CHAPTER 4: COMPLEX ANALYSIS

4.1: A COMPLEX-VALUED FUNCTION $f:\mathbb{C}\to\mathbb{C}$ AS A VECTOR FIELD

We can view a map $\mathbb{C}\to\mathbb{C}$ as a vector field $\mathbb{R}^2\to\mathbb{R}^2$ and vice versa i.e. if $z=x+iy$ and $f(z)=u(z)+iv(z)$ then we identify $z\mapsto f(z)$ with $(x,y)\mapsto(u(x,y),v(x,y))$.

Examples: (i) $f(z)=z^2$, $(x,y)\mapsto(x^2-y^2,2xy)$.

(ii)

$$f(x,y)=\begin{cases}\dfrac{x^2}{x^2+y^4} & (x,y)\neq 0\\ 0 & (x,y)=0\end{cases}$$

$$z\mapsto\frac{4\left(z^2+|z|^2+\bar{z}^2\right)}{4\left(z^2+|z|^2+\bar{z}^2\right)+\left(z^4-6|z|^2z^2+10|z|^4-6|z|z^3+\bar{z}^4\right)}$$

$$x=(z+\bar{z})/2$$
$$y=(z-\bar{z})/2i$$

All definitions of convergence, continuity etc. carry over to $f:\mathbb{C}\to\mathbb{C}$. We will take care with complex differentiation – limits in $\mathbb{C}$ can be strange.

Example: Let $g(x,y)=\dfrac{x^2y^2}{x^2+y^4}$ and $L_k=\{(t,kt)\mid t\in\mathbb{R}\}$. If we take the limit of g as we tend to 0 along any line L_k we get $g\to 0$. However, along the parabola $\{(t^2,t)\}$ we get that g is identically $1/2$. So $\lim_{(x,y)\to 0} g(x,y)$ does not exist.

4.2: COMPLEX DIFFERENTIATION

Definition: $f:\Omega\subset\mathbb{C}\to\mathbb{C}$ is *complex differentiable* at $z\in\Omega$ if

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$$

exists, in which case we denote the limit by $f'(z)$.

Viewed as a vector field, we have

$$\mathrm{D}\mathbf{f}=\begin{pmatrix}\partial u/\partial x & \partial u/\partial y\\ \partial v/\partial x & \partial v/\partial y\end{pmatrix}$$

Examples: (i) $z\mapsto z^n$ is everywhere complex differentiable with derivative nz^{n-1}:

$$\begin{aligned}\lim_{h\to 0}\frac{(z+h)^n - z^n}{h} &= \lim_{h\to 0}\left(nz^{n-1} + h\left(\frac{n(n-1)z^2}{2!} + \ldots + \frac{n!\,h^{k-1}z^{n-k}}{k!(n-k)!} + \ldots + h^{n-1}\right)\right)\\ &= nz^{n-1}\end{aligned}$$

(ii) $z \mapsto \bar{z}$ is nowhere complex differentiable, even though $(x,y) \mapsto (x,-y)$ is differentiable as a vector field in the plane:

$$\lim_{h\to 0}\frac{\overline{z+h} - \bar{z}}{h} = \lim_{h\to 0}\frac{\bar{h}}{h} = \begin{cases} 1 & h \text{ real} \\ -1 & h \text{ imaginary}\end{cases}$$

These 'limits' are different and so we conclude that $\lim_{h\to 0}\bar{h}/h$ does not exist.

(iii) $z \mapsto |z|^2$ is complex differentiable only at 0:

$$\begin{aligned}\lim_{h\to 0}\frac{|z+h|^2 - |z|^2}{h} &= \lim_{h\to 0}\frac{z\bar{h} + \bar{z}h + h\bar{h}}{h}\\ &= \lim_{h\to 0}\left(\bar{z} + z\frac{\bar{h}}{h}\right)\\ &= \bar{z} + z\lim_{h\to 0}\frac{\bar{h}}{h}\end{aligned}$$

and, by the previous example, this limit does not exist unless $z = 0$.

The moral here is that $\mathbf{f}:\mathbb{R}^2 \to \mathbb{R}^2$ differentiable $\not\Rightarrow$ $f:\mathbb{C}\to\mathbb{C}$ complex differentiable. Roughly speaking, f is not complex differentiable if it contains any instances of $\bar{z}$.

Definition: $f:\Omega\subset\mathbb{C}\to\mathbb{C}$ is *holomorphic* (on Ω) if f is complex differentiable at all points of Ω.

4.3: THE CAUCHY-RIEMANN EQUATIONS

Theorem: *$f:\Omega\subset\mathbb{C}\to\mathbb{C}$ is complex differentiable at $z_0\in\Omega$ if and only if $\mathbf{f}:\Omega\subset\mathbb{R}^2\to\mathbb{R}^2$ is differentiable at (x_0,y_0) and its derivative has the form* $\begin{pmatrix} a & -b \\ b & a\end{pmatrix}$.

These are the *Cauchy-Riemann equations*:

$$\left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} = \left.\frac{\partial v}{\partial y}\right|_{(x_0,y_0)} \quad\text{and}\quad \left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)} = -\left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)}$$

Proof: If f is complex differentiable at $z_0\in\Omega$ then

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

$$\Rightarrow \qquad \lim_{h \to 0} \left| \frac{f(z_0 + h) - f(z_0) - f'(z_0)h}{h} \right| = 0$$

$$\Rightarrow \qquad 0 = \lim_{(h_1, h_2)^{\mathrm{T}} \to 0} \frac{\left\| \begin{pmatrix} u(x_0 + h_1, y_0 + h_2) \\ v(x_0 + h_1, y_0 + h_2) \end{pmatrix} - \begin{pmatrix} u(x_0, y_0) \\ v(x_0, y_0) \end{pmatrix} - \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\|}{\left\| \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\|}$$

where we take $h = h_1 + ih_2$ and $f'(z_0) = a + ib$. But this is just saying that $\mathbf{f}$ is differentiable at (x_0, y_0) and that the Cauchy-Riemann equations hold.

Conversely, assume that the Cauchy-Riemann equations hold. Then $a = \partial u/\partial x$ and $b = -\partial u/\partial y$. Furthermore, since $\mathbf{f}$ is differentiable we know that

$$0 = \lim_{(h_1, h_2)^{\mathrm{T}} \to 0} \frac{\left\| \begin{pmatrix} u(x_0 + h_1, y_0 + h_2) \\ v(x_0 + h_1, y_0 + h_2) \end{pmatrix} - \begin{pmatrix} u(x_0, y_0) \\ v(x_0, y_0) \end{pmatrix} - \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\|}{\left\| \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\|}$$

which may be rewritten in complex notation as

$$0 = \lim_{h \to 0} \frac{|f(z_0 + h) - f(z_0) - (a + ib)h|}{|h|}$$

which says that

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = a + ib$$

and so f is complex differentiable at z_0 and $f'(z_0) = a + ib$.

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Corollary: *If* $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ *is complex differentiable at* $z_0 \in \Omega$ *and* $f'(z_0) \neq 0$ *then* f *is conformal at* z_0 *and* $\det(\mathrm{D}\mathbf{f}(z_0)) > 0$.

Corollary: *If* $f = u + iv$ *is holomorphic then* u, v *are harmonic.*

Warning: The converse is false!

Definition: Ω is *path-connected* if any two points in Ω can be joined by a path in Ω.

Proposition: *If* $f:\Omega\subset\mathbb{C}\to\mathbb{C}$ *is complex differentiable with* $f'(z)=0$ *for all* $z\in\Omega$ *and* Ω *is path-connected then* f *is constant.*

Proof: $f'(z)=0\Rightarrow \partial u/\partial x=\partial u/\partial y=0\Rightarrow \nabla u=0\Rightarrow u$ constant .Similarly for v. So f is constant.

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4.4: CONTOUR INTEGRALS

Definition: Given $f:\Omega\subset\mathbb{C}\to\mathbb{C}$ and a curve $C\subset\Omega$ parameterized by $z:[a,b]\to\Omega:t\mapsto z(t)$ we define the *contour integral* $\int_C f\,dz$ by

$$\int_C f\,dz=\int_a^b f(z(t))\frac{dz}{dt}\,dt$$

Similarly,

$$\int_C f\,|dz|=\int_a^b f(z(t))\left|\frac{dz}{dt}\right|dt\,.$$

Exercise: Check that these integrals are independent of the parameterization z of C.

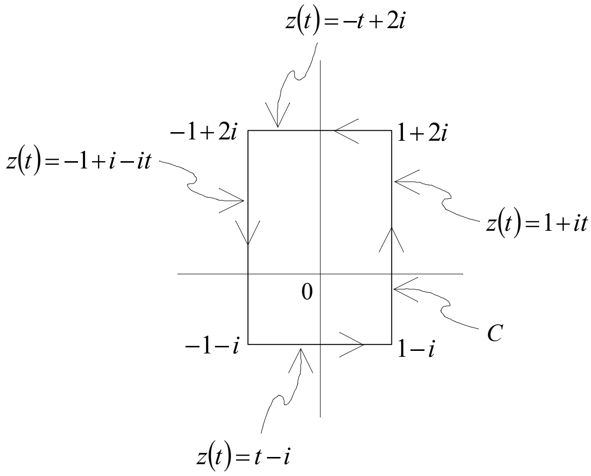
Examples: (i) $\Omega=\mathbb{C}\setminus\{0\}$, $f(z)=1/z$ and $C=\{z\mid |z|=R\}$.

$$\int_C f\,dz=\int_0^{2\pi}\frac{1}{Re^{it}}iRe^{it}\,dt=2\pi i$$

$$\int_C f\,|dz|=\int_0^{2\pi}\frac{1}{Re^{it}}R\,dt=0$$

Note that both of these integrals are independent of R.

(ii) $\Omega=\mathbb{C}\setminus\{0\}$, $f(z)=1/\bar{z}$ and C is the rectangle with vertices at $-1-i$, $1-i$, $1+2i$ and $-1+2i$.



$$\int_C \frac{dz}{\bar{z}} = \int_{-1}^{1} \frac{dt}{t+i} + \int_{-1}^{2} \frac{i\,dt}{1-it} + \int_{-1}^{1} \frac{-dt}{-t-2i} + \int_{-1}^{2} \frac{-i\,dt}{-1-i+it}$$

$$\int_{-1}^{1} \frac{dt}{t+i} = \log\frac{1+i}{1-i}$$

$$\int_{-1}^{2} \frac{i\,dt}{1-it} = \log\frac{1+i}{1-2i}$$

$$\vdots$$

$$\begin{aligned}\int_C \frac{dz}{\bar{z}} &= \log\left(\frac{1+i}{1-i} \times \frac{1+i}{1-2i} \times \frac{-1-2i}{1-2i} \times \frac{-1-2i}{-1+i}\right) \\ &= \log\frac{-3+4i}{3+4i} \\ &= i(\pi - 2\arccos(3/5)) \\ &= 2i\arcsin(4/5)\end{aligned}$$

Interpretation: We identify $f = u + iv$ with the vector field $\mathbf{f} = (u, -v)$.

$$\begin{aligned}\int_a^b f(z(t))\frac{dz}{dt}\,dt &= \int_a^b (u+iv)\left(\frac{dx}{dt} + i\frac{dy}{dt}\right)dt \\ &= \int_a^b \left(u\frac{dx}{dt} - v\frac{dy}{dt}\right)dt + i\int_a^b \left(u\frac{dy}{dt} + v\frac{dx}{dt}\right)dt \\ &= \int_a^b (u,-v)\cdot\left(\frac{dx}{dt}, \frac{dy}{dt}\right)dt - i\int_a^b (u,-v)\cdot\left(-\frac{dy}{dt}, \frac{dx}{dt}\right)dt\end{aligned}$$

$$\begin{aligned}\int_C f\,dz &= \int_C \mathbf{f}\cdot d\mathbf{r} + i\int_C \mathbf{f}\cdot\mathbf{n}\,ds \\ &= \begin{pmatrix}\text{tangential line integral} \\ \text{of } \mathbf{f} \text{ along } C\end{pmatrix} + i\begin{pmatrix}\text{flux of } \mathbf{f} \\ \text{across } C\end{pmatrix}\end{aligned}$$

If C is a closed curve then

$$\int_C f\,dz = \begin{pmatrix}\text{circulation of} \\ \mathbf{f} \text{ around } C\end{pmatrix} + i\begin{pmatrix}\text{flux of } \mathbf{f} \\ \text{across } C\end{pmatrix}$$

This is the link between complex analysis and 2D vector analysis.

4.5: CAUCHY'S THEOREM

Theorem: *Let $f:\Omega\subset\mathbb{C}\to\mathbb{C}$ be holomorphic and suppose $C\subset\Omega$ is a closed curve that bounds a region $E\subset\Omega$. Then $\int_C f\,dz=0$.*

Proof: Since f is holomorphic, by the Cauchy-Riemann equations,

$$\operatorname{curl}\mathbf{f}=\frac{\partial}{\partial x}(-v)-\frac{\partial}{\partial y}(u)=0$$
$$\nabla\cdot\mathbf{f}=\frac{\partial}{\partial x}(u)+\frac{\partial}{\partial y}(-v)=0$$

Thus,

$$\begin{aligned}\int_C f\,dz &= \int_{\partial E}\mathbf{f}\cdot d\mathbf{r}+i\int_{\partial E}\mathbf{f}\cdot\mathbf{n}\,ds\\ &= \underbrace{\iint_E \operatorname{curl}\mathbf{f}\,dxdy}_{\text{by Green}}+i\underbrace{\iint_E\nabla\cdot\mathbf{f}\,dxdy}_{\text{by Gauss}}\\ &= 0+i0=0\end{aligned}$$

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Remarks: (i) The geometric interpretation of the Cauchy-Riemann equations is that $\mathbf{f}$ is both curl-free and divergence-free i.e. irrotational and solenoidal.

(ii) We have already seen that $\int_{\{|z|=1\}}\frac{dz}{z}=2\pi i$. This does not contradict Cauchy's Theorem since $z\mapsto 1/z$ is defined on $\mathbb{C}\setminus\{0\}$ and $\{|z|=1\}$ does not bound a region in $\mathbb{C}\setminus\{0\}$.

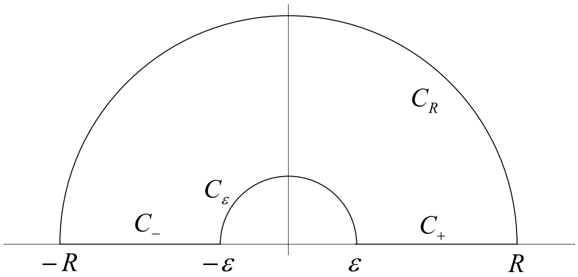
(iii) This is not the usual proof of Cauchy's Theorem, which does not require the continuity of f', as we have assumed in applying Green's and Gauss's Theorems.

Lemma: (The Estimation Lemma) *For any $f:\Omega\subset\mathbb{C}\to\mathbb{C}$ and $C\subset\Omega$ a curve parameterized by $z(t)$, $\left|\int_C f\,dz\right|\le\int_C|f||dz|$.*

(Compare this with the Triangle Inequality.)

Proposition: $\int_0^\infty\frac{\sin x}{x}\,dx=\frac{\pi}{2}$.

Proof: Let $f(z)=\frac{1}{z}e^{iz}$ on $\mathbb{C}\setminus\{0\}$.



Let $C = C_+ \cup C_- \cup C_R \cup C_\varepsilon$ so that $C = \partial E$ where $E = \{z \in \mathbb{C} \mid \varepsilon < |z| < R, \operatorname{Im} z > 0\}$. Note that $E \subset \mathbb{C} \setminus \{0\}$. Hence, by Cauchy's Theorem,

$$\int_C f\,dz = \int_{C_+} f\,dz + \int_{C_-} f\,dz + \int_{C_R} f\,dz + \int_{C_\varepsilon} f\,dz = 0\,.$$

$$\begin{aligned}\int_{C_+} f\,dz + \int_{C_-} f\,dz &= \int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,dx + \int_\varepsilon^R \frac{e^{ix}}{x}\,dx \\ &= 2i\int_\varepsilon^R \frac{\sin x}{x}\,dx\end{aligned}$$

So, $\int_0^\infty \frac{\sin x}{x}\,dx = -\frac{1}{2i}\left(\lim_{\varepsilon\to 0}\int_{C_\varepsilon} f\,dz + \lim_{R\to\infty}\int_{C_R} f\,dz\right)$.

$$\begin{aligned}\left|\int_{C_R} f\,dz\right| &\le \int_0^\pi \frac{\left|e^{i(R\cos t + iR\sin t)}\right|}{\left|Re^{it}\right|} R\,dt \\ &= \int_0^\pi e^{-R\sin t}\,dt \\ &= 2\int_0^{\pi/2} e^{-R\sin t}\,dt\end{aligned}$$

Now, for $0 \le t \le \pi/2$, $0 \le 2t/\pi \le \sin t$. So,

$$\begin{aligned}\int_0^{\pi/2} e^{-R\sin t}\,dt &\le \int_0^{\pi/2} e^{-2Rt/\pi}\,dt \\ &= -\frac{\pi}{2R} e^{-\frac{\pi}{2R}t}\Bigg|_0^{\pi/2}\end{aligned}$$

We can see that $\lim_{R\to\infty}\int_{C_R} f\,dz = 0$, but what is $\lim_{\varepsilon\to 0}\int_{C_\varepsilon} f\,dz$? Take $z(t) = \varepsilon e^{it}$ for $t \in [0,\pi]$ and consider

$$\begin{aligned}\int_{C_\varepsilon} \frac{dz}{z} &= \int_\pi^0 \frac{1}{z(t)}\frac{dz}{dt}\,dt \\ &= \int_\pi^0 \frac{1}{\varepsilon e^{it}} i\varepsilon e^{it}\,dt \\ &= -\pi i\end{aligned}$$

We prove this guess by proving that $\lim_{\varepsilon\to 0}\int_{C_\varepsilon}\frac{e^{iz}-1}{z}dz=0$. By the continuity of $z\mapsto e^{iz}$, given $\eta>0$ $\exists\delta>0$ such that if $0<\varepsilon<\delta$ and $|z|=\varepsilon$ then $\left|e^{iz}-e^{i0}\right|<\eta$. Then $0<\varepsilon<\delta$ and the Estimation Lemma together imply that

$$\begin{aligned}\left|\int_{C_\varepsilon}\frac{e^{iz}-1}{z}dz\right| &\le \int_{C_\varepsilon}\frac{\left|e^{iz}-1\right|}{|z|}|dz| \\ &\le \int_0^\pi \frac{\eta}{\varepsilon}\varepsilon\,dt \\ &= \pi\eta\end{aligned}$$

So $\lim_{\varepsilon\to 0}\int_{C_\varepsilon}\frac{e^{iz}-1}{z}dz=0$, and so $\lim_{\varepsilon\to 0}\int_{C_\varepsilon}\frac{e^{iz}}{z}dz=\lim_{\varepsilon\to 0}\int_{C_\varepsilon}\frac{dz}{z}=-i\pi$. Hence,

$$\int_0^\infty \frac{\sin x}{x}dx=-\frac{1}{2i}(-i\pi)=\frac{\pi}{2}.$$

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4.6: CAUCHY'S INTEGRAL FORMULA

Definitions: $\Omega\subset\mathbb{C}$ is *open* if $\forall z\in\Omega$ $\exists\delta>0$ such that $\{w\in\mathbb{C}\mid |w-z|<\delta\}\subset\Omega$.

$$\begin{aligned}\mathbb{B}(z,r)&=\{w\in\mathbb{C}\mid |w-z|<\delta\},\\ \overline{\mathbb{B}(z,r)}&=\{w\in\mathbb{C}\mid |w-z|\le\delta\},\\ \partial\mathbb{B}(z,r)&=\{w\in\mathbb{C}\mid |w-z|=\delta\}.\end{aligned}$$

Theorem: *Let* $\Omega\subset\mathbb{C}$ *be open and let* $f:\Omega\subset\mathbb{C}\to\mathbb{C}$ *be holomorphic. Suppose* $\overline{\mathbb{B}(a,R)}\subset\Omega$. *Then* $\forall w\in\mathbb{B}(a,R)$ *we have*

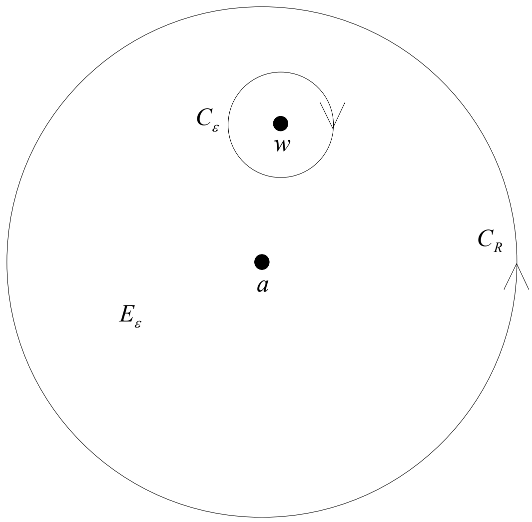
$$f(w)=\frac{1}{2\pi i}\int_{\{|z-a|=R\}}\frac{f(z)}{z-w}dz.$$

Remarks: (i) 'Action at a distance.' The values of f on $\{|z-a|=R\}$ determine the values of f at all points in the interior of this circle.

(ii) Cauchy's Integral Formula is, in fact, valid for any contour (curve) that winds once around w and that bounds a region in Ω. This will be made more precise in the Third Year course *Complex Analysis*.

Proof: Let $C_R=\{z\mid |z-a|=R\}=\partial\mathbb{B}(a,R)$. For $0<\varepsilon<R-|w-a|$ let

$$E_\varepsilon=\{z\mid |z-a|<R, |z-w|>\varepsilon\}=\mathbb{B}(a,R)\setminus\overline{\mathbb{B}(w,\varepsilon)}.$$



We now focus our attention on $\overline{E_\varepsilon}$, on which we define $g(z) = \dfrac{f(z)}{z-w}$. We can see that g is holomorphic on E_ε. By Cauchy's Theorem,

$$0 = \int_{\partial E_\varepsilon} g(z)dz = \int_{C_R} \frac{f(z)}{z-w} dz - \int_{C_\varepsilon} \frac{f(z)}{z-w} dz$$

where $C_\varepsilon = \partial\mathbb{B}(w, \varepsilon)$.

Our intuition is that for small ε, $\int_{C_\varepsilon} \frac{f(z)}{z-w} dz \approx \int_{C_\varepsilon} \frac{f(w)}{z-w} dz = f(w)\int_{C_\varepsilon} \frac{dz}{z-w} = 2\pi i f(w)$.

More precisely, by the continuity of f at w, given $\eta > 0$ $\exists \delta > 0$ such that if $0 < \varepsilon < \delta$ and $|z-w| = \varepsilon$ then $|f(z) - f(w)| < \eta$. Hence, by the Estimation Lemma,

$$\left| \int_{C_\varepsilon} \frac{f(z) - f(w)}{z-w} dz \right| \le \eta \int_{C_\varepsilon} \frac{|dz|}{\varepsilon} = \frac{\eta}{\varepsilon} 2\pi\varepsilon = 2\pi\eta$$

$$\Rightarrow \qquad \lim_{\varepsilon \to 0} \int_{C_\varepsilon} \frac{f(z) - f(w)}{z-w} dz = 0$$

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