MA244 ANALYSIS III
Revision Guide

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1 Uniform Distance and Continuity

Definition 1.1. If $f, g : [a, b] \to \mathbb{R}$ are both bounded then the *uniform distance*[1] between them is defined to be

$$d(f, g) := \sup_{x \in [a,b]} |f(x) - g(x)|.$$

Definition 1.2. A function $f : E \to \mathbb{R}$ is *uniformly continuous* on $E \subseteq \mathbb{R}$ if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } x, y \in E \text{ and } |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Uniform continuity is a strictly stronger condition on a function than ordinary continuity at a point: every uniformly continuous function is continuous, but a continuous function may fail to be uniformly continuous. For example, take $f : (0, 1) \to \mathbb{R}$, $f(x) := \frac{1}{x}$. However,

Theorem 1.3. *A continuous function on a closed, bounded interval*[2] *is uniformly continuous.*

2 Step Functions

Our intuitive notion of "integral" is that of "area under the graph": the easiest areas to calculate are those that can be decomposed into a finite number of rectangles. This leads us to the notion of a step function.

Definition 2.1. A *partition* of an interval $[a, b]$ is a finite set $P = \{p_j\}_{j=0}^{k}$ such that

$$a = p_0 < p_1 < \cdots < p_{k-1} < p_k = b.$$

Definition 2.2. Given two partitions P and Q of the same interval $[a, b]$, we can take their *common refinement*, $P \vee Q$, defined to be $P \cup Q$ with the points re-numbered in order.

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[1] This $d(\cdot, \cdot)$ is an example of a *metric*. See MA222 METRIC SPACES.

[2] Later, this will be known as being *compact*.

Definitions 2.3. A function $\phi : [a,b] \to \mathbb{R}$ is a *step function* if there exists a partition $P = \{p_j\}_{j=0}^k$ of $[a,b]$ such that ϕ is constant on (p_{j-1}, p_j) for $j = 1, \ldots, k$. Unless otherwise stated we shall take $\phi(x) = c_j$ for $x \in (p_{j-1}, p_j)$. Write

$$\mathscr{S}([a,b];\mathbb{R}) := \{\phi : [a,b] \to \mathbb{R} | \phi \text{ is a step function}\}.$$

Proposition 2.4. (Properties of step functions.)

(i) A step function $\phi \in \mathscr{S}([a,b];\mathbb{R})$ with partition $\{p_j\}_{j=0}^k$ is bounded and can take at most $2k+1$ values.

(ii) $\mathscr{S}([a,b];\mathbb{R})$ is an infinite-dimensional vector space over $\mathbb{R}$.

(iii) If $\phi, \psi \in \mathscr{S}([a,b];\mathbb{R})$ then $\phi \cdot \psi \in \mathscr{S}([a,b];\mathbb{R})$. (Recall that $(\phi \cdot \psi)(x) := \phi(x)\psi(x)$.)

We define the integral of a step function in the obvious way, as the sum of the areas of the rectangular strips between its graph and the axis:

Definition 2.5. If $\phi \in \mathscr{S}([a,b];\mathbb{R})$ with partition $\{p_j\}_{j=0}^k$ and values $\{c_j\}_{j=0}^k$ as above, then the *integral* of ϕ is defined by

$$\int_a^b \phi := \sum_{j=1}^k c_j(p_j - p_{j-1}).$$

Proposition 2.6. (Properties of the integral.) *Let $\phi \in \mathscr{S}([a,b];\mathbb{R})$.*

(i) $\int_a^b \phi$ is independent of the choice of partition for ϕ.

(ii) If $a \le u \le v \le w \le b$ then $\int_u^w \phi = \int_u^v \phi + \int_v^w \phi$.

(iii) The integral function $\int_a^b : \mathscr{S}([a,b];\mathbb{R}) \to \mathbb{R}$ is a linear map of vector spaces.

(iv) If $m \le \phi(x) \le M$ for all $x \in [a,b]$, then $m(b-a) \le \int_a^b \phi \le M(b-a)$.

Proof. (i) Suppose that P and Q are two suitable partitions: take R to be their common refinement. It suffices to show that $\int_a^b \phi$ is the same for P and R. Moreover, it suffices to show the addition of one point to P does not change $\int_a^b \phi$. Take $p_{j-1} < r < p_j$. Applying the definition of $\int_a^b \phi$ gives the required result. □

The following result, the Fundamental Theorem of Calculus (FTC), is immediately obvious, almost trivial, for step functions. However, when we look at more general regulated functions we shall see that it is a powerful tool of analysis. Without it and its generalizations, it is not clear how we will go about computing integrals.

Theorem 2.7. (FTC for Step Functions.) *If $\phi \in \mathscr{S}([a,b];\mathbb{R})$ with partition $\{p_j\}_{j=0}^k$ then $\Phi : [a,b] \to \mathbb{R} : x \mapsto \int_a^x \phi$ is differentiable on $\bigcup_{j=1}^k (p_{j-1}, p_j)$ with $\Phi'(x) = \phi(x)$.*

3 Regulated Functions

Definitions 3.1. $f : [a,b] \to \mathbb{R}$ is *regulated* if for all $\varepsilon > 0$ there is a $\phi \in \mathscr{S}([a,b];\mathbb{R})$ such that $d(f,\phi) < \varepsilon$.[3] Equivalently, f is regulated if there is a sequence of step functions $(\phi_n)_{n=1}^{\infty} \subset \mathscr{R}([a,b];\mathbb{R})$ such that $d(f,\phi_n) \to 0$ as $n \to \infty$. Write

$$\mathscr{R}([a,b];\mathbb{R}) := \{f : [a,b] \to \mathbb{R} | f \text{ is a regulated function}\}.$$

It is important to understand why these two definitions are equivalent, even though they are phrased slightly differently, and how to translate between them. In some proofs, one works with a single step function ϕ that is ε-close to f; in other proofs, one may need a whole sequence of step functions ϕ_n that converge to f uniformly.

Proposition 3.2. (Properties of regulated functions.)

(i) A regulated function $f \in \mathscr{R}([a,b];\mathbb{R})$ is bounded.

(ii) The space of continuous real-valued functions on a closed, bounded $[a,b]$, $C^0([a,b];\mathbb{R}) \subsetneq \mathscr{R}([a,b];\mathbb{R})$, i.e. every continuous function is regulated, but not vice-versa.

(iii) $\mathscr{R}([a,b];\mathbb{R})$ is an infinite-dimensional vector space over $\mathbb{R}$.

(iv) If $f, g \in \mathscr{R}([a,b];\mathbb{R})$ then $f \cdot g \in \mathscr{R}([a,b];\mathbb{R})$.

Proof. (ii) Let $f \in C^0([a,b];\mathbb{R})$ and fix $\varepsilon > 0$. We construct $\phi \in \mathscr{S}([a,b];\mathbb{R})$ with $d(f,\phi) < \varepsilon$. Since f is continuous on a closed, bounded interval, it is uniformly continuous, so choose $\delta > 0$ so that for all $x, y \in [a,b]$, $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$. Let $\{p_j\}_{j=0}^{k}$ be any partition of $[a,b]$ so that $|p_j - p_{j-1}| < \delta$ for $j = 1, \dots, k$. Define $\phi \in \mathscr{S}([a,b];\mathbb{R})$ by $\phi(p_j) := f(p_j)$ and $\phi|_{(p_{j-1},p_j)}(x) := f((p_{j-1} + p_j)/2)$. (You might like to sketch the graphs of f and ϕ.) Clearly $d(f,\phi) \le \frac{\varepsilon}{2} < \varepsilon$, so f is regulated. $\square$

Theorem 3.3. *If $f \in \mathscr{R}([a,b];\mathbb{R})$ and $(\phi_n)_{n=1}^{\infty} \subset \mathscr{S}([a,b];\mathbb{R})$ with $d(f,\phi_n) \to 0$ as $n \to \infty$ then the sequence $\left(\int_a^b \phi_n\right)_{n=1}^{\infty}$ converges in $\mathbb{R}$. If $(\psi_n)_{n=1}^{\infty} \subset \mathscr{S}([a,b];\mathbb{R})$ also satisfies $d(f,\psi_n) \to 0$ as $n \to \infty$ then $\lim_{n\to\infty} \int_a^b \phi_n = \lim_{n\to\infty} \int_a^b \psi_n$.*

Proof. Fix $\varepsilon > 0$. $\exists N_1 \in \mathbb{N}$ so that $n \ge N_1 \implies d(f,\phi_n) < \varepsilon$. So for $m, n > N_1$ and $x \in [a,b]$ we have $f(x) - \varepsilon < \phi_m(x), \phi_n(x) < f(x) + \varepsilon$, so $|\phi_m(x) - \phi_n(x)| < 2\varepsilon$. Thus,

$$\left| \int_a^b \phi_m - \int_a^b \phi_n \right| = \left| \int_a^b (\phi_m - \phi_n) \right| < 2\varepsilon(b-a),$$

so $\left(\int_a^b \phi_n\right)_{n=1}^{\infty}$ is Cauchy, and hence convergent, in $\mathbb{R}$.

Since ψ_n converges uniformly to f, $\exists N_2 \in \mathbb{N}$ so that $n \ge N_2, x \in [a,b] \implies f(x) - \varepsilon < \psi_n < f(x) + \varepsilon$. So if $n \ge \max\{N_1, N_2\}$, $|\phi_n(x) - \psi_n(x)| < 2\varepsilon$, and so $\left| \int_a^b \phi_n - \int_a^b \psi_n \right| < 2\varepsilon(b-a)$. Taking limits as $n \to \infty$ gives $\lim_{n\to\infty} \int_a^b \phi_n = \lim_{n\to\infty} \int_a^b \psi_n$. $\square$

Definitions 3.4. For $f \in \mathscr{R}([a,b];\mathbb{R})$, the *(definite) integral* of f is $\int_a^b f := \lim_{n\to\infty} \int_a^b \phi_n$ for any sequence of step functions $(\phi_n)_{n=1}^{\infty} \subset \mathscr{S}([a,b];\mathbb{R})$ with $d(f,\phi_n) \to 0$ as $n \to \infty$. By the previous result, this is well-defined. The *indefinite integral* of f is $F : [a,b] \to \mathbb{R} : x \mapsto \int_a^x f$.

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[3] In the language of metric spaces, $\mathscr{S}([a,b];\mathbb{R})$ is "dense" in $\mathscr{R}([a,b];\mathbb{R})$ with respect to the uniform metric.

Proposition 3.5. (Properties of the integral.) *Let* $f \in \mathscr{R}([a,b];\mathbb{R})$.

(i) *If* $a \le u \le v \le w \le b$ *then* $\int_u^w f = \int_u^v f + \int_v^w f$.

(ii) *The integral function* $\int_a^b : \mathscr{R}([a,b];\mathbb{R}) \to \mathbb{R}$ *is a linear map of vector spaces.*

(iii) *If* $m \le f(x) \le M$ *for all* $x \in [a,b]$, *then* $m(b-a) \le \int_a^b f \le M(b-a)$.

Proposition 3.6. *If* $f \in \mathscr{R}([a,b];\mathbb{R})$ *then* $F : [a,b] \to \mathbb{R}$ *is continuous.*

Unfortunately, the definition of $\int_a^b f$ for $f \in \mathscr{R}([a,b];\mathbb{R})$ is not very easy to work with if we wish to compute the value of the integral of a given function. The FTC offers an easy way to calculate integrals; so, we now generalize the FTC to regulated functions. There are two main ways of phrasing the result: both are important, and both are stated and proved here.

Theorem 3.7. (FTC Version I.) *If* $f \in \mathscr{R}([a,b];\mathbb{R})$ *is continuous at* $c \in [a,b]$ *then* F *is differentiable at* c *with* $F'(c) = f(c)$.

Proof. Let $h > 0$ and write $F(c+h) - F(c) = \int_c^{c+h} f$. Since f is continuous at c, $\forall \varepsilon > 0$, $\exists \delta > 0$ so that $|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$. Hence

$$h(f(c) - \varepsilon) < \int_c^{c+h} f < h(f(c) + \varepsilon),$$

and similarly for $h < 0$. Thus, as required,

$$|h| < \delta \implies \left| \frac{F(c+h) - F(c)}{h} - f(c) \right| < \varepsilon$$

□

Theorem 3.8. (FTC Version II.) *If* $f \in \mathscr{R}([a,b];\mathbb{R})$ *and if* $g : [a,b] \to \mathbb{R}$ *is differentiable with* $g' = f$ *then* $\int_a^b f = g(b) - g(a)$.

Proof. Fix $\varepsilon > 0$ and pick $\phi \in \mathscr{S}([a,b];\mathbb{R})$ with partition $\{p_j\}_{j=0}^k$ and $d(f,\phi) < \varepsilon$. Apply the Mean Value Theorem to $g|_{[p_{j-1},p_j]}$ to get a point $x_j \in (p_{j-1}, p_j)$ with

$$g'(x_j) = f(x_j) = \frac{g(p_j) - g(p_{j-1})}{p_j - p_{j-1}}.$$

Since $c_j - \varepsilon < f(x_j) < c_j + \varepsilon$ for each j,

$$(c_j - \varepsilon)(p_j - p_{j-1}) < g(p_j) - g(p_{j-1}) < (c_j + \varepsilon)(p_j - p_{j-1}).$$

Sum over j to get

$$\sum_{j=1}^k c_j(p_j - p_{j-1}) - \varepsilon(b-a) < g(b) - g(a) < \sum_{j=1}^k c_j(p_j - p_{j-1}) + \varepsilon(b-a).$$

Hence, $\left|(g(b) - g(a)) - \int_a^b \phi\right| < \varepsilon(b-a)$, and so $\left|(g(b) - g(a)) - \int_a^b f\right| < 2\varepsilon(b-a)$. Since $\varepsilon > 0$ is arbitrary, we are done. □

4 Convergence of Sequences of Functions

We have several notions of convergence of sequences of functions:

Definitions 4.1. Let $f : [a, b] \to \mathbb{R}$ and $f_n : [a, b] \to \mathbb{R}$ for each $n \in \mathbb{N}$.

(i) $(f_n)_{n=1}^{\infty}$ *converges pointwise* to f if $\forall x \in [a, b]$, $f_n(x) \to f(x)$ as $n \to \infty$. This is our first idea of convergence, and is very weak. It turns out that we need a far stronger kind of convergence to be able to deduce properties of the limit function from properties of the functions in the sequence (continuity, for example).

(ii) $(f_n)_{n=1}^{\infty}$ *converges uniformly* to f if $d(f, f_n) \to 0$ as $n \to \infty$. Pictorially, imagine drawing a "ribbon" of width 2ε around the graph of f. Eventually, the graphs of the f_n must all lie within this ribbon. As we have already noted, regulated functions are uniform limits of sequences of step functions.

(iii) $(f_n)_{n=1}^{\infty}$ is *uniformly Cauchy* if $d(f_m, f_n) \to 0$ as $m, n \to \infty$.

Theorem 4.2. *Let $(f_n)_{n=1}^{\infty} \subset \mathscr{R}([a, b]; \mathbb{R})$ with $f_n \to f$ uniformly as $n \to \infty$. Then $f \in \mathscr{R}([a, b]; \mathbb{R})$ and $\int_a^b f_n \to \int_a^b f$ as $n \to \infty$. I.e., "the uniform limit of a sequence of regulated functions is regulated and the limit of the integrals is the integral of the limit".*

Proof. Fix $\varepsilon > 0$ and pick $N \in \mathbb{N}$ so that $n \geq N \implies d(f, f_n) < \frac{\varepsilon}{2}$. Pick $\phi \in \mathscr{S}([a, b]; \mathbb{R})$ with $d(f_N, \phi) < \frac{\varepsilon}{2}$. By the Triangle Inequality, $d(f, \phi) < \varepsilon$, so $f \in \mathscr{R}([a, b]; \mathbb{R})$. So, for $n \geq N$,

$$\left| \int_a^b f_n - \int_a^b f \right| = \left| \int_a^b (f_n - f) \right| < \frac{\varepsilon(b - a)}{2}.$$

Since $\varepsilon > 0$ is arbitrary, $\int_a^b f_n \to \int_a^b f$ as $n \to \infty$. ☐

Theorem 4.3. *Let $(f_n)_{n=1}^{\infty} \subset C^0([a, b]; \mathbb{R})$ with $f_n \to f$ uniformly as $n \to \infty$. Then $f \in C^0([a, b]; \mathbb{R})$. I.e., "the uniform limit of a sequence of continuous functions is continuous".*

Proof. Fix $c \in [a, b]$ and $\varepsilon > 0$, and choose $N \in \mathbb{N}$ so that $n \geq N \implies d(f, f_n) < \frac{\varepsilon}{3}$. Since f_N is continuous at c, $\exists \delta > 0$ so that $|x - c| < \delta \implies |f_N(x) - f_N(c)| < \frac{\varepsilon}{3}$. Then for $|x - c| < \delta$,

$$|f(x) - f(c)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| < \varepsilon.$$ ☐

Theorem 4.4. *Let $(f_n)_{n=1}^{\infty} \subset C^1([a, b]; \mathbb{R})$ with $f_n \to f : [a, b] \to \mathbb{R}$ pointwise and $f_n' \to g : [a, b] \to \mathbb{R}$ uniformly as $n \to \infty$. Then $f \in C^1([a, b]; \mathbb{R})$ and $f' = g$. I.e., "when the derivatives converge uniformly, the pointwise limit is differentiable and equal to the limit of the derivatives".*

Proof. Fix $x \in [a, b]$. Since $f_n' \to g$ uniformly, $\int_a^x f_n' \to \int_a^x g$. By the FTC Version II and pointwise convergence of the f_n, $\int_a^x f_n' = f_n(x) - f_n(a) \to f(x) - f(a)$. So $\int_a^x g = f(x) - f(a)$ for all $x \in [a, b]$. By the FTC Version I, the map $x \mapsto f(x) = f(a) + \int_a^x g$ is differentiable with derivative g, as required. ☐

5 Series of Functions

By the limit of a series of functions we mean the limit of the sequence of partial sums, just as with series of real numbers. The following result, the Weierstrass M-test, is a very useful test for uniform convergence of series of functions. The idea is to bound each term, and that these bounds form a convergent series in $\mathbb{R}$.

Theorem 5.1. (Weierstrass M-test.) *Let $f_r : [a,b] \to \mathbb{R}$ for each $r \in \mathbb{N}$ and suppose that there are bounds $M_r \in \mathbb{R}$ with $\sum_{r=1}^{\infty} M_r < \infty$ and $|f_r(x)| < M_r$ for all $x \in [a,b]$ and $r \in \mathbb{N}$. Then $\sum_{r=1}^{\infty} f_r$ converges uniformly to some $f : [a,b] \to \mathbb{R}$.*

Proof. Take $s_n := \sum_{r=1}^{n} f_r$ and $t_n := \sum_{r=1}^{n} M_r$. By assumption, $(t_n)_{n=1}^{\infty}$ converges, and so is Cauchy. So, given $\varepsilon > 0$, $\exists Q \in \mathbb{N}$ so that $m \geq n \geq Q \implies |t_m - t_n| < \varepsilon$. So, for any $m \geq n \geq Q$ and $x \in [a,b]$,

$$|s_m(x) - s_n(x)| = \left| \sum_{r=n+1}^{m} f_r(x) \right| \leq \sum_{r=n+1}^{m} |f_r(x)| \leq \sum_{r=n+1}^{m} M_r = t_m - t_n < \varepsilon.$$

So $(s_n)_{n=1}^{\infty}$ is uniformly Cauchy, and hence uniformly convergent. □

Theorem 5.2. (A continuous nowhere-differentiable function.) *Let $f(x) := \min\{x - \lfloor x \rfloor, \lceil x \rceil - x\}$ = the distance from x to the nearest integer. Let $f_n(x) := 4^{-n} f(4^n x)$ and $F(x) := \sum_{n=1}^{\infty} f_n(x)$. Then F is everywhere continuous but nowhere differentiable.*

This is a useful exercise that applies much of what we have learned so far. *Hints:* Use the Weierstrass M-test to prove continuity. Observe that each f_n contributes ±1 to the slope of the graph of F. Assume that F is differentiable at an arbitrary $x \in \mathbb{R}$ and use this observation to derive a contradiction.

6 Power Series

We consider power series of the form $\sum_{n=0}^{\infty} a_n x^n$ with $a_n \in \mathbb{R}$.

Theorem 6.1. *Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges for some $x = x_0 \neq 0$ and that $0 < b < |x_0|$. Then $\sum_{n=0}^{\infty} a_n x^n$ and the derived series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ both converge uniformly on $[-b,b]$.*

Proof. Since $\sum_{n=0}^{\infty} a_n x_0^n$ converges, $a_n x_0^n \to 0$ as $n \to \infty$. Choose $K > 0$ so that $|a_n x_0^n| \leq K$ for all $n \in \mathbb{N}$. If $|x| < b$ then $|a_n x^n| \leq |a_n b^n| \leq K|\frac{b}{x_0}|^n$. The series $\sum_{n=0}^{\infty} K|\frac{b}{x_0}|^n = K/(1-|\frac{b}{x_0}|)$ and so $\sum_{n=0}^{\infty} a_n x_0^n$ converges uniformly on $[-b,b]$ by the Weierstrass M-test. By the Ratio Test, $\sum_{n=1}^{\infty} Kn|\frac{b}{x_0}|^{n-1}$ also converges. So, by the Weierstrass M-test,$\sum_{n=1}^{\infty} n a_n x^{n-1}$ also converges uniformly on $[-b,b]$. □

Definition 6.2. The *radius of convergence* of the power series $\sum_{n=0}^{\infty} a_n x^n$ is

$$R := \sup \left\{ |x_0| \,\middle|\, \sum_{n=0}^{\infty} a_n x_0^n \text{ converges} \right\}.$$

We know that the power series $\sum_{n=0}^{\infty} a_n x^n$ converges within the radius of convergence ($|x| < R$) and diverges outside it ($|x| > R$). However, we cannot be certain what will happen at the radius of convergence ($|x| = R$) without further examination.

Theorem 6.3. *Let* $\sum_{n=0}^{\infty} a_n x^n$ *converge pointwise on* $(-R, R)$. *Then the function* $f : (-R, R) \to \mathbb{R} : x \mapsto \sum_{n=0}^{\infty} a_n x^n$ *is continuous and differentiable on* $(-R, R)$ *with* $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$. *If* $[c, d] \subset (-R, R)$ *then* $f|_{[c,d]}$ *is regulated and*

$$\int_c^d f = \sum_{n=0}^{\infty} \frac{a_n(d^{n+1} - c^{n+1})}{n+1}.$$

Definitions 6.4. The following functions $\mathbb{C} \to \mathbb{C}$ are defined by power series:

$$\begin{gathered} \exp(x) = \textstyle\sum_{n=0}^{\infty} \frac{x^n}{n!}, \\ \cosh(x) = \textstyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \quad \sinh(x) = \textstyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \\ \cos(x) = \textstyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad \sin(x) = \textstyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}. \end{gathered}$$

Proposition 6.5. $\exp' = \exp$, $\cosh' = \sinh$, $\sinh' = \cosh$, $\cos' = -\sin$, $\sin' = \cos$.

Proposition 6.6. *For all* $x \in \mathbb{C}$,

$$\begin{aligned} \cosh(x) &= \tfrac{\exp(x)+\exp(-x)}{2}, \quad & \sinh(x) &= \tfrac{\exp(x)-\exp(-x)}{2}, \\ \cos(x) &= \tfrac{\exp(ix)+\exp(-ix)}{2}, \quad & \sin(x) &= \tfrac{\exp(ix)-\exp(-ix)}{2i}. \end{aligned}$$

For all $x, y \in \mathbb{C}$,

$$\begin{gathered} \exp(x + y) = \exp(x)\exp(y), \\ \cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y), \\ \sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y). \end{gathered}$$

Proposition 6.7. *If we define* $\log : (0, \infty) \to \mathbb{R}$ *by* $\log(x) := \int_1^x \frac{1}{t}\,\mathrm{d}t$, *then* $\exp : \mathbb{R} \to (0, \infty)$ *and* $\log$ *are mutually inverse bijections.*

Proof. First observe that $x > 0 \implies \exp(x) > 1 + x > 0$. We know that $\exp(-x)\exp(x) = \exp(0) = 1$, so $\exp(-x) > 0$. Thus $\exp' = \exp$ is positive on $\mathbb{R}$, so exp is strictly increasing by the Mean Value Theorem, and so is injective. Since $\exp(x) > 1 + x$ for $x > 0$, exp takes arbitrarily large positive values; since $\exp(-x) = \frac{1}{\exp(x)}$, exp takes arbitrarily small positive values. Hence, by the Intermediate Value Theorem, exp takes all values in $(0, \infty)$. So $\exp : \mathbb{R} \to (0, \infty)$ is a bijection.

If $0 < x < 1$ then $\log(x) = -\int_x^1 \frac{1}{t}\,\mathrm{d}t$. By the FTC, log is differentiable with $\log'(x) = \frac{1}{x}$. By the Chain Rule, for all $x \in \mathbb{R}$,

$$\begin{aligned} (\log \circ \exp)'(x) &= (\log' \circ \exp)(x)\exp'(x) \\ &= \frac{1}{\exp(x)}\exp(x) \\ &= 1 \end{aligned}$$

$(\log \circ \exp)(0) = \log(1) = \int_1^1 \frac{1}{t}\,\mathrm{d}t = 0$ and $(\log \circ \exp)(x) = \int_0^x (\log \circ \exp)' = \int_0^x 1 = x$. Thus, $\log \circ \exp = \mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$. Since $\log'(x) = \frac{1}{x} > 0$ for $x > 0$, log is strictly increasing and hence injective by the Mean Value Theorem. Since $\log \circ \exp = \mathrm{id}_{\mathbb{R}}$, log is surjective, hence bijective.

$$\exp = \mathrm{id}_{(0,\infty)} \circ \exp = \log^{-1} \circ \log \circ \exp = \log^{-1} \circ \mathrm{id}_{\mathbb{R}} = \log^{-1}.$$

So $\exp^{-1} = \log$ and $\exp = \log^{-1}$. □

Theorem 6.8. *There is a real number* $\pi > 0$ *such that* $\cos(x+\pi) = -\cos(x)$ *and* $\sin(x+\pi) = -\sin(x)$.

We say that cos and sin are 2π-*periodic*; exp is $2\pi i$-periodic.

7 Fourier Series

Fourier series are trigonometric series of the form

$$\frac{a_0}{2} + \sum_{r=1}^{\infty} (a_r \cos rx + b_r \sin rx)$$

Lemma 7.1. (Orthogonality relations.) *Let $j, k \in \mathbb{Z}$. Then*

$$\int_{-\pi}^{\pi} \cos(jx)\cos(kx)\ \mathrm{d}x = \begin{cases} 0 & j \neq k, \\ \pi & j = k \neq 0, \\ 2\pi & j = k = 0, \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(jx)\cos(kx)\ \mathrm{d}x = 0,$$

$$\int_{-\pi}^{\pi} \sin(jx)\sin(kx)\ \mathrm{d}x = \begin{cases} 0 & j \neq k, j = k = 0 \\ \pi & j = k \neq 0. \end{cases}$$

Proposition 7.2. *For $n \in \mathbb{N}$, define*

$$s_n(x) := \frac{a_0}{2} + \sum_{r=1}^{\infty} (a_r \cos(rx) + b_r \sin(rx))$$

and suppose that $s_n \to f$ uniformly on $\mathbb{R}$. Then

$$a_r = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(rx)\ \mathrm{d}x \text{ for } r \geq 0,$$

$$b_r := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(rx)\ \mathrm{d}x \text{ for } r \geq 1.$$

Proof. Fix $r \geq 0$. Then

$$\begin{aligned} d(s_n(\cdot)\cos(r\cdot), f(\cdot)\cos(r\cdot)) &= \sup_{x\in\mathbb{R}} |s_n(x)\cos(rx) - f(x)\cos(rx)| \\ &= \sup_{x\in\mathbb{R}} |s_n(x) - f(x)||\cos(rx)| \\ &\leq \sup_{x\in\mathbb{R}} |s_n(x) - f(x)| \\ &= d(s_n, f). \end{aligned}$$

As $n \to \infty$, $s_n \to f$ uniformly, so $s_n(\cdot)\cos(r\cdot) \to f(\cdot)\cos(r\cdot)$ uniformly. Hence, as a sequence in $\mathbb{R}$, $\int_a^b s_n(x)\cos(rx)\ \mathrm{d}x \to \int_a^b f(x)\cos(rx)\ \mathrm{d}x$ and so

$$\begin{aligned} \int_{-\pi}^{\pi} s_n \cos(rx)\ \mathrm{d}x = &\frac{a_0}{2} \int_{-\pi}^{\pi} \cos(rx)\ \mathrm{d}x \\ &+ \sum_{k=1}^{\infty} \left(a_k \int_{-\pi}^{\pi} \cos(kx)\cos(rx)\ \mathrm{d}x + b_k \int_{-\pi}^{\pi} \sin(kx)\cos(rx)\ \mathrm{d}x \right) \\ = &\ \pi a_r, \end{aligned}$$

which proves the claim for a_r. The proof for b_r is similar. ☐

Definition 7.3. To save space later, we shall say that $f : \mathbb{R} \to \mathbb{R}$, $a_r \in \mathbb{R}$ ($r \geq 0$) and $b_r \in \mathbb{R}$ ($r \geq 1$) *satisfy the standard hypotheses* if

(i) f is 2π-periodic;

(ii) $f|_{[-\pi,\pi]}$ is regulated;

(iii) $a_r := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(rx)\ \mathrm{d}x$;

(iv) $b_r := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(rx)\ \mathrm{d}x$.

We call the a_r and b_r the *Fourier coefficients* of f and call the series

$$\frac{a_0}{2} + \sum_{r=1}^{\infty} (a_r \cos(rx) + b_r \sin(rx))$$

the *Fourier series* of f. Write $s_n(x) := \frac{a_0}{2} + \sum_{r=1}^{n} (a_r \cos(rx) + b_r \sin(rx))$, the nth partial sum.

8 Orthogonality and Bessel's Inequality

Definitions 8.1. On the vector space

$$\mathscr{R}_{2\pi} := \{f : \mathbb{R} \to \mathbb{R} | f|_{[-\pi,\pi]} \text{ is regulated and } f \text{ is } 2\pi\text{-periodic}\}$$

we define the symmetric bilinear form[4]

$$\langle f, g\rangle_{L^2} := \frac{1}{\pi}\int_{-\pi}^{\pi} f \cdot g = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\ \mathrm{d}x.$$

We say that $f, g \in \mathscr{R}_{2\pi}$ are *orthogonal* if $\langle f, g\rangle_{L^2} = 0$. We write $\|f\|_{L^2}$ for $\sqrt{\langle f, f\rangle_{L^2}}$, the *$L^2$-norm* of f. We say that $f_n \in \mathscr{R}_{2\pi}$ converge to $f \in \mathscr{R}_{2\pi}$ in the *L^2 sense* if $\|f_n - f\|_{L^2} \to 0$ as $n \to \infty$.

Lemma 8.2. *$\langle \cdot, \cdot\rangle_{L^2}$ is a symmetric bilinear form on $\mathscr{R}_{2\pi}$.*

Theorem 8.3. *Assume the standard hypotheses on f and fix $N \in \mathbb{N}$. Then for any other choice of coefficients $A_0, \dots, A_N, B_1, \dots, B_N$,*

$$\|f - S_N\|_{L^2} \geq \|f - s_N\|_{L^2},$$

where $S_N(x) := \frac{A_0}{2} + \sum_{r=1}^{N} (A_r \cos(rx) + B_r \sin(rx))$ is the Nth partial sum for the coefficients A_r, B_r.

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[4] It is not an inner product, since $\langle f, f\rangle_{L^2} = 0$ does not imply that $f = 0$.

Proof.

$$\begin{aligned}
0 &\le \frac{1}{\pi}\int_{-\pi}^{\pi}\left(f(x)-S_N(x)\right)^2\ \mathrm{d}x\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f^2 - A_0\frac{1}{\pi}\int_{-\pi}^{\pi} f - 2\frac{1}{\pi}\sum_{r=1}^{N}\left(A_r\int_{-\pi}^{\pi} f(x)\cos(rx)\ \mathrm{d}x + B_r\int_{-\pi}^{\pi} f(x)\sin(rx)\ \mathrm{d}x\right)\\
&\qquad + \frac{1}{\pi}\int_{-\pi}^{\pi} S_N(x)^2\ \mathrm{d}x\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f^2 - A_0a_0 - 2\sum_{r=1}^{N}(A_ra_r+B_rb_r) + \frac{A_0^2}{2} + \sum_{r=1}^{N}(A_r^2+B_r^2)\\
&= \underbrace{\frac{1}{2}(a_0-A_0)^2 + \sum_{r=1}^{N}\left((a_r-A_r)^2+(b_r-B_r)^2\right)}_{(*)} + \frac{1}{\pi}\int_{-\pi}^{\pi} f^2 - \left(\frac{a_0^2}{2}+\sum_{r=1}^{N}(a_r^2+b_r^2)\right)
\end{aligned}$$

and this distance^2 is least when $(*) = 0$, i.e., when $A_r \equiv a_r$ and $B_r \equiv b_r$. □

Corollary 8.4. (Bessel's Inequality.) *Assume the standard hypotheses on f. Then*

$$\frac{a_0^2}{2} + \sum_{r=1}^{\infty}(a_r^2+b_r^2) \le \|f\|_{L^2}^2.$$

Corollary 8.5. *Assume the standard hypotheses on f. Then $a_r, b_r \to 0$ as $r \to \infty$.*

This corollary may also be deduced from

Lemma 8.6. (Riemann-Lebesgue.) *Let $f \in \mathscr{R}([-\pi,\pi];\mathbb{R})$. Then $\int_{-\pi}^{\pi} f(x)\cos(tx)\ \mathrm{d}x \to 0$ and $\int_{-\pi}^{\pi} f(x)\sin(tx)\ \mathrm{d}x \to 0$ as $t \to \infty$ in $\mathbb{R}$.*

Proof. Since f, cos and sin are regulated (and so are their products), the integrals exist. Let $\phi \in \mathscr{S}([-\pi,\pi];\mathbb{R})$ with partition $\{p_j\}_{j=0}^{k}$. Then

$$\begin{aligned}
\int_{-\pi}^{\pi}\phi(x)\cos(tx)\ \mathrm{d}x &= \sum_{j=1}^{k} c_j\int_{p_{j-1}}^{p_j}\cos(tx)\ \mathrm{d}x\\
&= \sum_{j=1}^{k}\frac{c_j}{t}(\sin(tp_j)-\sin(tp_{j-1}))\\
&\to 0 \text{ as } t \to \infty.
\end{aligned}$$

For $\varepsilon > 0$ take ϕ so that $d(f,\phi) < \frac{\varepsilon}{4\pi}$ and $T > 0$ so that $\left|\int_{-\pi}^{\pi}\phi(x)\cos(tx)\ \mathrm{d}x\right| < \frac{\varepsilon}{2}$ for $t > T$. Then, for $t > T$,

$$\begin{aligned}
\left|\int_{-\pi}^{\pi} f(x)\cos(tx)\ \mathrm{d}x\right| &\le \left|\int_{-\pi}^{\pi}(f(x)-\phi(x))\cos(tx)\ \mathrm{d}x\right| + \left|\int_{-\pi}^{\pi}\phi(x)\cos(tx)\ \mathrm{d}x\right|\\
&< \frac{\varepsilon}{4\pi}2\pi + \frac{\varepsilon}{2}\\
&= \varepsilon.
\end{aligned}$$

Similarly for sin. □

9 Completeness and Convergence of Fourier Series

Definition 9.1. We say that the set of functions $\{1, \cos(r\cdot), \sin(r\cdot) | r \in \mathbb{N}\}$ is *complete* if for every $f \in \mathscr{R}_{2\pi}$, the Fourier series for f converges to f in the L^2 sense, i.e.

$$\|f - s_N\|_{L^2} \to 0 \text{ as } N \to \infty.$$

Proposition 9.2. (Parseval's Formula.) *If $\{1, \cos(r\cdot), \sin(r\cdot) | r \in \mathbb{N}\}$ is complete and f satisfies the standard hypotheses then*

$$\frac{a_0^2}{2} + \sum_{r=1}^{\infty} (a_r^2 + b_r^2) = \|f\|_{L^2}^2.$$

It can be shown that uniform convergence implies pointwise convergence and that uniform convergence implies L^2 convergence. However, there is no implication relating pointwise convergence and L^2 convergence.

Lemma 9.3. *For all $y \in \mathbb{R}$,*

$$\frac{1}{2} + \sum_{r=1}^{N} \cos(ry) = \frac{1}{2} \frac{\sin((N+1/2)y)}{\sin(y/2)}$$

where the right hand side is interpreted as being $N + \frac{1}{2}$ for $y \in 2\pi\mathbb{Z}$. Also,

$$\int_0^{\pi} \frac{\sin((N+1/2)y)}{\sin(y/2)} \, \mathrm{d}y = \frac{1}{2} \int_{-\pi}^{\pi} \frac{\sin((N+1/2)y)}{\sin(y/2)} \, \mathrm{d}y = \pi.$$

Theorem 9.4. *Assume the standard hypotheses on f and that f is differentiable. Then the Fourier series for f converges to f pointwise.*

Theorem 9.5. *Assume the standard hypotheses on f and that $f \in C^2([-\pi, \pi]; \mathbb{R})$. Then there is a $K > 0$ such that $|a_r|, |b_r| < \frac{K}{r^2}$ for $r \geq 1$.*

Proof. f, f' and f'' are all continuous and 2π-periodic and so bounded. Let $r \geq 1$.

$$\begin{aligned}
\pi a_r &= \int_{-\pi}^{\pi} f(x) \cos(rx) \, \mathrm{d}x \\
&= \frac{1}{r} f(x) \sin(rx) \Big|_{-\pi}^{\pi} - \frac{1}{r} \int_{-\pi}^{\pi} f'(x) \sin(rx) \, \mathrm{d}x \\
&= \frac{1}{r^2} f'(x) \cos(rx) \Big|_{-\pi}^{\pi} - \frac{1}{r^2} \int_{-\pi}^{\pi} f''(x) \cos(rx) \, \mathrm{d}x \\
&= -\frac{1}{r^2} \int_{-\pi}^{\pi} f''(x) \cos(rx) \, \mathrm{d}x.
\end{aligned}$$

So

$$\begin{aligned}
|a_r| &\leq \frac{1}{\pi r^2} \int_{-\pi}^{\pi} |f''(x) \cos(rx)| \, \mathrm{d}x \\
&\leq \frac{1}{\pi r^2} 2\pi \sup_{x \in [-\pi, \pi]} |f''(x)| \\
&= \frac{K}{r^2},
\end{aligned}$$

where $K := 2 \sup_{x \in [-\pi, \pi]} |f''(x)|$. Similarly for $|b_r|$. □

Repeated integration by parts leads to the general rule that if f is C^k then the Fourier coefficients for f decay as $\frac{K}{r^k}$.

Theorem 9.6. *Assume the standard hypotheses on f and that $f \in C^2([-\pi,\pi];\mathbb{R})$. Then the Fourier series for f converges to f uniformly (and, hence, also in the L^2 sense).*

Proof. We know there is a $K > 0$ such that $|a_r|, |b_r| < \frac{K}{r^2}$ for $r \geq 1$, so

$$|a_r \cos(rx) + b_r \sin(rx)| \leq |a_r| + |b_r| \leq \frac{2K}{r^2}.$$

Hence, by the Weierstrass M-test, $\frac{a_0}{2} + \sum_{r=1}^{\infty} (a_r \cos(rx) + b_r \sin(rx))$ converges uniformly to some continuous $g \in \mathscr{R}_{2\pi}$; by the pointwise convergence in the differentiable case, $g = f$. $\square$

Lemma 9.7. *The polynomial $h : \mathbb{R} \to \mathbb{R}$, $h(x) := \frac{x^5}{5} - \frac{2v^2x^3}{3} + v^4x$ satisfies $h'(\pm v) = 0$, $h''(\pm v) = 0$ and $h(\pm v) = \pm\frac{8v^5}{15}$.*

Theorem 9.8. *Assume the standard hypotheses on f. Then the Fourier series for f converges to f in the L^2 sense.*

Proof. Fix $\varepsilon > 0$ and let $\phi \in \mathscr{S}([-\pi,\pi];\mathbb{R})$ with partition $\{p_j\}_{j=0}^{k}$ and $d(f,\phi) < \frac{\varepsilon}{4}$. As usual, $x \in (p_{j-1}, p_j) \implies \phi(x) = c_j$; for convenience, let $c_0 = c_k$ and $c_{k+1} = c_1$. Choose $v < \frac{1}{2}\min_{1\leq j\leq k} |p_j - p_{j-1}|$. We construct $g \in C^2([-\pi,\pi];\mathbb{R})$ by setting $g(x) := \phi(x)$ for $x \in [-\pi,\pi] \setminus \bigcup_{j=0}^{k}(p_j - v, p_j + v)$ and $g(p_j + t) := \frac{c_j + c_{j+1}}{2} + \frac{15}{16v^5}(c_{j+1} - c_j)h(t)$ for $t \in [-v, v]$. By the uniform convergence theorem for C^2 functions, $\exists N \in \mathbb{N}$ such that the Nth partial sum of the Fourier series for g has $d(s_N, g) < \frac{\varepsilon}{4}$. For $x \in [-\pi,\pi] \setminus \bigcup_{j=0}^{k}(p_j - v, p_j + v)$,

$$|s_N(x) - f(x)| \leq |s_N(x) - g(x)| + |g(x) - f(x)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2};$$

and for $x \in [-\pi,\pi] \cap \bigcup_{j=0}^{k}(p_j - v, p_j + v)$,

$$\begin{aligned} |s_N(x) - f(x)| &\leq |s_N(x) - g(x)| + |g(x) - \phi(x)| + |\phi(x) - f(x)| \\ &< \frac{\varepsilon}{4} + \max_{0\leq j\leq k} |c_{j+1} - c_j| + \frac{\varepsilon}{4} \\ &=: J. \end{aligned}$$

Hence, for $v \leq \frac{\varepsilon^2\pi}{4J^2k}$,

$$\begin{aligned} \pi\|s_N - f\|_{L^2}^2 &= \int_{-\pi}^{\pi} (s_N(x) - f(x))^2 \, \mathrm{d}x \\ &\leq \left(\frac{\varepsilon}{2}\right)^2 2\pi + J^2 2vk \\ &< \left(\frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}\right)\pi \\ &= \varepsilon^2\pi. \end{aligned}$$

So $\|s_N - f\|_{L^2} < \varepsilon$, as required. $\square$

Lemma 9.9. *(i) If $x_n \to x$ in $\mathbb{R}$ as $n \to \infty$, then $\frac{x_0 + x_1 + \cdots + x_{n-1}}{n} \to x$ as $n \to \infty$.*

(ii) If $s_n := \sum_{r=0}^{n} x_r \to s$ in $\mathbb{R}$ as $n \to \infty$, then $\sigma_n := \frac{s_0+s_1+\cdots+s_{n-1}}{n} \to s$ as $n \to \infty$.

σ_n is known as the nth *Cesáro mean*. It is possible that the Cesáro means of a series might converge even if the partial sums do not.

Proposition 9.10. *If $f \in \mathscr{R}([a,b];\mathbb{R})$ and $q \in (a,b)$ then $f(q+) := \lim_{x \searrow q} f(x)$ and $f(q-) := \lim_{x \nearrow q} f(x)$ both exist, as do $f(a+)$ and $f(b-)$.*

Lemma 9.11. *For all $N \in \mathbb{N}$,*

$$\sum_{n=0}^{N-1} \frac{\sin((n+1/2)u)}{\sin(u/2)} = \begin{cases} \left(\frac{\sin(Nu/2)}{\sin(u/2)}\right)^2 & u \notin 2\pi\mathbb{Z} \\ N^2 & u \in 2\pi\mathbb{Z} \end{cases}$$

$$\int_0^\pi \left(\frac{\sin(Nu/2)}{\sin(u/2)}\right)^2 \mathrm{d}u = N\pi.$$

Theorem 9.12. (Fejér's Theorem.) *Assume the standard hypotheses on f and fix $x \in \mathbb{R}$. As usual, let s_n be the nth partial sum of the Fourier series for f and let σ_N be the Nth Cesáro mean of the sequence $(s_n)_{n=0}^\infty$. Then*

$$\sigma_N(x) \to \frac{f(x+)+f(x-)}{2} \text{ as } N \to \infty.$$

(This limit is $f(x)$ if f is continuous at x.)

Proof. We take

$$s_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+u)\frac{\sin((n+1/2)u)}{\sin(u/2)}\,\mathrm{d}u.$$

From the above lemma,

$$\begin{aligned}
\sigma_N(x) &= \frac{1}{2N\pi}\int_{-\pi}^{\pi} f(x+u)\left(\frac{\sin(Nu/2)}{\sin(u/2)}\right)^2 \mathrm{d}u \\
&= \frac{1}{2N\pi}\left(\int_{-\pi}^{0} f(x+u)\left(\frac{\sin(Nu/2)}{\sin(u/2)}\right)^2 \mathrm{d}u + \int_0^\pi f(x+u)\left(\frac{\sin(Nu/2)}{\sin(u/2)}\right)^2 \mathrm{d}u\right) \\
&= \frac{1}{2N\pi}\left(\int_{\pi}^{0} f(x-v)\left(\frac{\sin(-Nv/2)}{\sin(-v/2)}\right)^2 (-dv) + \int_0^\pi f(x+u)\left(\frac{\sin(Nu/2)}{\sin(u/2)}\right)^2 du\right) \\
&= \frac{1}{2N\pi}\int_0^\pi (f(x+u)+f(x-u))\left(\frac{\sin(Nu/2)}{\sin(u/2)}\right)^2 \mathrm{d}u
\end{aligned}$$

and so, if we write

$$h(x,u) = (f(x+u) - f(x+) + f(x-u) - f(x-))\left(\frac{\sin(Nu/2)}{\sin(u/2)}\right)^2,$$

$$\begin{aligned}
\sigma_N(x) - \frac{f(x+)+f(x-)}{2} &= \frac{1}{2N\pi}\int_0^\pi h(x,u)\,\mathrm{d}u \\
&= \frac{1}{2N\pi}\left(\int_0^\delta h(x,u)\,\mathrm{d}u + \int_\delta^\pi h(x,u)\,\mathrm{d}u\right)
\end{aligned}$$

By definition of $f(x+), f(x-)$, $\forall \varepsilon > 0$, $\exists \delta > 0$ so that $0 < u < \delta \implies |f(x+u) - f(x+)| < \frac{\varepsilon}{2}$ and $|f(x-u) - f(x-)| < \frac{\varepsilon}{2}$. Now

$$\begin{aligned}
&\left|\frac{1}{2N\pi}\int_0^\delta h(x,u)\,\mathrm{d}u\right| \\
&\quad\le \sup_{0\le u\le\delta} |f(x+u) - f(x+) + f(x-u) - f(x-)|\,\frac{1}{2N\pi}\int_0^\delta \left(\frac{\sin(Nu/2)}{\sin(u/2)}\right)^2\,\mathrm{d}u \\
&\quad\le \frac{\varepsilon}{2N\pi}\int_0^\delta \left(\frac{\sin(Nu/2)}{\sin(u/2)}\right)^2\,\mathrm{d}u \\
&\quad< \frac{\varepsilon}{2}
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{1}{2N\pi}\int_\delta^\pi h(x,u)\,\mathrm{d}u\right| \\
&\quad\le \sup_{\delta\le u\le\pi} \left(f(x+u) - f(x+) + f(x-u) - f(x-)\right)\frac{1}{2N\pi}\int_\delta^\pi \left(\frac{\sin(Nu/2)}{\sin(u/2)}\right)^2\,\mathrm{d}u \\
&\quad\le 4\sup_y |f(y)|\frac{1}{2N\pi}\int_\delta^\pi \frac{1}{\sin^2(\delta/2)}\,\mathrm{d}u \\
&\quad= 2\sup_y |f(y)|\frac{\pi-\delta}{N\pi}\frac{1}{\sin^2(\delta/2)} \\
&\quad< \frac{\varepsilon}{2}
\end{aligned}$$

for suitably large N. $\square$