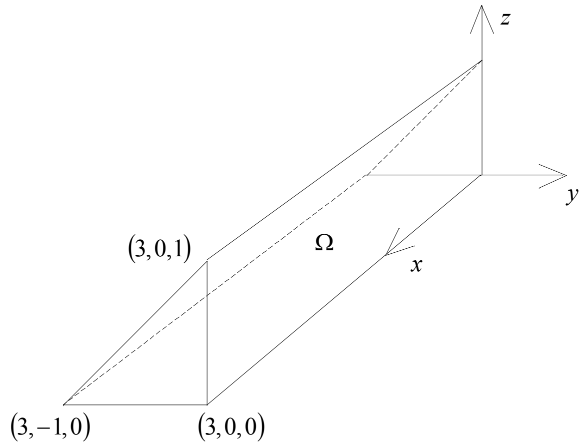
$$\mathbf{r}(\theta,\phi)=\left((2+\cos\theta)\cos\phi,(2+\cos\theta)\sin\phi,\sin\theta\right) \text{ for } 0\le\theta,\phi<2\pi .$$

$$\begin{aligned}\operatorname{Area}(\mathbf{r}([0,2\pi)\times[0,2\pi))) &= \int_0^{2\pi}\int_0^{2\pi}\left\|\tfrac{d\mathbf{r}}{d\theta}\times\tfrac{d\mathbf{r}}{d\phi}\right\| d\theta\, d\phi \\ &= \int_0^{2\pi}\int_0^{2\pi}(2+\cos\theta)\, d\theta\, d\phi \\ &= 8\pi^2\end{aligned}$$

Definition: The *flux* of $\mathbf{v}:\mathbb{R}^3\to\mathbb{R}^3$ across S parameterized by $\mathbf{r}:E\to\mathbb{R}^3$ with unit normal $\mathbf{n}$ is

$$\begin{aligned}\Phi(\mathbf{v}) &= \iint_S \mathbf{v}\cdot\mathbf{n}\, dA \\ &= \iint_E \mathbf{v}(\mathbf{r}(x,y))\cdot\left(\tfrac{d\mathbf{r}}{dx}\times\tfrac{d\mathbf{r}}{dy}\right) dx dy\end{aligned}$$

Example: Calculate the volume integral of $f(x,y,z)=2ze^{x-(y+1)^3}$ over the prism Ω in $\mathbb{R}^3$:



$$\begin{aligned}\iiint_\Omega f\, dV &= \int_0^3\int_{-1}^0\int_0^{y+1} 2ze^{x-(y+1)^3}\, dz dy dx \\ &= \int_0^3\int_{-1}^0 (y+1)^2 e^{x-(y+1)^3}\, dy dx \\ &= \int_0^3 -\tfrac{1}{3}e^{x-(y+1)^3}\Big|_{y=-1}^{0} dx \\ &= \int_0^3 -\tfrac{1}{3}e^{x-1}+\tfrac{1}{3}e^x\, dx \\ &= \tfrac{1}{3}\left(e^3-e^2-1+e^{-1}\right)\end{aligned}$$

Theorem: (Gauss' Divergence Theorem in $\mathbb{R}^3$) *Let* $\Omega\subset\mathbb{R}^3$ *be a (piecewise) region and* $\mathbf{v}:\overline{\Omega}\to\mathbb{R}^3$ *a vector field. Then*

$$\iiint_\Omega \nabla\cdot\mathbf{v}\,dV = \iint_{\partial\Omega} \mathbf{v}\cdot\mathbf{n}\,dA$$

where $\mathbf{n}$ *is the outward unit normal to* Ω.

2.5: CHANGE OF VARIABLES AND THE SPHERICAL MEAN

Example: Let E be the ellipse $\{(x,y) \mid x^2/a^2 + y^2/b^2 < 1\}$ and B^2 the unit disc centred on the origin, $B^2 = \{(u,v) \mid u^2+v^2<1\}$. Let $\boldsymbol{\varphi}(u,v) = (au, bv)$.

$$\begin{aligned} \mathrm{D}\boldsymbol{\varphi} &= \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \\ \det(\mathrm{D}\boldsymbol{\varphi}) &= ab \end{aligned}$$

So for some function $f : E \to \mathbb{R}$

$$\iint_E f(x,y)\,dxdy = \iint_{B^2} f(u(x,y), v(x,y))ab\,dudv$$

$\boldsymbol{\varphi}$ simplifies the geometry of the ellipse to that of the circle. For example, by taking $f=1$,

$$\begin{aligned} \mathrm{Area}(E) &= \iint_E 1\,dxdy \\ &= \iint_{B^2} ab\,dudv \\ &= ab\,\mathrm{Area}(B^2) \\ &= \pi ab \end{aligned}$$

Example: (Spherical polars) S^2 is usually parameterized by

$$\boldsymbol{\omega}(\theta,\phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \text{ for } 0\le\theta<\pi\ ,\ 0\le\phi<2\pi\ .$$

We can extend this to almost all of $\mathbb{R}^3$ by $\boldsymbol{\psi} : \mathbb{R}_{>0}\times S^2 \to \mathbb{R}^3 : (r,\boldsymbol{\omega}) \mapsto r\boldsymbol{\omega}$.

$$\begin{aligned} dxdydz &= dV \\ &= |\det(\mathrm{D}\boldsymbol{\psi})|\,drd\theta d\phi \\ &= r^2\sin\theta\; drd\theta d\phi \\ &= r^2\,drdA \\ &= drdA_r \end{aligned}$$

where dA_r is the elemental area on a sphere of radius r.

This coordinate system leads us to the notion of *spherical mean*: given a function $f:\mathbb{R}^3\to\mathbb{R}$ we define a new function $\bar{f}:\mathbb{R}_{>0}\to\mathbb{R}$ by

$$\bar{f}(r)=\frac{\iint_{S^2(r)} f(r\boldsymbol{\omega})\,dA_r}{\iint_{S^2(r)} 1\,dA_r}$$

where $S^2(r)$ denotes the sphere of radius r.

$$\begin{aligned}\bar{f}(r)&=\tfrac{1}{4\pi r^2}\iint_{S^2(r)} f(r\boldsymbol{\omega})\,dA_r\\&=\tfrac{1}{4\pi}\iint_{S^2} f(r\boldsymbol{\omega})\,dA\end{aligned}$$

Definition: The *spherical mean* of $f:\mathbb{R}^3\to\mathbb{R}$ is $\bar{f}:\mathbb{R}_{\geq 0}\to\mathbb{R}$ given by

$$\bar{f}(r)=\tfrac{1}{4\pi}\iint_{S^2} f(r\boldsymbol{\omega})\,dA.$$

Question: What is $d\bar{f}/dr$?

$$\begin{aligned}\tfrac{d\bar{f}}{dr}&=\tfrac{1}{4\pi}\iint_{S^2} f(r\boldsymbol{\omega})\,dA\\&=\tfrac{1}{4\pi}\iint_{S^2} \nabla f(r\boldsymbol{\omega})\cdot\boldsymbol{\omega}\,dA\\&=\tfrac{1}{4\pi r^2}\iint_{S^2(r)} \nabla f(r\boldsymbol{\omega})\cdot\boldsymbol{\omega}\,dA_r\\&=\tfrac{1}{4\pi r^2}\iint_{S^2(r)} \nabla f(r\boldsymbol{\omega})\cdot\mathbf{n}(r\boldsymbol{\omega})\,dA_r\end{aligned}$$

But $S^2(r)=\partial B^3(r)$, and so we can apply the Divergence Theorem to get

$$\frac{d\bar{f}}{dr}=\frac{1}{4\pi r^2}\iiint_{B^3(r)} \nabla\cdot(\nabla f)\,dV.$$

Definition: The *Laplacian* of $f:\mathbb{R}^n\to\mathbb{R}$ is denoted by Δf and is

$$\Delta f=\sum_{i=1}^{n}\frac{\partial^2 f}{\partial x_i^2}$$

It is very easy to check that $\Delta f=\nabla\cdot\nabla f$, so

$$\frac{d\bar{f}}{dr}=\frac{1}{4\pi r^2}\iiint_{B^3(r)} \Delta f\,dV.$$

Definition: We say that $f:\mathbb{R}^n\to\mathbb{R}$ is *harmonic* if $\Delta f=0$.

Theorem: *If $f:\mathbb{R}^3\to\mathbb{R}$ is harmonic then for all $r>0$ $\bar{f}(r)=f(0)$.*

I.e. the average value on a sphere of any size is the same as the value at the centre of the sphere. There is some kind of 'balance,' which is important in many situations.

Proof: We know that $d\bar{f}/dr = 0$ and so $\bar{f}$ is constant. Hence, $\bar{f}(r) = \bar{f}(0)$.

$$\begin{aligned}\bar{f}(0) &= \tfrac{1}{4\pi}\iint_{S^2} f(0\boldsymbol{\omega})\,dA \\ &= f(0)\tfrac{1}{4\pi}\iint_{S^2} 1\,dA \\ &= f(0)\end{aligned}$$

■

2.6: THE CONTINUITY EQUATION

We wish to study the motion of a fluid. Let $\mathbf{v}(\mathbf{x},t)$ and $\rho(\mathbf{x},t)$ be the velocity and density of the fluid at a point $\mathbf{x}$ at time t respectively. What is the mass of fluid in some region $\Omega \subset \mathbb{R}^3$?

$$M_\Omega(t) = \iiint_\Omega \rho(\mathbf{x},t)\,dV$$

$$\frac{dM_\Omega}{dt} = \iiint_\Omega \frac{\partial\rho}{\partial t}\,dV$$

The outflow of mass from Ω is

$$-\frac{dM_\Omega}{dt} = -\iiint_\Omega \frac{\partial\rho}{\partial t}\,dV\,.$$

Another way of calculating the efflux of mass from Ω would be to study the flow across $\partial\Omega$. In a small time δt the outflow across $\partial\Omega$ would be

$$\left(\iint_{\partial\Omega} \rho\mathbf{v}\cdot\mathbf{n}\,dA\right)\delta t\,.$$

By the principle of conservation of mass and the Divergence Theorem,

$$-\iiint_\Omega \frac{\partial\rho}{\partial t}\,dV = \iint_{\partial\Omega} \rho\mathbf{v}\cdot\mathbf{n}\,dA = \iiint_\Omega \nabla\cdot(\rho\mathbf{v})\,dV$$

So, for all $\Omega \subset \mathbb{R}^3$,

$$\iiint_\Omega \left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right)dV = 0\,. \qquad (*)$$

Lemma: *If* $f:\mathbb{R}^3 \to \mathbb{R}$ *is continuous and* $\iiint_\Omega f\,dV = 0$ *for all* $\Omega \subset \mathbb{R}^3$ *then* $f = 0$.

Proof: Assume otherwise. Then there is a point $\mathbf{x}_0 \in \mathbb{R}^3$ such that $f(\mathbf{x}_0) \neq 0$; without loss of generality, assume $f(\mathbf{x}_0) > 0$. By continuity, there exists an $\varepsilon > 0$ such that $\|\mathbf{x}-\mathbf{x}_0\| < \varepsilon \Rightarrow f(\mathbf{x}) > \tfrac{1}{2}f(\mathbf{x}_0) > 0$. Now take $\Omega = \{\mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x}-\mathbf{x}_0\| < \varepsilon\}$. Then

$$\iiint_\Omega f\,dV > \tfrac{1}{2} f(\mathbf{x}_0)\text{Volume}(\Omega) \neq 0 .$$

This is a contradiction, so $f = 0$.

■

Applying the lemma to $(*)$ gives the *continuity equation*:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 .$$

An important special case is that of an incompressible fluid such as water. For such fluids ρ is constant and so $\partial\rho/\partial t = \nabla\rho = 0$, so the continuity equation becomes $\nabla \cdot \mathbf{v} = 0$. Important examples of vector fields for which $\nabla \cdot \mathbf{v} = 0$ are given by $\mathbf{v} = \nabla f$ for some harmonic f.

2.7: STOKES' THEOREM

Consider a parallelogram $P \subset \mathbb{R}^3$, its edges being the two vectors $\boldsymbol{\xi}, \boldsymbol{\eta}$. Let $\mathbf{p}_0$ be one corner of P. We can parameterize P by

$$\mathbf{r}(u,v) = \mathbf{p}_0 + u\boldsymbol{\xi} + v\boldsymbol{\eta} \text{ for } 0 \le u, v \le 1 .$$

Given a vector field $\mathbf{v} = (v_1, v_2, v_3)$ we calculate the circulation $\oint_{\partial P} \mathbf{v} \cdot d\mathbf{r}$, noting that the unit normal is $\boldsymbol{\xi} \times \boldsymbol{\eta} / \|\boldsymbol{\xi} \times \boldsymbol{\eta}\|$.

$$\oint_{\partial P} \mathbf{v} \cdot d\mathbf{r} = \int_0^1 \int_0^1 \underbrace{\begin{pmatrix} \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \end{pmatrix}}_{\nabla \times \mathbf{v}} \cdot \underbrace{\begin{pmatrix} \xi_2\eta_3 - \xi_3\eta_2 \\ \xi_3\eta_1 - \xi_1\eta_3 \\ \xi_1\eta_2 - \xi_2\eta_1 \end{pmatrix}}_{\boldsymbol{\xi} \times \boldsymbol{\eta}} du\,dv$$

This is Stokes' Theorem for the parallelogram P.

Definition: If $\mathbf{v} = (v_1, v_2, v_3)$ is a vector field on $\mathbb{R}^3$ then

$$\nabla \times \mathbf{v} = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right).$$

With this notation we get

$$\oint_{\partial P} \mathbf{v} \cdot d\mathbf{r} = \iint_P (\nabla \times \mathbf{v}) \cdot \mathbf{n}\, dA$$

$$\mathbf{n} = \frac{\boldsymbol{\xi} \times \boldsymbol{\eta}}{\|\boldsymbol{\xi} \times \boldsymbol{\eta}\|}$$

$$dA = \left\| \tfrac{\partial \mathbf{r}}{\partial u} \times \tfrac{\partial \mathbf{r}}{\partial v} \right\| du dv = \|\boldsymbol{\xi} \times \boldsymbol{\eta}\| \, du dv$$

Remark: $\operatorname{curl} \mathbf{v} = (\nabla \times \mathbf{v}) \cdot \mathbf{k}$.

We now generalize from the parallelogram to arbitrary surfaces. To do this we need to exclude 'non-orientable' surfaces the Möbius band, on which we cannot define a continuously varying normal vector. Non-orientable surfaces are called *one-sided.* Orientable surfaces have two choices of normal vector.

Definition: A surface S is *oriented* if it admits a continuous global choice of unit normal.

Definition: If S is an oriented surface with boundary and $\mathbf{n}$ is a chosen unit normal to S then the unit tangent $\boldsymbol{\tau}$ to ∂S is *positively oriented with respect to* $\mathbf{n}$ if $\mathbf{n} \times \boldsymbol{\tau}$ is in inward-pointing unit tangent to S along ∂S.

A simple way of remembering this is that if you are walking along ∂S with $\mathbf{n}$ as 'up' then S is on your left.

Theorem: (Stokes' Theorem) *Let $S \subset \mathbb{R}^3$ be an oriented surface with boundary. Let $\mathbf{n}$ be a unit normal to S and $\boldsymbol{\tau}$ a unit tangent to ∂S, positively oriented with respect to $\mathbf{n}$. Let $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field. Then*

$$\iint_S (\nabla \times \mathbf{v}) \cdot \mathbf{n} \, dA = \oint_{\partial S} \mathbf{v} \cdot \boldsymbol{\tau} \, ds .$$

The idea of the proof is to approximate S by a union of small parallelograms on each of which Stokes' Theorem holds, then take the limit as the size of the parallelograms goes to zero.

Definition: We say that a vector field $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ is *axially symmetric* about the z-axis if for all $\mathbf{x} \in \mathbb{R}^3$ and $0 \le \theta < 2\pi$, $R_\theta \mathbf{v}(\mathbf{x}) = \mathbf{v}(R_\theta \mathbf{x})$, where

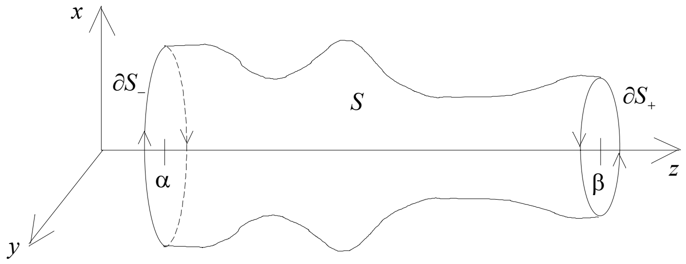
$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

It can be shown that such a vector field must be of the form

$$\mathbf{v}(x, y, z) = \big(a(z)x + b(z)y, -b(z)x + a(z)y, h(z, \rho)\big),$$

where $\rho = \sqrt{x^2 + y^2}$.

Example: Let S be the graph of $f(z)$, $\alpha < z < \beta$, rotated about the z-axis. Verify Stokes' Theorem for a vector field that is axially symmetric about the z-axis and such a surface S.



Parameterize S by $\mathbf{r}(z,\theta) = (f(z)\cos\theta, f(z)\sin\theta, z)$ for $\alpha < z < \beta$, $0 \le \theta < 2\pi$. On the upper boundary, ∂S_+, the positively oriented tangent is $-\partial\mathbf{r}/\partial\theta$; on the lower boundary, ∂S_-, it is $\partial\mathbf{r}/\partial\theta$. The unit normal to S is

$$\mathbf{n}(z,\theta) = \tfrac{\partial\mathbf{r}}{\partial z} \times \tfrac{\partial\mathbf{r}}{\partial\theta} \Big/ \left\| \tfrac{\partial\mathbf{r}}{\partial z} \times \tfrac{\partial\mathbf{r}}{\partial\theta} \right\|.$$

Now,

$$\nabla \times \mathbf{v} = \left(\tfrac{y}{\rho}\tfrac{\partial h}{\partial\rho} + b'(z)x - a'(y)y, -\tfrac{x}{\rho}\tfrac{\partial h}{\partial\rho} + a'(z)x + b'(z)y, -b(z) - b(z) \right)$$

$$\iint_S (\nabla \times \mathbf{v}) \cdot \mathbf{n}\, dA = \int_0^{2\pi} \int_\alpha^\beta (\nabla \times \mathbf{v}(\mathbf{r}(z,\theta))) \cdot \left(\tfrac{\partial\mathbf{r}}{\partial z} \times \tfrac{\partial\mathbf{r}}{\partial\theta} \right) dz\, d\theta$$

$$\begin{aligned} \frac{\partial\mathbf{r}}{\partial z} \times \frac{\partial\mathbf{r}}{\partial\theta} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f'(z)\cos\theta & f'(z)\sin\theta & 0 \\ -f(z)\sin\theta & f(z)\cos\theta & 0 \end{vmatrix} \\ &= (-f(z)\cos\theta, -f(z)\sin\theta, f(z)f'(z)) \end{aligned}$$

So,

$$\begin{aligned} (\nabla \times \mathbf{v}) \cdot \left(\tfrac{\partial\mathbf{r}}{\partial z} \times \tfrac{\partial\mathbf{r}}{\partial\theta} \right) &= \begin{pmatrix} \tfrac{\partial h}{\partial\rho}\sin\theta + b'(z)f(z)\cos\theta - a'(z)f(z)\sin\theta \\ -\tfrac{\partial h}{\partial\rho}\cos\theta + a'(z)f(z)\cos\theta + b'(z)f(z)\sin\theta \\ -2b(z) \end{pmatrix} \cdot f(z) \begin{pmatrix} -\cos\theta \\ -\sin\theta \\ f'(z) \end{pmatrix} \\ &= -b'(z)f(z)^2\left(\cos^2\theta + \sin^2\theta\right) - 2b(z)f(z)f'(z) \\ &= -b'(z)f(z)^2 - 2b(z)f(z)f'(z) \\ &= -\tfrac{d}{dz}\left(b(z)f(z)^2\right) \end{aligned}$$

So,

$$\begin{aligned} \iint_S (\nabla \times \mathbf{v}) \cdot \mathbf{n}\, dA &= -\int_0^{2\pi} \int_\alpha^\beta \tfrac{d}{dz}\left(b(z)f(z)^2\right) dz\, d\theta \\ &= 2\pi\left(b(\alpha)f(\alpha)^2 - b(\beta)f(\beta)^2\right) \end{aligned}$$

Now to calculate the circulation of $\mathbf{v}$ on ∂S. First parameterize the upper boundary by $\mathbf{r}(\beta,\theta)=(f(\beta)\cos\theta, f(\beta)\sin\theta, \beta)$ for $0\le\theta<2\pi$. So the circulation on ∂S_+ is

$$\begin{aligned}\oint_{\partial S_+}\mathbf{v}\cdot d\mathbf{r} &= \int_0^{2\pi}\left(a(\beta)x+b(\beta)y,-b(\beta)x+a(\beta)y,h(\beta,\rho)\right)\cdot(-y,x,0)\,d\theta\\ &= \int_0^{2\pi}\left(-b(\beta)y^2-b(\beta)x^2\right)d\theta\\ &= -2\pi b(\beta)f(\beta)^2\end{aligned}$$

Similarly, $\oint_{\partial S_-}\mathbf{v}\cdot d\mathbf{r} = 2\pi b(\alpha)f(\alpha)^2$. Hence,

$$\begin{aligned}\oint_{\partial S}\mathbf{v}\cdot d\mathbf{r} &= 2\pi\left(b(\alpha)f(\alpha)^2-b(\beta)f(\beta)^2\right)\\ &= \iint_S(\nabla\times\mathbf{v})\cdot\mathbf{n}\,dA\end{aligned}$$

as required.