CHAPTER 5: POWER SERIES EXPANSIONS OF HOLOMORPHIC FUNCTIONS

5.1: TAYLOR'S THEOREM

Theorem: *Let* $f:\Omega\subset\mathbb{C}\to\mathbb{C}$ *be holomorphic and suppose that* $\overline{\mathbb{B}(a,R)}\subset\Omega$. *For each* $n\geq 0$ *define* a_n *by*

$$a_n=\frac{1}{2\pi i}\int_{\{|z-a|=R\}}\frac{f(z)}{(z-a)^{n+1}}dz.$$

Then the power series $\sum_{n=0}^{\infty}a_n(w-a)^n$ *converges for all* $w\in\mathbb{B}(a,R)$ *and we have*

$$f(w)=\sum_{n=0}^{\infty}a_n(w-a)^n.$$

In particular, f *is infinitely complex differentiable and* $f^{(n)}(a)=n!a_n$.

Proof: See handout.

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5.2: BINOMIAL SERIES

Recall that $\mathrm{Log}:\mathbb{C}\setminus(-\infty,0]\to\mathbb{C}$ is defined by

$$\mathrm{Log}(z)=\log(r)+i\theta$$
$$\text{for } z=re^{i\theta},\ r\geq 0,\ -\pi<\theta\leq\pi.$$

Log is differentiable and $\frac{d}{dz}\mathrm{Log}(z)=\frac{1}{z}$. This follows from the formula for the differentiation of inverse functions:

$$\left(f^{-1}\right)'(z)=1/f'\left(f^{-1}(z)\right),$$
$$\mathrm{Log}=\exp^{-1},\ \exp'=\exp.$$

$$(1+z)^a=\exp(a\,\mathrm{Log}(1+z))\text{ for } z\in\mathbb{C}\setminus(-\infty,0].$$

So for $z\in\mathbb{C}\setminus(-\infty,0]$ we have

$$\frac{d}{dz}(1+z)^a=\frac{d}{dz}\exp(a\,\mathrm{Log}(1+z))=\frac{a}{1+z}=a(1+z)^{a-1}.$$

If follows that $\frac{d^n}{dz^n}(1+z)^a = a(a-1)\dots(a-n+1)(1+z)^{a-n}$. So, if we let $f(z)=(1+z)^a$ then $f^{(n)}(0)=a(a-1)\dots(a-n+1)$. Observe that f is holomorphic on the unit disc $\mathbb{B}(0,1)$. Hence, by Taylor's Theorem, if $|z|<1$ we have

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)z^n}{n!}$$

$$(1+z)^a = 1+az+\frac{a(a-1)}{2!}z^2+\frac{a(a-1)(a-2)}{3!}z^3+\dots+\frac{a(a-1)\dots(a-n+1)}{n!}z^n+\dots$$

This is our *binomial series*. The radius of convergence R is given by

$$\begin{aligned} R &= \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \\ &= \lim_{n\to\infty}\left|\frac{a(a-1)\dots(a-n+1)}{n!}\frac{(n+1)!}{a(a-1)\dots(a-n)}\right| \\ &= \lim_{n\to\infty}\left|\frac{n+1}{a-n}\right| \\ &= 1 \end{aligned}$$

for $a\notin\mathbb{N}$. This is consistent with the fact that f is not defined (and so not holomorphic) on $\mathbb{B}(0,\rho)$ for $\rho>1$.

5.3: EXAMPLES OF TAYLOR SERIES

(i)

$$\begin{aligned} f(z) &= \frac{1}{\sqrt{4+9z^2}} \\ &= 4^{-1/2}\left(1+9z^2/4\right)^{-1/2} \\ &= \frac{1}{2}\left(1+(-1/2)\left(\frac{9z^2}{4}\right)+\frac{(-1/2)(-3/2)}{2!}\left(\frac{9z^2}{4}\right)^2+\dots+\frac{(-1/2)\dots(1/2-k)}{k!}\left(\frac{9z^2}{4}\right)^k+\dots\right) \end{aligned}$$

Let us simplify the general term:

$$\begin{aligned} \underbrace{\frac{(-1)}{2}\frac{(-3)}{2}\dots\frac{(1-2k)}{2}}_{k \text{ factors}}\frac{9^k z^{2k}}{k!4^k} &= \frac{(-1)^k}{2^k}\frac{1\times2\times3\times\dots\times(2k-1)\times 2k}{2\times4\times6\times\dots\times 2k}\frac{9^k z^{2k}}{k!4^k} \\ &= \frac{(-1)^k(2k)!9^k z^{2k}}{(k!)^2 16^k} \end{aligned}$$

Therefore, the Taylor series of f about $z=0$ is

$$f(z)=\frac{1}{2}\left(1+\sum_{k=1}^{\infty}\frac{(-1)^k(2k)!9^k}{(k!)^2 16^k}z^{2k}\right)=\sum_{n=0}^{\infty}a_n z^n$$

where $a_0=1/2$, $a_{2k}=\dfrac{(-1)^k(2k)!9^k}{2(k!)^2 16^k}$, $a_{2k+1}=0$.

As for the radius of convergence, we have applied the binomial series to $\left(1+9z^2/4\right)^{-1/2}$, valid when $\left|9z^2/4\right|<1$ i.e. $|z|<2/3$, so $R=2/3$.

(ii) Let $p(z)$ be a polynomial and suppose $p(0)=1$. What is the Taylor series of $1/p(z)$ about $z=0$? For example, let $p(z)=1+4z+z^2$:

$$\begin{aligned}\frac{1}{p(z)}&=1-\left(4z+z^2\right)+\left(4z+z^2\right)^2-\left(4z+z^2\right)^3+\ldots+(-1)^k\left(4z+z^2\right)^k\\&=1-4z-z^2+16z^2+8z^3+z^4-64z^3-48z^4-12z^5-z^6+\ldots\\&=1-4z+15z^2-56z^3+O\left(z^4\right)\end{aligned}$$

This method works for any $1/g(z)$ when g is holomorphic and $g(0)\neq 0$. For example, let $g(z)=\cos(z)$:

$$\begin{aligned}\frac{1}{\cos(z)}&=\frac{1}{1+\left(-z^2/2+z^4/4!-z^6/6!+\ldots\right)}\\&=1-\left(-\frac{z^2}{2!}+\frac{z^4}{4!}-\ldots\right)+\left(-\frac{z^2}{2}+\frac{z^4}{4!}-\ldots\right)^2-\ldots\\&=1+\left(z^2/2-z^4/4!+z^6/6!-\ldots\right)+\left(z^4/4-z^6/6!+\ldots\right)+\ldots\\&=1+\frac{z^2}{2}+\frac{5}{24}z^4+O\left(z^6\right)\end{aligned}$$

(iii) The Taylor series of e^z about $z=i$:

$$e^z=e^i e^{z-i}=\sum_{n=0}^{\infty}\frac{e^i}{n!}(z-i)^n$$

The expression $a_n=\dfrac{1}{2\pi i}\displaystyle\int_{\{|z-a|=R\}}\frac{f(z)}{(z-a)^{n+1}}dz$ turns out to be useful for proving many theorems of complex analysis:

Theorem: (Liouville's Theorem) *If $f:\mathbb{C}\to\mathbb{C}$ is holomorphic and bounded then f is constant.*

Proof: By Taylor's Theorem,

$$f'(a) = \frac{1}{2\pi i}\int_{\{|z-a|=R\}} \frac{f(z)}{(z-a)^2}\,dz.$$

By definition, f is bounded if and only if $\exists M > 0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. By the Estimation Lemma,

$$\begin{aligned}|f'(a)| &\leq \frac{M}{2\pi}\int_{\{|z-a|=R\}} \frac{|dz|}{R^2} \\ &= \frac{M}{2\pi R^2} 2\pi R \\ &= \frac{M}{R} \to 0 \text{ as } R \to \infty\end{aligned}$$

Hence, $f'(a) = 0$ for all $a \in \mathbb{C}$, and so f is constant.

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5.3: LAURENT SERIES

Definition: A *Laurent series* is a series of the form $\sum_{n\in\mathbb{Z}} a_n z^n$.

It is often convenient to split a Laurent series into two parts: $\sum_{n=1}^{\infty} a_{-n} z^{-n}$ and $\sum_{n=0}^{\infty} a_n z^n$. If we set $z = 1/\zeta$ in the first sum we get the series $\sum_{n=1}^{\infty} a_{-n}\zeta^n$; let this series have radius of convergence ρ. Then $\sum_{n=1}^{\infty} a_{-n} z^{-n}$ converges when $|z| > R_1$ where $R_1 = 1/\rho$. Let the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ be R_2. If $R_1 < R_2$ then $\sum_{n\in\mathbb{Z}} a_n z^n$ converges on the annulus for which $R_1 < |z| < R_2$.

Theorem: (Laurent's Theorem) *Suppose* $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ *is holomorphic on the annulus* $\mathbb{A}(a, R_1, R_2) = \{z \in \mathbb{C} \mid R_1 < |z-a| < R_2\} \subset \Omega$. *For* $\rho \in (R_1, R_2)$ *and* $n \in \mathbb{Z}$ *let*

$$a_n = \frac{1}{2\pi i}\int_{\{|z-a|=\rho\}} \frac{f(z)}{(z-a)^{n+1}}\,dz.$$

Then the Laurent series $\sum_{n\in\mathbb{Z}} a_n (w-a)^n$ *converges to* $f(w)$ *for all* $w \in \mathbb{A}(a, R_1, R_2)$.

Remarks: (i) It makes no sense now to claim $f^{(n)}(a) = n!a_n$ because f may not actually be defined at a.

(ii) Laurent's Theorem tacitly asserts that the value of a_n is independent of the value of ρ as long as $R_1 < \rho < R_2$.

Proof: Similar to Taylor's Theorem.

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5.4: RESIDUE THEORY

Definition: The coefficient a_{-1} in the Laurent expansion of f about a is called the *residue* of f at a and is denoted $\operatorname{Res}(f;a)$.

Observe that $a_{-1}=\frac{1}{2\pi i}\int_{\{|z-a|=\rho\}}f(z)dz$. Thus, a_{-1} is what is 'left over' in the evaluation of $\frac{1}{2\pi i}\int_{\{|z-a|=\rho\}}f(z)dz$, thereby causing Cauchy's Theorem to fail on the disc $\mathbb{B}(a,R_2)$.

Definitions: We say that f has a *point singularity* at $a\in\mathbb{C}$ if f is holomorphic on the punctured disc $\mathbb{B}(a,R)\setminus\{a\}=\{z\in\mathbb{C}\mid 0<|z-a|<R\}$. This happens when $\sum_{n=1}^{\infty}a_{-n}\zeta^n$ converges for all $\zeta\in\mathbb{C}$ in the Laurent expansion of f; $\zeta=1/(w-a)$.

The simplest type of point singularity occurs when, in the Laurent expansion of f, there exists an $N\in\mathbb{N}$ such that $a_{-N}\neq 0$ but $n>N\Rightarrow a_{-n}=0$. In this case we say that f is *meromorphic* on the disc $\mathbb{B}(a,R)$ and that it has a *pole of order* N at a. A pole of order 1 is called a *simple pole*, a pole of order 2 a *double pole*, etc.

If for all $N\in\mathbb{N}$ $\exists n>N$ such that $a_{-n}\neq 0$ we say that f has an *essential singularity* at a. For example, $\exp(1/z)$ has an essential singularity at 0.

Theorem: (The Residue Theorem) *Let* Ω *be open in* $\mathbb{C}$ *and* $f:\Omega\to\mathbb{C}$ *holomorphic on* $\Omega\setminus\{z_1,\dots,z_k\}$. *Let* $E\subset\Omega$ *be a region such that* $\partial E\subset\Omega$ *and* $\{z_1,\dots,z_k\}\subset E$. *Then*

$$\frac{1}{2\pi i}\int_{\partial E}f(z)dz=\sum_{j=1}^{k}\operatorname{Res}(f;z_j).$$

Proof: Let $\overline{\mathbb{B}_1},\dots,\overline{\mathbb{B}_k}$ be pairwise disjoint closed discs centred on $z_1,\dots,z_k$ such that each $\overline{\mathbb{B}_j}\subset E$. Then f is holomorphic on $E\setminus\bigcup_{j=1}^{k}\overline{\mathbb{B}_j}$ and so, by Cauchy's Theorem,

$$\frac{1}{2\pi i}\int_{\partial\left(E\setminus\bigcup\overline{\mathbb{B}_j}\right)}f(z)dz=0$$

$$\Rightarrow\qquad \frac{1}{2\pi i}\int_{\partial E}f(z)dz=\frac{1}{2\pi i}\sum_{j}\int_{\partial\mathbb{B}_j}f(z)dz=\sum_{j}\frac{1}{2\pi i}\int_{\partial\mathbb{B}_j}f(z)dz$$

$$\Rightarrow\qquad \frac{1}{2\pi i}\int_{\partial E}f(z)dz=\sum_{j=1}^{k}\operatorname{Res}(f;z_j)$$

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Theorem: (Picard's Theorem) If f has an essential singularity at 0 then for all $\varepsilon > 0$, $f|_{\mathbb{B}(0,\varepsilon)\setminus\{0\}}$ assumes every value infinitely often, except at most one value.

5.5: CALCULATING AND USING RESIDUES

Proposition: *Suppose* g *is a holomorphic function on* $\mathbb{B}(a,\rho)$ *for* $\rho > 0$ *satisfying* $g(a) \neq 0$. *Fix* $N \in \mathbb{N}$ *and define* f *by* $f(z) = g(z)/(z-a)^N$. *Then*

(i) f *has a pole of order* N *at* a;

(ii) $\operatorname{Res}(f;a) = a_{N-1}$, *where* $g(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$.

Proof:

$$g(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + a_{N-1}(z-a)^{N-1} + a_N(z-a)^N + \ldots$$

$$f(z) = \underbrace{\frac{a_0}{(z-a)^N}}_{\text{pole of order } N} + \ldots + \underbrace{\frac{a_{N-1}}{(z-a)}}_{\substack{\text{contribution}\\ \text{to residue}}} + a_N + a_{N+1}(z-a) + \ldots$$

provided $g(a) \neq 0$.

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Example: Let $f(z) = \frac{1}{1+z^4}$. Classify all the poles of f, i.e. locate them, state their order and calculate the residue of f at each.

$$\begin{aligned} z^4 + 1 &= (z^2 + i)(z^2 - i) \\ &= (z + i\sqrt{i})(z - i\sqrt{i})(z - \sqrt{i})(z + \sqrt{i}) \end{aligned}$$

So f has poles at $\pm\sqrt{i}, \pm i\sqrt{i}$. Consider the pole at $\sqrt{i}$:

$$\begin{gathered} g(z) = 1/(z + i\sqrt{i})(z - i\sqrt{i})(z + \sqrt{i}) \\ f(z) = g(z)/(z - \sqrt{i}) \\ g(\sqrt{i}) = 1/4i\sqrt{i} \end{gathered}$$

So f has a simple pole at $\sqrt{i}$ and $\operatorname{Res}(f;\sqrt{i}) = g(\sqrt{i}) = \sqrt{i}/4$. Similarly we see that all the other three poles are simple and have residues

$$\begin{gathered} \operatorname{Res}(f; -\sqrt{i}) = \sqrt{i}/4, \\ \operatorname{Res}(f; i\sqrt{i}) = -i\sqrt{i}/4, \\ \operatorname{Res}(f; -i\sqrt{i}) = i\sqrt{i}/4. \end{gathered}$$

We can also use residues to calculate integrals:

Example: Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$ by considering $\int_{C_R} \frac{dz}{1+z^4}$, where $C_R = (-R, R) \cup C_R^+$, $R > 1$.

$$\frac{1}{2\pi i}\int_{C_R} \frac{dz}{1+z^4} = \operatorname{Res}\left(f; \sqrt{i}\right) + \operatorname{Res}\left(f; i\sqrt{i}\right)$$
$$= -\sqrt{i}/4 - i\sqrt{i}/4$$

By the Estimation Lemma,

$$\left|\int_{C_R^+} \frac{dz}{1+z^4}\right| \le \frac{\pi R}{R^4+1} \to 0 \text{ as } R \to \infty .$$

So, $\int_{C_R} \frac{dz}{1+z^4} = \int_{-R}^{R} \frac{dx}{1+x^4} + \int_{C_R^+} \frac{dz}{1+z^4}$. As $R \to \infty$,

$$-\frac{2\pi i\sqrt{i}(1+i)}{4} = \int_{-\infty}^{\infty} \frac{dx}{1+x^4}$$
$$= -\frac{\pi}{2}\frac{i(1+i)(1+i)}{\sqrt{2}}$$
$$= \pi/2$$

5.6: THE FUNDAMENTAL THEOREM OF ALGEBRA

Theorem: *Let* p *be a polynomial of degree* $n > 0$ *over* $\mathbb{C}$*, i.e.*

$$p(z) = a_n z^n + \ldots + a_1 z + a_0$$

with $a_i \in \mathbb{C}$ *and* $a_n \neq 0$. *Then there are* $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ *so that*

$$p(z) = a_n (z - \lambda_1) \ldots (z - \lambda_n).$$

Proof: The proof has three steps, of which Step 1 is the most important. In Step 1 we establish the existence of a $\lambda_1 \in \mathbb{C}$ such that $p(\lambda_1) = 0$. In Step 2 we use Taylor's Theorem to show that $p(z) = (z - \lambda_1) p_1(z)$, where p_1 is a polynomial of degree $n-1$. In Step 3 we repeat Steps 1 and 2 $n-1$ more times.

Step 1: Suppose $p(z) \neq 0$ for all $z \in \mathbb{C}$. Let $f(z) = 1/p(z)$. Then f is holomorphic on $\mathbb{C}$ and $\lim_{|z|\to\infty} f(z) = 0$:

$$f(z) = \frac{1}{z^n}\frac{1}{a_n + \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n}}$$

and so for $|z| > \frac{n-1}{|a_n|}\left(|a_{n-1}| + \ldots + |a_0|\right)$ we have

$$|f(z)| < \frac{1}{|a_n z^n|} \frac{1}{1 - \frac{n}{n+1}} = \frac{n+1}{|a_n z^n|} \to 0 \text{ as } |z| \to \infty .$$

In particular, f is bounded and so, by Liouville's Theorem, constant (in fact, this constant is 0 since $\lim_{|z|\to\infty} f(z) = 0$). But p is not constant and so neither is f. This contradicts the assumption that $p(z) \neq 0$ for all $z \in \mathbb{C}$ and so there is a $\lambda_1 \in \mathbb{C}$ such that $p(\lambda_1) = 0$.

Step 2: Now apply Taylor's Theorem to p at λ_1:

$$p(z) = p(\lambda_1) + (z - \lambda_1)p'(\lambda_1) + (z - \lambda_1)^2 p''(\lambda_1) + \ldots + \frac{1}{n!}(z - \lambda_1)^n p^{(n)}(\lambda_1) = (z - \lambda_1)p_1(z)$$

where p_1 is a polynomial of degree $n-1$.

Step 3: If $n = 1$ we are done. If $n > 1$ we apply Steps 1 and 2 to the polynomial p_1 to obtain the existence of $\lambda_2 \in \mathbb{C}$ and a polynomial p_2 of degree $n-2$ such that $p_1(\lambda_2) = 0$ and $p_1(z) = (z - \lambda_2)p_2(z)$. It follows that $p(z) = (z - \lambda_1)(z - \lambda_2)p_2(z)$. Carrying out Steps 1 and 2 a total of n times yields

$$p(z) = (z - \lambda_1)\ldots(z - \lambda_n)p_n(z)$$

where p_n is a polynomial of degree 0 i.e. a constant. Equating coefficients yields $p_n(z) = a_n$, establishing the result.

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