9. The Root System and the Weyl Group

$\{h_\alpha \mid \alpha \in \Phi\}$ spans H, so we can find a subset $\{h_{\alpha_1},\ldots,h_{\alpha_l}\}$ that forms a basis for H; $\dim(H)=l$.

Proposition 9.1. *Let $\alpha \in \Phi$. Then $h_\alpha = \sum_{i=1}^{l}\mu_i h_{\alpha_i}$ for some $\mu_i \in \mathbb{Q}$.*

Proof. We know this for $\mu_i \in \mathbb{C}$. Let $\langle h_{\alpha_i}, h_{\alpha_j}\rangle = \xi_{ij} \in \mathbb{Q}$. The matrix $\Xi = (\xi_{ij})$ is non-singular, since if it were singular we would have $\eta_1,\ldots,\eta_l$ not all zero such that $\sum_{i=1}^{l}\eta_i\xi_{ij} = 0$. Then

$$\left\langle \sum_{i=1}^{l}\eta_i h_{\alpha_i}, h_{\alpha_i}\right\rangle = \sum_{i=1}^{l}\eta_i\xi_{ij} = 0$$

So $\left\langle \sum_{i=1}^{l}\eta_i h_{\alpha_i}, x\right\rangle = 0$ for all $x \in H$. $\langle\cdot,\cdot\rangle$ is non-degenerate, so all $\eta_i = 0$, which is a contradiction.

$$\begin{aligned}\langle h_\alpha, h_{\alpha_1}\rangle &= \mu_1\xi_{11} + \ldots + \mu_l\xi_{l1}\\ &\vdots\\ \langle h_\alpha, h_{\alpha_l}\rangle &= \mu_1\xi_{1l} + \ldots + \mu_l\xi_{ll}\end{aligned}$$

We have l linear equations in l unknowns with a non-singular coefficient matrix, all the entries of which are rational. Hence, by Cramer's Rule, there is a unique solution $(\mu_i) \in \mathbb{Q}^l$

■

Let $H_{\mathbb{Q}}$ be the set of all $\sum_{i=1}^{l}\mu_i h_{\alpha_i}$, $\mu_i \in \mathbb{Q}$. $\dim_{\mathbb{Q}}(H_{\mathbb{Q}}) = l$. $H_{\mathbb{Q}}$ is independent of the choice of basis; all $h_\alpha \in H_{\mathbb{Q}}$.

Let $H_{\mathbb{R}}$ be the set of all $\sum_{i=1}^{l}\mu_i h_{\alpha_i}$, $\mu_i \in \mathbb{R}$. $\dim_{\mathbb{R}}(H_{\mathbb{R}}) = l$.

Proposition 9.2. *Let $x \in H_{\mathbb{R}}$. Then $\langle x,x\rangle \in \mathbb{R}_{\geq 0}$ and $\langle x,x\rangle = 0 \Leftrightarrow x = 0$.*

Proof. Let $x \in H_{\mathbb{R}}$, $x = \sum_{i=1}^{l}\mu_i h_{\alpha_i}$.

$$\begin{aligned}\langle x,x\rangle &= \sum_i \sum_j \mu_i\mu_j \langle h_{\alpha_i}, h_{\alpha_j}\rangle \\ &= \sum_i \sum_j \mu_i\mu_j \operatorname{tr}\left(\operatorname{ad} h_{\alpha_i} \operatorname{ad} h_{\alpha_j}\right) \\ &= \sum_i \sum_j \mu_i\mu_j \sum_{\alpha\in\Phi} \alpha\left(h_{\alpha_i}\right)\alpha\left(h_{\alpha_j}\right) \\ &= \sum_\alpha \sum_i \sum_j \mu_i\mu_j \alpha\left(h_{\alpha_i}\right)\alpha\left(h_{\alpha_j}\right) \\ &= \sum_\alpha \left(\sum_i \mu_i \alpha\left(h_{\alpha_i}\right)\right)^2\end{aligned}$$

So $\langle x,x\rangle \in \mathbb{R}$ and $\langle x,x\rangle \geq 0$. Suppose $\langle x,x\rangle = 0$. Then for all $\alpha \in \Phi$, $\sum_i \mu_i \alpha\left(h_{\alpha_i}\right) = 0$. In particular, $\sum_i \mu_i \alpha_j\left(h_{\alpha_i}\right) = 0$ for $j = 1,\ldots,l$; $\sum_i \mu_i \langle h_{\alpha_i}, h_{\alpha_j}\rangle = \sum_i \mu_i \xi_{ij} = 0$ for all j. Ξ is non-singular, so $\mu_i = 0$ for all i, so $x = 0$.

■

So all $h_\alpha \in H_{\mathbb{R}}$; $\dim_{\mathbb{R}}(H_{\mathbb{R}}) = l$. We introduce a total order on $H_{\mathbb{R}}$: let $x \in H_{\mathbb{R}}$, $x = \sum_i \mu_i h_{\alpha_i}$. If $x \neq 0$ we say $x \succ 0$ if the first non-zero μ_i is positive; if $x \neq 0$ we say $x \prec 0$ if the first non-zero μ_i is negative. We have trichotomy: for each $x \in H_{\mathbb{R}}$ precisely one of $x = 0$, $x \prec 0$, $x \succ 0$ is true.

So, for $\alpha \in \Phi$, $h_\alpha \prec 0$ or $h_\alpha \succ 0$. Define $\alpha \prec 0$ if $h_\alpha \prec 0$ and $\alpha \succ 0$ if $h_\alpha \succ 0$. Define

$$\Phi^+ = \{\alpha \in \Phi \mid \alpha \succ 0\}, \text{ the positive roots, and}$$
$$\Phi^- = \{\alpha \in \Phi \mid \alpha \prec 0\}, \text{ the negative roots.}$$

Clearly, $\Phi = \Phi^+ \cup \Phi^-$.

A *fundamental root* is a positive root that is not the sum of two positive roots. Let Π be the set of fundamental roots.

Proposition 9.3. *(i) Every positive root is a sum of fundamental roots.*
(ii) $\{h_\alpha \mid \alpha \in \Pi\}$ *is a basis of* $H_{\mathbb{R}}$.
(iii) If $\alpha, \beta \in \Pi$ *and* $\alpha \neq \beta$ *then* $\langle h_\alpha, h_\beta\rangle \leq 0$.

Proof. (i) Let $\alpha \in \Phi^+$. If $\alpha \in \Pi$ we are done. If $\alpha \notin \Pi$ the there exist $\beta, \gamma \in \Phi^+$ such that $\alpha = \beta + \gamma$ with $\beta, \gamma \prec \alpha$. Repeat to get the result.

(iii) Let $\alpha, \beta \in \Pi$ with $\alpha \neq \beta$. Then $\alpha - \beta \notin \Phi$ since if not

$$\alpha = (\alpha - \beta) + \beta \text{ or } \beta = (\beta - \alpha) + \alpha$$

so either α or β would be a sum of positive roots. Consider the α-chain of roots through β:

$$\beta, \alpha+\beta, \ldots, q\alpha+\beta$$

$$\Rightarrow \qquad 2\frac{\langle h_\alpha, h_\beta\rangle}{\langle h_\alpha, h_\alpha\rangle} = p - q = -q \qquad \text{by 8.11.}$$

$$\Rightarrow \qquad \langle h_\alpha, h_\beta\rangle \leq 0 \qquad \text{since } \langle h_\alpha, h_\alpha\rangle \geq 0 \text{ by 9.2.}$$

(ii) By (i), the h_α for $\alpha \in \Phi$ span H. We show the h_α are linearly independent. Suppose not: then there exist $\mu_i \in \mathbb{R}$ not all zero such that

$$\sum_{\alpha_i \in \Pi} \mu_i h_{\alpha_i} = 0$$

Rearrange this sum, taking all the positive μ_i to one side. Then

$$x = \mu_{i_1} h_{\alpha_{i_1}} + \ldots + \mu_{i_r} h_{\alpha_{i_r}} = \mu_{j_1} h_{\alpha_{j_1}} + \ldots + \mu_{j_s} h_{\alpha_{j_s}}$$
$$\mu_{i_u}, \mu_{j_v} > 0,\ i_u, j_v \text{ distinct for } 1 \leq u \leq r,\ 1 \leq v \leq s.$$

Then

$$\langle x, x\rangle = \left\langle \mu_{i_1} h_{\alpha_{i_1}} + \ldots + \mu_{i_r} h_{\alpha_{i_r}}, \mu_{j_1} h_{\alpha_{j_1}} + \ldots + \mu_{j_s} h_{\alpha_{j_s}} \right\rangle \leq 0$$

by (iii). So $x = 0$, a contradiction.

■

Note. Φ^+ can be chosen in many different ways. However, Π is determined by Φ^+ and Φ^+ is determined by Π.

Example. Let $L = \mathfrak{sl}_n(\mathbb{C})$. The roots are

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \mapsto \lambda_j - \lambda_i \text{ for } j \neq i$$

Define Φ^+ to be the roots with $j > i$. Then the fundamental roots are

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \mapsto \lambda_{i+1} - \lambda_i \text{ for } 1 \le i \le n-1$$

$$\lambda_j - \lambda_i = (\lambda_{i+1} - \lambda_i) + (\lambda_{i+2} - \lambda_{i+1}) + \ldots + (\lambda_j - \lambda_{j-1})$$

$$\dim(H) = n - 1 = l\text{, the rank of } L.$$

For each $\alpha \in \Phi$ we define $s_\alpha : H_{\mathbb{R}} \to H_{\mathbb{R}}$ by

$$s_\alpha(x) = x - 2\frac{\langle h_\alpha, x\rangle}{\langle h_\alpha, h_\alpha\rangle} h_\alpha$$

s_α is linear and $s_\alpha(h_\alpha) = -h_\alpha$. The set of x such that $\langle x, x\rangle = 0$ forms a hyperplane i.e. a subspace of codimension 1. s_α is the reflection of $H_{\mathbb{R}}$ in the hyperplane orthogonal to h_α.

$$s_\alpha^2 = \mathrm{id}$$
$$s_\alpha = s_{-\alpha}$$

Let W be the group of all non-singular linear maps $H_{\mathbb{R}} \to H_{\mathbb{R}}$ generated by $\{ s_\alpha \mid \alpha \in \Phi \}$. W is called the *Weyl group*.†

Proposition 9.4. *(i) W is a finite group.*
(ii) W is a group of isometries, i.e. for all $x, y \in H_{\mathbb{R}}$, $w \in W$, $\langle w(x), w(y)\rangle = \langle x, y\rangle$.
(iii) For each $\alpha \in \Phi$ and $w \in W$ there is a $\beta \in \Phi$ such that $w(h_\alpha) = h_\beta$.

Proof. (ii) Let $x, y \in H_{\mathbb{R}}$. Then

$$\begin{aligned} \langle s_\alpha(x), s_\alpha(y)\rangle &= \left\langle x - 2\frac{\langle h_\alpha, x\rangle}{\langle h_\alpha, h_\alpha\rangle} h_\alpha, y - 2\frac{\langle h_\alpha, y\rangle}{\langle h_\alpha, h_\alpha\rangle} h_\alpha \right\rangle \\ &= \langle x, y\rangle - 4\frac{\langle h_\alpha, x\rangle\langle h_\alpha, y\rangle}{\langle h_\alpha, h_\alpha\rangle} - 4\frac{\langle h_\alpha, x\rangle\langle h_\alpha, y\rangle}{\langle h_\alpha, h_\alpha\rangle^2}\langle h_\alpha, h_\alpha\rangle \\ &= \langle x, y\rangle \end{aligned}$$

So s_α is an isometry; so w is an isometry for all $w \in W$.

(iii) Now consider $s_\alpha(h_\beta)$:

† After Hermann Weyl.

$$s_\alpha(h_\alpha) = h_{-\alpha}$$
$$s_\alpha(h_{-\alpha}) = h_\alpha$$

So suppose $\beta \neq \pm\alpha$. Consider the α-chain of roots through β,

$$-p\alpha + \beta, \ldots, \beta, \ldots, q\alpha + \beta$$

$$\begin{aligned} s_\alpha(h_\beta) &= h_\beta - 2\frac{\langle h_\alpha, h_\beta \rangle}{\langle h_\alpha, h_\alpha \rangle} h_\alpha \\ &= h_\beta - (p-q)h_\alpha \\ &= h_{\beta + (q-p)\alpha} \end{aligned}$$

Now $\beta + (q-p)\alpha \in \Phi$ since $-p \leq p - q \leq q$. So s_α permutes the h_β for $\beta \in \Phi$. Hence $w \in W$ permutes the h_β for $\beta \in \Phi$. Note that

$$\beta + ((q-p)\alpha + \beta) = (-p\alpha + \beta) + (q\alpha + \beta)$$

so s_α inverts the h_β in a given α-chain.

(i) We have a homomorphism from W to the group of permutations of the h_α for $\alpha \in \Phi$. Φ is finite, so the image of this homomorphism is finite. If $w \in W$ is in the kernel then $w(h_\alpha) = h_\alpha$ for all $\alpha \in \Phi$. Since the h_α span $H_{\mathbb{R}}$, $w = \mathrm{id}$. Hence, W is finite.
■

Proposition 9.5. *Given any root $\alpha \in \Phi$ there exists a fundamental root $\alpha_i \in \Pi$ and a $w \in W$ such that $h_\alpha = w(h_{\alpha_i})$.*

Proof. Each $\alpha \in \Phi$ has the form $\alpha = n_1\alpha_1 + \ldots + n_l\alpha_l$, $n_i \in \mathbb{Z}$. If $\alpha \in \Phi^+$ then all $n_i \geq 0$; if $\alpha \in \Phi^-$ then all $n_i \leq 0$. We may assume $\alpha \in \Phi^+$ since if $\alpha \in \Phi^-$ then use $h_\alpha = s_\alpha(h_{-\alpha})$. The quantity $n_1 + \ldots + n_l$ is called the *height* of α, $\mathrm{ht}(\alpha)$. We use induction on $\mathrm{ht}(\alpha)$. If $\mathrm{ht}(\alpha) = 1$ we are done, so assume $\mathrm{ht}(\alpha) > 1$. By 8.12, at least two $n_i > 0$.

$$0 < \langle h_\alpha, h_\alpha \rangle = \sum_i n_i \langle h_{\alpha_i}, h_\alpha \rangle$$

All $n_i \geq 0$, so there exists i such that $\langle h_{\alpha_i}, h_\alpha \rangle > 0$. Let $s_{\alpha_i}(h_\alpha) = h_\beta$.

$$h_\beta = h_\alpha - 2\frac{\langle h_{\alpha_i}, h_\alpha \rangle}{\langle h_{\alpha_i}, h_{\alpha_i} \rangle} h_{\alpha_i}$$
$$\beta = \alpha - 2\frac{\langle h_{\alpha_i}, h_\alpha \rangle}{\langle h_{\alpha_i}, h_{\alpha_i} \rangle} \alpha_i$$

So $\mathrm{ht}(\beta) < \mathrm{ht}(\alpha)$. Passing from α to β changes only one n_i, hence β has at least one $n_j > 0$, so $\beta \in \Phi^+$. By induction, $\beta = w'(h_{\alpha_j})$ for some $w' \in W$ and some $\alpha_j \in \Pi$. Thus, taking $w = s_{\alpha_i} w' \in W$,

$$h_\alpha = s_{\alpha_i}(h_\beta) = s_{\alpha_i} w'(h_{\alpha_j}) = w(h_{\alpha_j}).$$

■

Proposition 9.6. *The Weyl group W is generated by $s_{\alpha_1}, \ldots, s_{\alpha_l}$ for $\Pi = \{\alpha_1, \ldots, \alpha_l\}$.*

Proof. Suppose W_0 is the subgroup generated by $\{s_{\alpha_i} \mid \alpha_i \in \Pi\}$. To show $W = W_0$ we show $s_\alpha \in W_0$ for all $\alpha \in \Phi$. The proof of 9.5 shows that $h_\alpha = w(h_{\alpha_i})$ for some $\alpha_i \in \Pi$ and some $w \in W$. Consider $ws_{\alpha_i} w^{-1} \in W_0$.

$$ws_{\alpha_i} w^{-1}(h_\alpha) = ws_{\alpha_i}(h_{\alpha_i}) = w(-h_{\alpha_i}) = -h_\alpha$$

Let $x \in H_{\mathbb{R}}$ be such that $\langle h_\alpha, x \rangle = 0$. Then

$$\Rightarrow \quad \langle w^{-1}(h_\alpha), w^{-1}(x) \rangle = 0$$
$$\Rightarrow \quad \langle h_{\alpha_i}, w^{-1}(x) \rangle = 0$$
$$\Rightarrow \quad ws_{\alpha_i} w^{-1}(x) = ww^{-1}(x) = x$$

Hence, $ws_{\alpha_i} w^{-1} = s_\alpha$. Then $s_\alpha \in W_0$, so $W = W_0$.

■

Example. $L = \mathfrak{sl}_3(\mathbb{C})$; $\dim(L) = 8$.

$$H = \left\{ \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \middle| \; \lambda_1 + \lambda_2 + \lambda_3 = 0 \right\}$$

$\dim(H) = 2$.

$$L = H \oplus \mathbb{C}E_{12} \oplus \mathbb{C}E_{23} \oplus \mathbb{C}E_{13} \oplus \mathbb{C}E_{21} \oplus \mathbb{C}E_{32} \oplus \mathbb{C}E_{31}$$

Let

$$h = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$
$$[E_{ij}h] = (\lambda_j - \lambda_i)E_{ij}$$

The roots are

$$\alpha_1 : h \mapsto \lambda_2 - \lambda_1 \qquad \alpha_2 : h \mapsto \lambda_3 - \lambda_2 \qquad \alpha_1 + \alpha_2 : h \mapsto \lambda_3 - \lambda_1$$
$$-\alpha_1 : h \mapsto \lambda_1 - \lambda_2 \qquad -\alpha_2 : h \mapsto \lambda_2 - \lambda_3 \qquad -\alpha_1 - \alpha_2 : h \mapsto \lambda_1 - \lambda_3$$
$$\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2)\}$$
$$\Pi = \{\alpha_1, \alpha_2\}$$

Consider the corresponding vectors $h_\alpha \in H$. Let

$$h = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \in H, \; h' = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix} \in H$$

$$\begin{aligned}
\langle h, h' \rangle &= \operatorname{tr}(\operatorname{ad} h \operatorname{ad} h') \\
&= 2(\lambda_2 - \lambda_1)(\mu_2 - \mu_1) + 2(\lambda_3 - \lambda_2)(\mu_3 - \mu_2) + 2(\lambda_3 - \lambda_1)(\mu_3 - \mu_1) \\
&= 2(2\lambda_1\mu_1 + 2\lambda_2\mu_2 + 2\lambda_3\mu_3 - (\lambda_1\mu_2 + \lambda_1\mu_3 + \lambda_2\mu_1 + \lambda_2\mu_3 + \lambda_3\mu_1 + \lambda_3\mu_2)) \\
&= 4(\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3) - 2(\lambda_1 + \lambda_2 + \lambda_3)(\mu_1 + \mu_2 + \mu_3) + 2(\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3) \\
&= 6(\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3) \\
&= 6\operatorname{tr}(hh')
\end{aligned}$$

h_{α_1} satisfies $\langle h_{\alpha_1}, h \rangle = \alpha_1(h) = \lambda_2 - \lambda_1$, so

$$h_{\alpha_1} = \frac{1}{6}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Similarly,

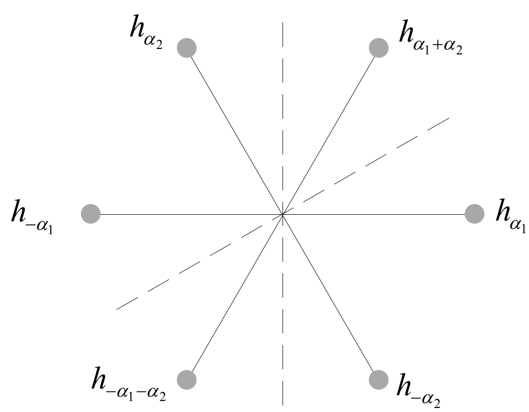
$$h_{\alpha_2} = \frac{1}{6}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For $x \in H_{\mathbb{R}}$ define $|x| = \sqrt{\langle x, x \rangle}$. With this notation, $|h_{\alpha_1}| = |h_{\alpha_2}| = 1/\sqrt{3}$ and

$$\langle h_{\alpha_1}, h_{\alpha_2} \rangle = 6\tfrac{1}{6}\tfrac{1}{6}(-1) = -\tfrac{1}{6}$$

The angle between h_{α_1} and h_{α_2} is given by the cosine formula:

$$\langle h_{\alpha_1}, h_{\alpha_2} \rangle = |h_{\alpha_1}||h_{\alpha_2}|\cos\theta$$
$$\theta = 2\pi/3$$



$$W = \{\mathrm{id}, s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_1}s_{\alpha_2}, s_{\alpha_2}s_{\alpha_1}, s_{\alpha_1}s_{\alpha_2}s_{\alpha_1} = s_{\alpha_2}s_{\alpha_1}s_{\alpha_2} = s_{\alpha_1+\alpha_2}\}$$

10. THE DYNKIN DIAGRAM

We shall consider the geometrical properties of the h_α for $\alpha \in \Phi$.

Proposition 10.1. *Let $\alpha, \beta \in \Phi$, $\beta \neq \pm\alpha$. Then*
(i) the angle between α and β is one of

$$\pi/6,\ \pi/4,\ \pi/3,\ \pi/2,\ 2\pi/3,\ 3\pi/4,\ 5\pi/6;$$

(ii) if the angle is $\pi/3$ or $2\pi/3$, h_α and h_β have the same length;
(iii) if the angle is $\pi/4$ or $3\pi/4$, the ratio of the lengths of h_α and h_β is $\sqrt{2}$;
(iv) if the angle is $\pi/6$ or $5\pi/6$, the ratio of the lengths of h_α and h_β is $\sqrt{3}$.

Proof. Let $\theta_{\alpha\beta}$ be the angle between h_α and h_β. We have

$$\langle h_\alpha, h_\beta \rangle = |h_\alpha||h_\beta|\cos\theta_{\alpha\beta}$$

$$\Rightarrow \qquad \langle h_\alpha, h_\beta \rangle^2 = \langle h_\alpha, h_\alpha \rangle\langle h_\beta, h_\beta \rangle \cos^2\theta_{\alpha\beta}$$

$$\Rightarrow \qquad 4\cos^2\theta_{\alpha\beta} = \frac{2\langle h_\alpha, h_\beta \rangle}{\langle h_\alpha, h_\alpha \rangle}\frac{2\langle h_\beta, h_\alpha \rangle}{\langle h_\beta, h_\beta \rangle}$$

By 8.11, both factors on the RHS are integers, so $4\cos^2\theta_{\alpha\beta} \in \mathbb{Z}$. $0 \le \cos^2\theta_{\alpha\beta} < 1$, so $0 \le 4\cos^2\theta_{\alpha\beta} < 4$, so $4\cos^2\theta_{\alpha\beta} \in \{0,1,2,3\}$.

$$\Rightarrow \qquad \cos^2\theta_{\alpha\beta} \in \left\{0, \pm\tfrac{1}{2}, \pm\tfrac{1}{\sqrt{2}}, \pm\tfrac{\sqrt{3}}{2}\right\}$$

$$\Rightarrow \qquad \theta_{\alpha\beta} \in \{\pi/2, \pi/3, 2\pi/3, \pi/4, 3\pi/4, \pi/6, 5\pi/6\}$$

$$4\cos^2\theta_{\alpha\beta} = \left(2\tfrac{\langle h_\alpha, h_\beta \rangle}{\langle h_\alpha, h_\alpha \rangle}\right)\left(2\tfrac{\langle h_\beta, h_\alpha \rangle}{\langle h_\beta, h_\beta \rangle}\right)$$

Suppose $\theta_{\alpha\beta}$ is $\pi/3$ or $2\pi/3$, so $4\cos^2\theta_{\alpha\beta} = 1$. $1 = 1.1 = (-1).(-1)$, so $|h_\alpha| = |h_\beta|$.

Suppose $\theta_{\alpha\beta}$ is $\pi/4$ or $3\pi/4$, so $4\cos^2\theta_{\alpha\beta} = 2$. $2 = 1.2 = 2.1 = (-1).(-2) = (-2).(-1)$. So one of $\langle h_\alpha, h_\alpha \rangle$ and $\langle h_\beta, h_\beta \rangle$ is twice the other, so one of $|h_\alpha|$, $|h_\beta|$ is $\sqrt{2}$ times the other.

Suppose $\theta_{\alpha\beta}$ is $\pi/6$ or $5\pi/6$, so $4\cos^2\theta_{\alpha\beta} = 3$. $3 = 1.3 = 3.1 = (-1).(-3) = (-3).(-1)$. So, as above, one of $|h_\alpha|$, $|h_\beta|$ is $\sqrt{3}$ times the other.

■

Proposition 10.2. *Let $\alpha \in \Phi$. Then every α - chain of roots has at most four roots in it.*

Proof. Consider the α - chain of roots through β with β as the first root:

$$\beta, \alpha+\beta, \ldots, q\alpha+\beta$$

By 8.11, $2\langle h_\alpha, h_\beta \rangle / \langle h_\alpha, h_\alpha \rangle = -q$. The LHS is $0, -1, -2$ or -3 by 10.1. So $q \le 3$. So the length of the α -chain is at most 4.

■

Let

$$a_{ij} = 2\frac{\langle h_{\alpha_i}, h_{\alpha_j} \rangle}{\langle h_{\alpha_i}, h_{\alpha_i} \rangle}$$

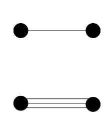
and $A = (a_{ij})$. A is called the *Cartan matrix*; the a_{ij} are the *Cartan integers*.

Proposition 10.3. *The Cartan matrix has the following properties:*
(i) for each i, $a_{ii} = 2$;
(ii) for $i \ne j$, $a_{ij} \in \{0, -1, -2, -3\}$;
(iii) $a_{ij} = -2 \Rightarrow a_{ji} = -1$; $a_{ij} = -3 \Rightarrow a_{ji} = -1$;
(iv) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$.

Proof. If $i \ne j$ then $4\cos^2\theta_{\alpha\beta} = a_{ij}a_{ji}$.
(i) Clear.
(ii) Follows from 10.1, 9.3.
(iii) Follows from 10.1.
(iv) Clear.

■

We incorporate this information into a graph. The *Dynkin*[†] *diagram* is a graph Δ with l vertices, one for each fundamental root. If $i \ne j$ then vertices i, j are joined by $n_{ij} = a_{ij}a_{ji}$ edges, $0 \le n_{ij} \le 3$. The Dynkin diagram may be disconnected, as in



[†] After E.B. Dynkin. Also due to H.S.M. Coxeter.

It splits into connected components and the Cartan matrix splits into corresponding blocks; off-diagonal blocks are zero:

$$A = \left(\begin{array}{c|c|c} * & & 0 \\ \hline & \ddots & \\ \hline 0 & & * \end{array}\right)$$

We define a corresponding quadratic form Q:

$$Q(x_1,\ldots,x_l) = \sum_{i=1}^{l} 2x_i^2 - \sum_{1\le i\ne j\le l} \sqrt{n_{ij}}\, x_i x_j$$

Recall the correspondence between quadratic forms on $\mathbb{R}$ and real symmetric matrices:

$$\begin{aligned} M &= (m_{ij}) \text{ symmetric} \\ xMx^{\mathrm{T}} &= \sum_{i,j} m_{ij} x_i x_j \end{aligned}$$

The matrix of $Q(x_1,\ldots,x_l)$ is

$$\begin{pmatrix} 2 & -\sqrt{n_{12}} & -\sqrt{n_{13}} & & \\ -\sqrt{n_{12}} & 2 & -\sqrt{n_{23}} & \ddots & \\ -\sqrt{n_{13}} & -\sqrt{n_{23}} & 2 & \ddots & \\ & \ddots & \ddots & \ddots & \\ & & & & 2 \end{pmatrix}$$

Proposition 10.4. *The quadratic form $Q(x_1,\ldots,x_l)$ is positive definite, i.e. $Q(x_1,\ldots,x_l)\ge 0$ and $Q(x_1,\ldots,x_l)=0 \Leftrightarrow x_1=\ldots=x_l=0$.*

Proof.

$$\begin{aligned} 4\cos^2\theta_{ij} &= a_{ij}a_{ji} = n_{ij} \\ 2\cos\theta_{ij} &= -\sqrt{n_{ij}} \\ \left\langle h_{\alpha_i}, h_{\alpha_j}\right\rangle &= \left|h_{\alpha_i}\right|\left|h_{\alpha_j}\right|\cos\theta_{ij} \end{aligned}$$

$$Q(x_1,\ldots,x_l) = \sum_{i,j=1}^{l} 2\frac{\left\langle h_{\alpha_i}, h_{\alpha_j}\right\rangle}{\left|h_{\alpha_i}\right|\left|h_{\alpha_j}\right|} x_i x_j = 2\left\langle \sum_i \frac{x_i}{\left|h_{\alpha_i}\right|} h_{\alpha_i}, \sum_j \frac{x_j}{\left|h_{\alpha_j}\right|} h_{\alpha_j} \right\rangle = 2\langle y, y\rangle$$

where $y = \sum_i x_i h_{\alpha_i} / \left|h_{\alpha_i}\right|$. So $Q(x_1,\ldots,x_l)\ge 0$. If $Q(x_1,\ldots,x_l)=0$ then $\langle y,y\rangle = 0$, so $y=0$, so all $x_i = 0$. The converse is clear.

■

Recall. Any quadratic form can be diagonalized; there exists a non-singular real $l \times l$ matrix P such that $PMP^{\mathrm{T}} = D$, a diagonal matrix. Let $y = xP^{-1}$; then $xMx^{\mathrm{T}} = yDy^{\mathrm{T}}$.

Proposition 10.5. *Let $M = (m_{ij})$ be an $l \times l$ real symmetric matrix. Then the associated quadratic form $\sum_{i,j} m_{ij} x_i x_j$ is positive definite if and only if all leading minors of M have positive determinant. (The leading minors are*

$$(m_{11}), \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \ldots, M \text{ .)}$$

Proof. We use induction on l. Assume the quadratic form is positive definite. If $l = 1$, $M = (m_{11})$. $m_{11}x^2 > 0 \Leftrightarrow m_{11} > 0$. Suppose $l > 1$. $\sum_{i,j=1}^{l} m_{ij} x_i x_j$ is still positive definite as it is the original with $x_l = 0$. By induction, the first $l-1$ leading minors of M have positive determinant; we require that $\det(M) > 0$. $xMx^{\mathrm{T}} = yDy^{\mathrm{T}}$, D diagonal with entries $d_1, \ldots, d_l > 0$. Now if $PMP^{\mathrm{T}} = D$,

$$\det(P)^2 \det(M) = \det(D) > 0 .$$

Conversely, suppose that all leading minors of M have positive determinant. The same is true of the smaller $(l-1) \times (l-1)$ leading minor. By induction, $\sum_{i,j=1}^{l-1} m_{ij} x_i x_j$ is positive definite. So we have a diagonal form in new coordinates $y_1, \ldots, y_l$:

$$\sum_{i,j=1}^{l-1} m_{ij} x_i x_j = \sum_{k=1}^{l-1} d_k x_k^2 \text{ with } d_k > 0 .$$
$$\sum_{i,j=1}^{l} m_{ij} x_i x_j = \sum_{k=1}^{l-1} d_k x_k^2 + 2e_1 y_1 x_l + \ldots + 2e_{l-1} y_{l-1} x_l + e x_l^2$$

This may be diagonalized by a further transformation of coordinates:

$$z_i = y_i + \tfrac{e_i}{d_i} x_i$$

We get $d_1 z_1^2 + \ldots d_{l-1} z_{l-1}^2 + f x_l^2$. So there is a non-singular P such that

$$PMP^{\mathrm{T}} = \begin{pmatrix} d_1 & & & 0 \\ & \ddots & & \\ & & d_{l-1} & \\ 0 & & & f \end{pmatrix}$$
$$\det(P)^2 \det(M) = f \prod_{i=1}^{l-1} d_i$$

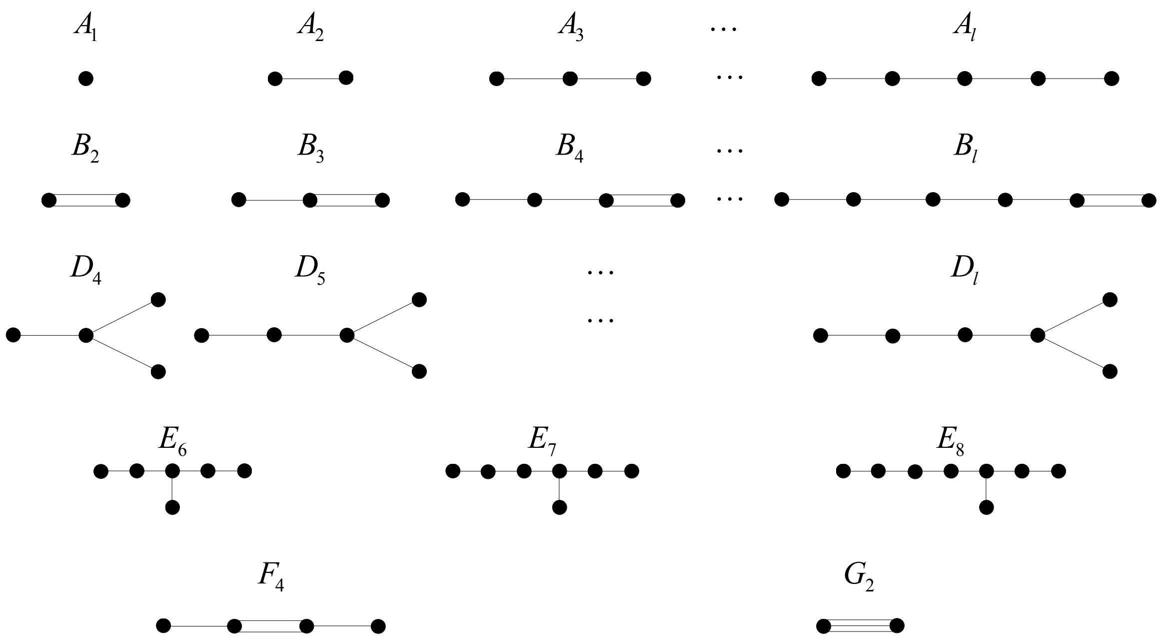
We assume $\det(M) > 0$, so $f\prod_{i=1}^{l-1} d_i > 0$, so $f > 0$. Thus, the form is positive definite.

■

We consider graphs with the following properties:
(i) the graph is connected;
(ii) any two distinct vertices are joined by 0, 1, 2 or 3 edges;
(iii) the associated quadratic form is positive definite.

The Dynkin diagram of a semisimple Lie algebra has connected components satisfying (i)-(iii). It is possible to determine all graphs satisfying (i)-(iii).

Theorem 10.6. *The only graphs satisfying (i)-(iii) are*



Proof. The given graphs clearly satisfy (i) and (ii). We show that they satisfy (iii). We show $Q(x_1,\ldots,x_l)$ is positive definite by induction on l. If $l=1$ we have $Q(x_1)=2x_1^2$, which is positive definite. Suppose $l>1$. There is a vertex l such that when it is removed we have another graph on the list. By induction, $Q(x_1,\ldots,x_{l-1})$ is positive definite, so all leading minors of the matrix of $Q(x_1,\ldots,x_{l-1})$ have positive determinant. To complete the induction we show that the matrix of $Q(x_1,\ldots,x_l)$ has positive determinant.

Let Y_l be a graph of l vertices and y_l the determinant of the matrix of the associated quadratic form. In the case $l=1$, $a_1 = |2| = 2$. In the case $l=2$ we have

$$a_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

$$b_2 = \begin{vmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{vmatrix} = 2$$

$$c_2 = \begin{vmatrix} 2 & -\sqrt{3} \\ -\sqrt{3} & 2 \end{vmatrix} = 1$$

Suppose $l \geq 3$. Remove a vertex l joined to just one other vertex $l-1$ by a single edge. If Y_l is the given graph, let Y_{l-1} be the graph with vertex l removed in this way, and let Y_{l-2} be the graph with vertices l and $l-1$ removed in this way.

$$y_l = \det(Y_l) = \begin{vmatrix} & & & & 0 \\ & & & & \vdots \\ & & & & 0 \\ & & & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{vmatrix} = 2y_{l-1} - (-1)(-1)y_{l-2} = 2y_{l-1} - y_{l-2}$$

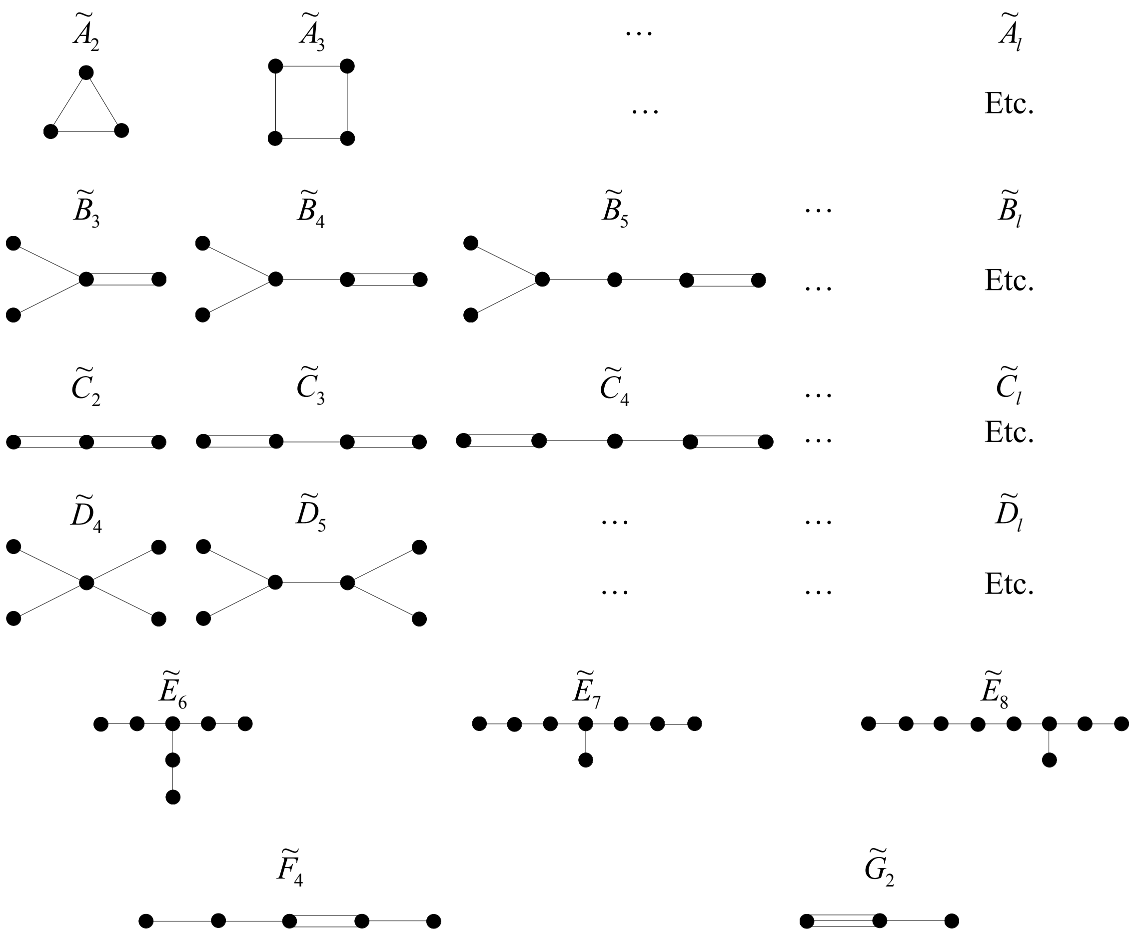
Hence:

Type A_l	$a_l = 2a_{l-1} - a_{l-2} \Rightarrow a_l = l+1$
Type B_l	$b_l = 2b_{l-1} - b_{l-2} \Rightarrow b_l = 2$
Type D_l	$d_4 = 2a_3 - a_1^2 = 4$
	$d_5 = 2d_4 - a_3 = 4$
	$\Rightarrow d_l = 4$ by induction
Type E_6	$e_6 = 2d_5 - a_4 = 3$
Type E_7	$e_7 = 2e_6 - d_5 = 2$
Type E_8	$e_8 = 2e_7 - e_6 = 1$
Type F_4	$f_4 = 2b_3 - a_2 = 1$
Type G_2	$g_2 = 1$

Hence, $Q(x_1, \ldots, x_l)$ is positive definite in each case.

In order to show the converse, i.e. that the graphs on our list are the only possible ones, we shall first require some additional results.

Proposition 10.7. *For each of the following graphs the corresponding quadratic form* $Q(x_1, \ldots, x_l)$ *has determinant zero.*



Proof. In most cases we can calculate the determinant as before, but not in types $\widetilde{A}_l$, $\widetilde{C}_l$.

Type $\widetilde{A}_l$

$$\begin{vmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & 0 & \\ & -1 & \ddots & & \\ & 0 & & \ddots & -1 \\ -1 & & & -1 & 2 \end{vmatrix} = 0$$

since the row sum is $(0,\dots,0)$

Type $\widetilde{C}_l$

$$\begin{vmatrix} & & & & 0 \\ & & & & \vdots \\ & & & & 0 \\ & & & 2 & -\sqrt{2} \\ 0 & \cdots & 0 & -\sqrt{2} & 2 \end{vmatrix} = 2b_l - \left(-\sqrt{2}\right)\left(-\sqrt{2}\right)b_{l-1} = 0$$

Type $\widetilde{B}_l$ $\qquad \widetilde{b}_3 = 2b_3 - a_1^2 = 2.2 - 2^2 = 0$

Type $\widetilde{D}_l$ $\qquad \widetilde{d}_4 = 2d_4 - a_1^3 = 2.4 - 2^3 = 0$

Type $\widetilde{E}_6$ $\qquad \widetilde{e}_6 = 2e_6 - a_5 = 2.3 - 6 = 0$

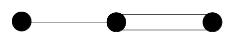
Type $\widetilde{E}_7$	$\widetilde{e}_7 = 2e_7 - d_6 = 2.2 - 4 = 0$
Type $\widetilde{E}_8$	$\widetilde{e}_8 = 2e_8 - e_7 = 2.1 - 2 = 0$
Type $\widetilde{F}_4$	$\widetilde{f}_4 = 2f_4 - b_3 = 2.1 - 2 = 0$
Type $\widetilde{G}_2$	$\widetilde{g}_2 = 2g_2 - a_1 = 2.1 - 2 = 0$

■

Lemma 10.8. *Let Y be a graph in which any two vertices are joined by at most three edges. Suppose the corresponding quadratic form is positive definite. Suppose Y' is a graph obtained from Y by omitting some of the vertices, or by reducing the number of edges, or both. Then the quadratic form for Y' is also positive definite.*

We call Y' a *subgraph* of Y.

Example.

$Y' =$  is a subgraph of $Y =$ 

Proof. The quadratic form for Y is

$$Q(x_1,\ldots,x_l) = \sum_{i=1}^{l} 2x_i^2 - \sum_{1\le i\ne j\le l} \sqrt{n_{ij}}\, x_i x_j \; .$$

The quadratic form for Y' is

$$Q'(x_1,\ldots,x_m) = \sum_{i=1}^{m} 2x_i^2 - \sum_{1\le i\ne j\le m} \sqrt{n'_{ij}}\, x_i x_j \; ,$$

with $m \le l$ and $n'_{ij} \le n_{ij}$. Suppose Q' is not positive definite. Then there exists $(y_1,\ldots,y_m) \ne 0$ with $Q'(y_1,\ldots,y_m) \le 0$. Consider $Q(|y_1|,\ldots,|y_m|,0,\ldots,0)$. This is

$$\begin{aligned}
\sum_{i=1}^{m} 2y_i^2 - \sum_{1\le i\ne j\le m} \sqrt{n_{ij}}\,|y_i||y_j| &\le \sum_{i=1}^{m} 2y_i^2 - \sum_{1\le i\ne j\le m} \sqrt{n'_{ij}}\,|y_i||y_j| \\
&\le \sum_{i=1}^{m} 2y_i^2 - \sum_{1\le i\ne j\le m} \sqrt{n_{ij}}\, y_i y_j \\
&\le Q'(y_1,\ldots,y_m)
\end{aligned}$$

So $Q(|y_1|,\ldots,|y_m|,0,\ldots,0) \le Q'(y_1,\ldots,y_m) \le 0$. So $Q(x_1,\ldots,x_l)$ is not positive definite, a contradiction.

■

We now return to the proof of 10.6.

Suppose Y is some graph satisfying conditions. (i)-(iii). By 10.7 and 10.8 we know that no graph of the form $\widetilde{A},\widetilde{B},\widetilde{C},\widetilde{D},\widetilde{E},\widetilde{F}$ or $\widetilde{G}$ can be obtained as a subgraph of Y.

(a) Y contains no cycles, for otherwise Y would have a subgraph of the form $\widetilde{A}_l$.

(b) If Y has a triple edge then $Y = G_2$, for otherwise Y would have $\widetilde{G}_2$ as a subgraph.

(c) Suppose Y has no triple edge. Then Y can have no more than one double edge, for otherwise Y has a subgraph of type $\widetilde{C}_l$.

(d) Suppose Y has one double edge. Then Y has no branch point, for otherwise Y has $\widetilde{B}_l$ as a subgraph.

(e) If the double edge is not at one end then $Y = F_4$, for otherwise Y has a subgraph $\widetilde{F}_4$. If the double edge is at one end, $Y = B_l$.

(f) Now suppose Y has only single edges. Then Y cannot have a branch point with four or more branches, for otherwise Y has $\widetilde{D}_4$ as a subgraph.

(g) Y can have no more than one branch point, for otherwise Y has a subgraph $\widetilde{D}_l$, $l \geq 5$.

(h) If Y has no branch points, $Y = A_l$. So suppose Y has just one branch point with three branches of lengths $l_1 \leq l_2 \leq l_3$, $l_1 + l_2 + l_3 + 1 = l$. Then $l_1 = 1$, for otherwise Y would have $\widetilde{E}_6$ as a subgraph.

(i) If $l_1 = l_2 = 1$, $Y = D_l$. Also, $l_2 \leq 2$, for otherwise Y has $\widetilde{E}_7$ as a subgraph.

(j) So assume $l_1 = 1$, $l_2 = 2$. Then $l_3 \leq 4$, for otherwise Y has $\widetilde{E}_8$ as a subgraph.

$$\begin{aligned} l_3 = 2 &\Rightarrow Y = E_6 \\ l_3 = 3 &\Rightarrow Y = E_7 \\ l_3 = 4 &\Rightarrow Y = E_8 \end{aligned}$$

■

Corollary 10.9. *Every Dynkin diagram of a semisimple Lie algebra has connected components of type* $A_l, B_l, D_l, E_6, E_7, E_8, F_4$ *and* G_2.

The Cartan matrix $A = (a_{ij})$ determines the Dynkin diagram since $n_{ij} = a_{ij}a_{ji}$. However, the Dynkin diagram does not always determine the Cartan matrix. Recall that the a_{ij} satisfy

$$a_{ii} = 2$$
$$a_{ij} \in \{0,1,2,3\} \text{ for } i \neq j$$

If $n_{ij} = 1$ then $a_{ij} = a_{ji} = 1$. If $n_{ij} = 2$ then $(a_{ij}, a_{ji}) = (-2,-1)$ or $(-1,-2)$. If $n_{ij} = 3$ then $(a_{ij}, a_{ji}) = (-3,-1)$ or $(-1,-3)$. In this last case the Dynkin diagram is G_2. We have

$$A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \text{ or } \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

and one is obtained from the other by re-labeling the vertices.

Suppose $n_{ij} = 2$. If $l = 2$ the possibilities for the Cartan matrix are

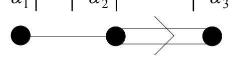
$$A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \text{ or } \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix},$$

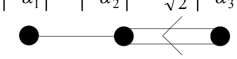
again obtainable from one another by re-labeling. If $l \geq 3$ there are two possible Cartan matrices:

$$B_l = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -2 & 2 \end{pmatrix} \qquad C_l = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -2 \\ & & & -1 & 2 \end{pmatrix}$$

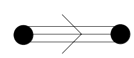
$$B_3 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix} \qquad C_3 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$

$$|h_{\alpha_1}| = |h_{\alpha_2}| = \sqrt{2}|h_{\alpha_3}| \qquad |h_{\alpha_1}| = |h_{\alpha_2}| = \tfrac{1}{\sqrt{2}}|h_{\alpha_3}|$$





We place an arrow on the Dynkin diagram when we have a double or triple edge; the arrow points from the longer root to the shorter one. For example, with G_2:



long short

Theorem 10.10. *The possible Cartan matrices with connected Dynkin diagrams are (up to permutation of the numbering of the vertices):*

$$A_l = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \qquad B_l = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -2 & 2 \end{pmatrix}$$

$$C_l = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -2 \\ & & & -1 & 2 \end{pmatrix} \qquad D_l = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & 2 & -1 & -1 \\ & & & -1 & 2 & 0 \\ & & & -1 & 0 & 2 \end{pmatrix}$$

$$E_6 = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & 0 & -1 \\ & & -1 & 2 & -1 & \\ & & 0 & -1 & 2 & \\ & & -1 & & & 2 \end{pmatrix} \qquad E_7 = \begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & -1 & 2 & -1 & 0 & -1 \\ & & & -1 & 2 & -1 & \\ & & & 0 & -1 & 2 & \\ & & & -1 & & & 2 \end{pmatrix}$$

$$E_8 = \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & 0 & -1 \\ & & & & -1 & 2 & -1 & \\ & & & & 0 & -1 & 2 & \\ & & & & -1 & & & 2 \end{pmatrix}$$

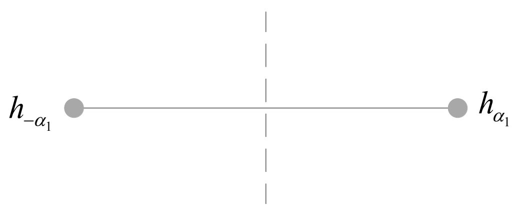
$$F_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \qquad G_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

11. THE INDECOMPOSABLE ROOT SYSTEMS

A root system is called *indecomposable* if it has a connected Dynkin diagram.

Case $l = 1$. We have only one possibility, A_1:

A_1

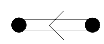


$$\Phi = \{\pm\alpha_1\}$$
$$W = \langle s_{\alpha_1} \rangle;\ |W| = 2$$

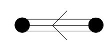
Case $l = 2$. Here we have three possibilities

A_2

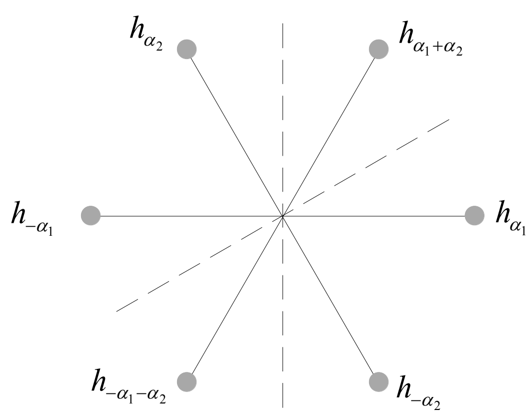
B_2



G_2

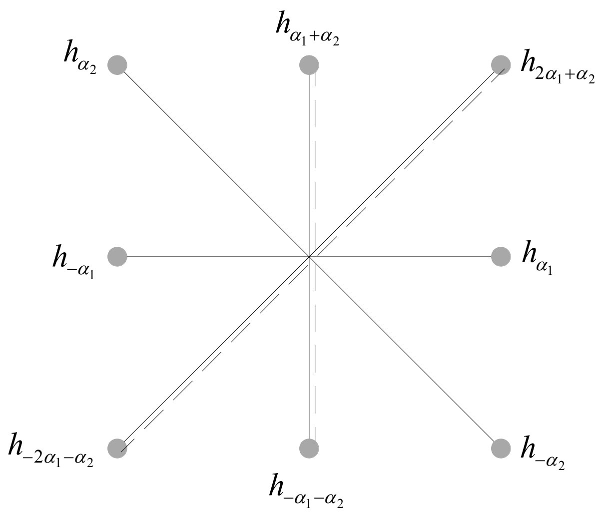


Type A_2.



$$\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2)\}$$

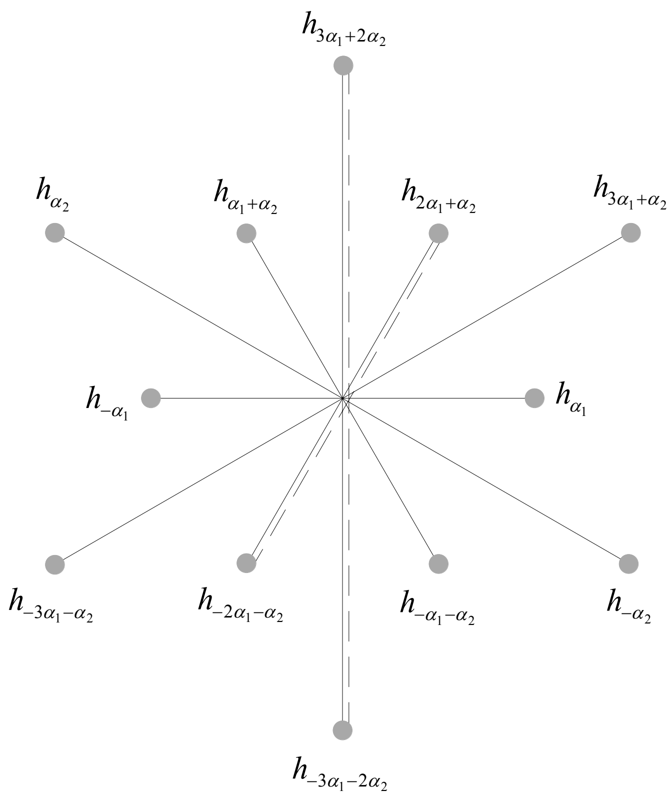
Type B_2.



$$\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2), \pm(2\alpha_1+\alpha_2)\}$$

(In the above diagram, the dashed lines indicating the reflection axes are shown slightly offset for clarity.)

Type G_2.

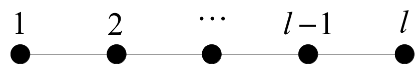


$$\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2), \pm(2\alpha_1+\alpha_2), \pm(3\alpha_1+\alpha_2), \pm(3\alpha_1+2\alpha_2)\}$$

(Again, the dashed lines indicating the reflection axes are shown slightly offset.)

Case $l \geq 3$. *Type* A_l. It is convenient to describe the root system of type A_l in a Euclidean space of dimension $l+1$.

Let V be an $(l+1)$-dimensional Euclidean space. Let $\{\varepsilon_0,\ldots,\varepsilon_n\}$ be an orthogonal basis of vectors of the same length, so $\langle \varepsilon_i, \varepsilon_j \rangle = K\delta_{ij}$ for some $K > 0$.



Define $h_{\alpha_1} = \varepsilon_0 - \varepsilon_1$, $h_{\alpha_2} = \varepsilon_1 - \varepsilon_2$, …, $h_{\alpha_l} = \varepsilon_{l-1} - \varepsilon_l$. The h_{α_i} are linearly independent.

$$\langle h_{\alpha_i}, h_{\alpha_i} \rangle = 2K$$
$$\langle h_{\alpha_i}, h_{\alpha_j} \rangle = 0 \text{ if } j \neq i-1, i, i+1$$
$$\langle h_{\alpha_i}, h_{\alpha_{i+1}} \rangle = -K$$
$$a_{i,i+1} = 2\frac{\langle h_{\alpha_i}, h_{\alpha_{i+1}} \rangle}{\langle h_{\alpha_i}, h_{\alpha_i} \rangle} = \frac{-2K}{2K} = -1$$

Thus for suitable K the h_{α_i} form a fundamental system of roots of type A_l. Let V_0 be the subspace spanned by these vectors; $\dim(V_0) = l$. Consider the map $V \to V$ given by $\varepsilon_0 \leftrightarrow \varepsilon_1$, $\varepsilon_i \mapsto \varepsilon_i$ for $i \geq 2$.

$$h_{\alpha_1} \mapsto -h_{\alpha_1}$$
$$h_{\alpha_2} \mapsto h_{\alpha_1} + h_{\alpha_2}$$
$$h_{\alpha_i} \mapsto h_{\alpha_i} \text{ for } i \geq 2$$

This is s_{α_1}. Similarly, the linear map $V \to V$ such that $\varepsilon_{i-1} \leftrightarrow \varepsilon_i$, all others fixed, is s_{α_i}.

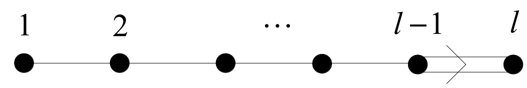
W is generated by $s_{\alpha_1},\ldots,s_{\alpha_l}$. The group generated by all transpositions $(\varepsilon_{i-1}\varepsilon_i)$ is isomorphic to S_{l+1}. So we have a homomorphism $S_{l+1} \to V$. This map is surjective; it is also injective, since any permutation of $\{\varepsilon_0,\ldots,\varepsilon_n\}$ that fixes each h_{α_i} is the identity. Hence, $W \cong S_{l+1}$.

Each h_α has the form $h_\alpha = w(h_{\alpha_i})$ for some $w \in W$ and some i. Hence

$$\Phi = \{\varepsilon_i - \varepsilon_j \mid 0 \leq i \neq j \leq l\}.$$

So $|\Phi| = l(l+1)$.

Type B_l.



Let V be a Euclidean space of dimension l with basis $\{\varepsilon_1,\ldots,\varepsilon_l\}$ such that $\langle \varepsilon_i,\varepsilon_j\rangle = K\delta_{ij}$. Define

$$
\begin{aligned}
h_{\alpha_1} &= \varepsilon_1 - \varepsilon_2 \\
h_{\alpha_2} &= \varepsilon_2 - \varepsilon_3 \\
&\vdots \\
h_{\alpha_{l-1}} &= \varepsilon_{l-1} - \varepsilon_l \\
h_{\alpha_l} &= \varepsilon_l
\end{aligned}
$$

These form a fundamental system of vectors of type B_l.

$$
\begin{gathered}
|h_{\alpha_1}| = \ldots = |h_{\alpha_{l-1}}| = \sqrt{2}|h_{\alpha_l}| \\
\langle h_{\alpha_i}, h_{\alpha_l}\rangle = 0 \text{ for } 1 \le i \le l-2 \\
\langle h_{\alpha_{l-1}}, h_{\alpha_l}\rangle = -K \\
2\frac{\langle h_{\alpha_{l-1}}, h_{\alpha_l}\rangle}{\langle h_{\alpha_{l-1}}, h_{\alpha_{l-1}}\rangle} = \frac{-2K}{2K} = -1
\end{gathered}
$$

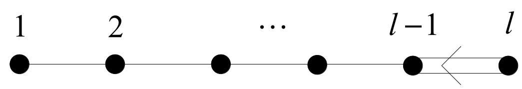
$$
\begin{gathered}
s_{\alpha_1} : \varepsilon_1 \leftrightarrow \varepsilon_2 \text{ and leaves others fixed,} \\
s_{\alpha_2} : \varepsilon_2 \leftrightarrow \varepsilon_3 \text{ and leaves others fixed,} \\
\vdots \\
s_{\alpha_{l-1}} : \varepsilon_{l-1} \leftrightarrow \varepsilon_l \text{ and leaves others fixed,} \\
s_{\alpha_l} : \varepsilon_l \mapsto -\varepsilon_l \text{ and leaves others fixed.}
\end{gathered}
$$

$W = \langle s_{\alpha_1},\ldots,s_{\alpha_l}\rangle$; for $w \in W$, $w(\varepsilon_i) = \pm\varepsilon_j$. $|W| = 2^l l!$. Each h_α has the form $h_\alpha = w(h_{\alpha_i})$ for some $w \in W$ and some i. Hence,

$$\Phi = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \le i \ne j \le l\} \cup \{\pm\varepsilon_i \mid 1 \le i \le l\}$$

$$|\Phi| = 2^2\binom{l}{2} + 2l = 2l(l-1) + 2l = 2l^2$$

Type C_l.



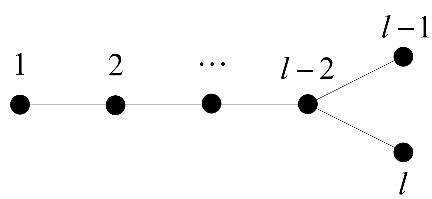
Let V be a Euclidean space of dimension l with basis $\{\varepsilon_1,\ldots,\varepsilon_l\}$ such that $\langle \varepsilon_i, \varepsilon_j \rangle = K\delta_{ij}$. Define

$$\begin{aligned} h_{\alpha_1} &= \varepsilon_1 - \varepsilon_2 \\ &\vdots \\ h_{\alpha_{l-1}} &= \varepsilon_{l-1} - \varepsilon_l \\ h_{\alpha_l} &= 2\varepsilon_l \end{aligned}$$

W is the same as for B_l. For $w \in W$, $w(\varepsilon_i) = \pm\varepsilon_j$. $|W| = 2^l l!$. The h_α are the vectors $\pm\varepsilon_i \pm \varepsilon_j$ (for $i \neq j$) and $\pm 2\varepsilon_i$. Hence,

$$|\Phi| = 2^2\binom{l}{2} + 2l = 2l(l-1) + 2l = 2l^2$$

Type D_l.



Let V be a Euclidean space of dimension l with basis $\{\varepsilon_1,\ldots,\varepsilon_l\}$ such that $\langle \varepsilon_i, \varepsilon_j \rangle = K\delta_{ij}$. Define

$$\begin{aligned} h_{\alpha_1} &= \varepsilon_1 - \varepsilon_2 \\ &\vdots \\ h_{\alpha_{l-1}} &= \varepsilon_{l-1} - \varepsilon_l \\ h_{\alpha_l} &= \varepsilon_{l-1} + \varepsilon_l \end{aligned}$$

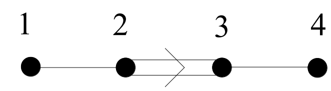
This is a fundamental system of roots of type D_l.

$$s_{\alpha_1} : \varepsilon_1 \leftrightarrow \varepsilon_2 \text{ and leaves others fixed,}$$
$$\vdots$$
$$s_{\alpha_{l-1}} : \varepsilon_{l-1} \leftrightarrow \varepsilon_l \text{ and leaves others fixed,}$$
$$s_{\alpha_l} : \pm\varepsilon_{l-1} \mapsto \mp\varepsilon_l \text{ and leaves others fixed.}$$

For $w \in W$, $w(\varepsilon_i) = \pm\varepsilon_j$. There will be an even number of sign changes, so $|W| = 2^{l-1} l!$. The h_α have the form $\pm\varepsilon_i \pm \varepsilon_j$ for $i \neq j$. So

$$|\Phi| = 2^2 \binom{l}{2} = 4\,\tfrac{l(l-1)}{2} = 2l(l-1)$$

Type F_4.



Let V be a 4-dimensional Euclidean space with basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}, \langle \varepsilon_i, \varepsilon_j \rangle = K\delta_{ij}$.

$$\begin{aligned} h_{\alpha_1} &= \varepsilon_1 - \varepsilon_2 \\ h_{\alpha_2} &= \varepsilon_2 - \varepsilon_3 \\ h_{\alpha_3} &= \varepsilon_3 \\ h_{\alpha_4} &= \tfrac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4) \end{aligned}$$

This is a fundamental system of vectors of type F_4. $s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_3}$ permute $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and change signs arbitrarily.

$$s_{\alpha_4} : \begin{cases} \varepsilon_1 \mapsto \tfrac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4) = \varepsilon_1 + h_{\alpha_4} \\ \varepsilon_2 \mapsto \tfrac{1}{2}(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4) = \varepsilon_2 + h_{\alpha_4} \\ \varepsilon_3 \mapsto \tfrac{1}{2}(-\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4) = \varepsilon_3 + h_{\alpha_4} \\ \varepsilon_4 \leftrightarrow \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4) \\ \varepsilon_1 + \varepsilon_2 \mapsto -\varepsilon_3 + \varepsilon_4 \end{cases}$$

Let $S = \{ h_\alpha \mid \alpha \in \Phi \}$.

$$\{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i \neq j \leq 3\} \subseteq S$$
$$\{\pm\varepsilon_i \mid 1 \leq i \leq 3\} \subseteq S$$

$$\tfrac{1}{2}\left(\pm\varepsilon_1 \pm\varepsilon_2 \pm\varepsilon_3 \pm\varepsilon_4\right) \in S$$
$$\left\{\pm\varepsilon_i \pm\varepsilon_4 \mid 1 \le i \le 3\right\} \subseteq S$$
$$\pm\varepsilon_4 \in S$$

So S contains

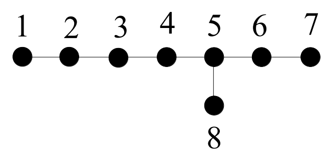
$$\pm\varepsilon_i \pm\varepsilon_j \text{ for } 1 \le i \ne j \le 4$$
$$\pm\varepsilon_i \text{ for } 1 \le i \le 4$$
$$\tfrac{1}{2}\left(\pm\varepsilon_1 \pm\varepsilon_2 \pm\varepsilon_3 \pm\varepsilon_4\right)$$

This collection of vectors is closed under the actions of $s_{\alpha_1},\ldots,s_{\alpha_4}$.

$$|\Phi| = 2^2\binom{4}{2} + 2.4 + 2^4 = 48$$

(We have 24 short roots and 24 long ones.)

Type E_8.



Let V be a Euclidean space of dimension 8 with basis $\{\varepsilon_1,\ldots,\varepsilon_8\}$ such that $\langle\varepsilon_i,\varepsilon_j\rangle = K\delta_{ij}$. Define

$$\left.\begin{aligned} h_{\alpha_1} &= \varepsilon_1 - \varepsilon_2 \\ &\vdots \\ h_{\alpha_6} &= \varepsilon_6 - \varepsilon_7 \\ h_{\alpha_7} &= \varepsilon_6 + \varepsilon_7 \end{aligned}\right\}\text{type } D_7$$
$$h_{\alpha_8} = -\tfrac{1}{2}\left(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6+\varepsilon_7+\varepsilon_8\right)$$
$$\left|h_{\alpha_7}\right| = \left|h_{\alpha_8}\right|$$
$$\left\langle h_{\alpha_7}, h_{\alpha_8}\right\rangle = -K$$
$$2\frac{\left\langle h_{\alpha_7}, h_{\alpha_8}\right\rangle}{\left\langle h_{\alpha_7}, h_{\alpha_7}\right\rangle} = \frac{-2K}{2K} = -1$$

These vectors form a fundamental system of type E_8. $s_{\alpha_1},\ldots,s_{\alpha_7}$ permute $\varepsilon_1,\ldots,\varepsilon_7$ and change an even number of signs.

$$s_{\alpha_8} : \varepsilon_i \mapsto \tfrac{1}{4}\left(-\varepsilon_1 + \ldots + 3\varepsilon_i + \ldots - \varepsilon_8\right) = \varepsilon_i + \tfrac{1}{2} h_{\alpha_8}$$

Let $S = \left\{ h_\alpha \mid \alpha \in \Phi \right\}$. S contains $\pm\varepsilon_i \pm \varepsilon_j$ for $1 \le i \ne j \le 7$. S contains $\frac{1}{2}\sum_{i=1}^{8} \varepsilon_i$ so S contains $\frac{1}{2}\left(-\varepsilon_1 - \ldots - \varepsilon_6 + \varepsilon_7 + \varepsilon_8\right) = s_{\alpha_8}\left(\varepsilon_7 + \varepsilon_8\right)$. So $\varepsilon_7 + \varepsilon_8 \in S$. So S contains $\pm\varepsilon_i \pm \varepsilon_8$ for $1 \le i \le 7$. S also contains $\frac{1}{2}\left(\pm\varepsilon_1 \pm \ldots \pm \varepsilon_8\right)$ with an even number of negative signs. So S contains

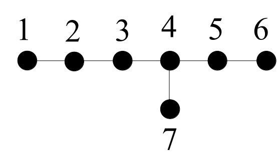
$$\pm\varepsilon_i \pm \varepsilon_j \text{ for } 1 \le i \ne j \le 8$$
$$\tfrac{1}{2}\left(\pm\varepsilon_1 \pm \ldots \pm \varepsilon_8\right) \text{ with an even number of negative signs}$$

This is the whole of S, for it is invariant under $s_{\alpha_1},\ldots,s_{\alpha_8}$: this is clear for $s_{\alpha_1},\ldots,s_{\alpha_7}$ but requires a little work to check for s_{α_8}. So the roots of E_8 are

$$\pm\varepsilon_i \pm \varepsilon_j \text{ for } 1 \le i \ne j \le 8$$
$$\tfrac{1}{2}\sum_{i=1}^{8} \pm\varepsilon_i \text{ where } \Pi(\pm) = 1$$

$$|\Phi| = 2^2\binom{8}{2} + 2^7 = 240$$

Type E_7.



Take V as before – $\dim(V) = 8$. Take V_0 to be the subspace of V perpendicular to $\varepsilon_1 - \varepsilon_8$. $\dim(V_0) = 7$ and $h_{\alpha_2},\ldots,h_{\alpha_8}$ form a basis of V_0. This is a fundamental system of type E_7. Consider $S = \left\{ h_\alpha \mid \alpha \in \Phi(E_7) \right\}$. This set lies in $\left\{ h_\alpha \mid \alpha \in \Phi(E_8) \right\} \cap V_0$. This intersection is

$$\pm\varepsilon_i \pm \varepsilon_j \text{ for } 2 \le i \ne j \le 7$$
$$\pm\left(\varepsilon_1 + \varepsilon_8\right)$$
$$\tfrac{1}{2}\left(\varepsilon_1 \pm \varepsilon_2 \pm \ldots \pm \varepsilon_7 + \varepsilon_8\right) \text{ where } \Pi(\pm) = 1$$
$$\tfrac{1}{2}\left(-\varepsilon_1 \pm \varepsilon_2 \pm \ldots \pm \varepsilon_7 - \varepsilon_8\right) \text{ where } \Pi(\pm) = 1$$

All of these can be obtained from $h_{\alpha_2},\ldots,h_{\alpha_8}$ by means of $s_{\alpha_2},\ldots,s_{\alpha_8}$. This is obvious except for $\pm(\varepsilon_1+\varepsilon_8)$.

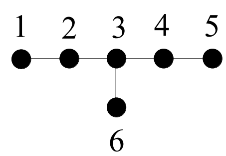
$$s_{\alpha_8}:\varepsilon_1+\varepsilon_2 \leftrightarrow \tfrac{1}{2}(\varepsilon_1-\varepsilon_2-\ldots-\varepsilon_7+\varepsilon_8)$$

So $\pm(\varepsilon_1+\varepsilon_8)\in S$. So S is

$$\begin{gathered}\pm\varepsilon_i\pm\varepsilon_j \text{ for } 2\le i\ne j\le 7\\ \pm(\varepsilon_1+\varepsilon_8)\\ \tfrac{1}{2}(\varepsilon_1\pm\varepsilon_2\pm\ldots\pm\varepsilon_7+\varepsilon_8) \text{ where } \Pi(\pm)=1\\ \tfrac{1}{2}(-\varepsilon_1\pm\varepsilon_2\pm\ldots\pm\varepsilon_7-\varepsilon_8) \text{ where } \Pi(\pm)=1\end{gathered}$$

$$|\Phi(E_7)|=2^2\binom{6}{2}+2+2^5+2^5$$

Type E_6.



We proceed as before. $h_{\alpha_3},\ldots,h_{\alpha_8}$ form a fundamental system of vectors of type E_6. Let V_0 be the subspace of V for E_8 that is orthogonal to $\varepsilon_1-\varepsilon_8$ and $\varepsilon_2-\varepsilon_8$. $\dim(V_0)=6$ and $h_{\alpha_3},\ldots,h_{\alpha_8}$ form a basis of this space.

$$\{h_\alpha \mid \alpha\in\Phi(E_6)\}\subseteq\{h_\alpha \mid \alpha\in\Phi(E_8)\}\cap V_0$$

The h_α in V_0 are

$$\begin{gathered}\pm\varepsilon_i\pm\varepsilon_j \text{ for } 3\le i\ne j\le 7\\ \tfrac{1}{2}(\varepsilon_1+\varepsilon_2\pm\varepsilon_3\pm\ldots\pm\varepsilon_7+\varepsilon_8) \text{ where } \Pi(\pm)=1\\ \tfrac{1}{2}(-\varepsilon_1-\varepsilon_2\pm\varepsilon_3\pm\ldots\pm\varepsilon_7-\varepsilon_8) \text{ where } \Pi(\pm)=1\end{gathered}$$

All of these are obtainable from $h_{\alpha_3},\ldots,h_{\alpha_8}$ by $s_{\alpha_3},\ldots,s_{\alpha_8}$.

$$|\Phi(E_6)|=2^2\binom{5}{2}+2^4+2^4=72$$

Theorem 11.1. *The number of roots in each of the indecomposable root systems is*

A_l	B_l	C_l	D_l	E_6	E_7	E_8	F_4	G_2
$l(l+2)$	$l(2l+1)$	$l(2l+1)$	$l(2l-1)$	78	133	248	52	14

12. THE SEMISIMPLE LIE ALGEBRAS

Theorem 12.1. *(a) If a semisimple Lie algebra L has connected Dynkin diagram Δ then L is simple.*
(b) If L is a semisimple Lie algebra whose Dynkin diagram Δ has connected components $\Delta_1,\ldots,\Delta_r$ then $L = L_1 \oplus \cdots \oplus L_r$ where L_i is a simple Lie algebra with Dynkin diagram Δ_i.

Proof. (a) Let $L = H \oplus \left(\oplus_{\alpha\in\Phi} L_\alpha\right)$ be a Cartan decomposition with connected Dynkin diagram Δ. Let $0 \neq I \lhd L$. We first show that $I \cap H \neq 0$. Suppose not, i.e. that $I \cap H = 0$. Let $0 \neq x \in I$ with $x = h + \sum_\alpha \mu_\alpha e_\alpha$ and the number of non-zero μ_α as small as possible. Let $\mu_\beta \neq 0$.

$$[xh_\beta] = \sum_\alpha \mu_\alpha [e_\alpha h_\beta] = \sum_\alpha \mu_\alpha \alpha(h_\beta) e_\alpha$$

By 8.7 we can choose $e_{-\beta}$ with $[e_{-\beta}e_\beta] = h_\beta$.

$$[[xh_\beta]e_{-\beta}] = -\mu_\beta \beta(h_\beta) h_\beta + \sum_{\substack{\alpha\in\Phi \\ \alpha-\beta\in\Phi}} \mu_\alpha \alpha(h_\beta) N_{\alpha,-\beta} e_{\alpha-\beta}$$

$[[xh_\beta]e_{-\beta}] \in I$ is non-zero since

$$-\mu_\beta \beta(h_\beta) h_\beta = -\underbrace{\mu_\beta}_{\neq 0} \underbrace{\langle h_\beta, h_\beta\rangle}_{\neq 0} h_\beta \neq 0$$

The number of non-zero μ_α with $\alpha - \beta \in \Phi$ is less than before, a contradiction. Hence, $I \cap H \neq 0$.

We next show that $I \supseteq H$. Suppose not. Then $0 \subset I \cap H \subset H$. $I \cap H$ is not orthogonal to all h_{α_i}, $\alpha_i \in \Pi$. For suppose $I \cap H$ is not orthogonal to h_{α_i}. Let $x \in I \cap H$ be such that $\langle x, h_{\alpha_i}\rangle \neq 0$. Then

$$[e_{\alpha_i} x] = \alpha_i(x) e_{\alpha_i} = \langle h_{\alpha_i}, x\rangle e_{\alpha_i} \in I$$

So $e_{\alpha_i} \in I$. So $[e_{-\alpha_i} e_{\alpha_i}] = h_{\alpha_i} \in I$. So for each $\alpha_i \in \Pi$ either $\langle I \cap H, h_{\alpha_i}\rangle = 0$ or $h_{\alpha_i} \in I$. Both classes are non-empty. Choose $h_{\alpha_j} \notin I$; then $\langle I \cap H, h_{\alpha_j}\rangle = 0$. This means Δ is disconnected, a contradiction. Hence, $H \subseteq I$.

Now let $\alpha \in \Phi$.

$$[e_\alpha h_\alpha] = \alpha(h_\alpha)e_\alpha = \underbrace{\langle h_\alpha, h_\alpha \rangle}_{\neq 0} e_\alpha$$

So $e_\alpha \in I$. Hence I contains H and all e_α. So $I = L$; L is simple.

(b) Suppose Δ is the disjoint union of connected components $\Delta_1, \ldots, \Delta_r$. Then Π is the union of orthogonal components $\Pi_1, \ldots, \Pi_r$. Let H_i be the subspace spanned by $\{h_\alpha \mid \alpha \in \Pi_i\}$. Then $H = H_1 \oplus \ldots \oplus H_r$ and the H_i are mutually orthogonal. Now consider s_α for some $\alpha \in \Pi_i$. Then α transforms H_i into itself and fixes each vector in H_j for $j \neq i$; $s_\alpha(H_j) = H_j$. Since the s_α for $\alpha \in \Pi$ generate W, $w(H_j) = H_j$ for each $w \in W$.

For each $\alpha \in \Phi$, $h_\alpha = w(h_{\alpha_i})$ for some $w \in W$ and some i. So $h_\alpha \in H_i$ for some i. Let $\Phi_i = \{\alpha \in \Phi \mid h_\alpha \in H_i\}$. Then $\Phi = \Phi_1 \cup \ldots \cup \Phi_r$. Let L_i be the subspace of L spanned by H_i and the L_α with $\alpha \in \Phi_i$. We see that $L = L_1 \oplus \ldots \oplus L_r$ as a direct sum of vector spaces.

To see that L_i is a subalgebra of L it is sufficient to show that $\alpha, \beta \in \Phi_i \Rightarrow [e_\alpha e_\beta] \in L_i$. If $\alpha + \beta \notin \Phi$ then $[e_\alpha e_\beta] = 0$. If $\beta = -\alpha$ then $[e_\alpha e_{-\alpha}] = -h_\alpha \in H_i$. If $\alpha + \beta \in \Phi$ then $\alpha + \beta \in \Phi_i$ and $h_{\alpha+\beta} = h_\alpha + h_\beta \in H_i$. So L_i is a subalgebra.

We next check that $i \neq j \Rightarrow [L_i L_j] = 0$. Let $\alpha \in \Phi_i$, $\beta \in \Phi_j$.

$$[h_\alpha h_\beta] = 0$$
$$[h_\alpha e_\beta] = 0 \text{ since } \langle h_\alpha, h_\beta \rangle = 0$$
$$[e_\alpha h_\beta] = 0 \text{ since } \langle h_\alpha, h_\beta \rangle = 0$$
$$[e_\alpha e_\beta] = 0 \text{ since } \alpha + \beta \notin \Phi$$

$h_\alpha + h_\beta$ does not lie in any H_k, $h_\alpha \in H_i$, $h_\beta \in H_j$. So $[L_i L_j] = 0$ for $i \neq j$:

$$[x_1 + \ldots + x_r, y_1 + \ldots + y_r] = [x_1 y_1] + \ldots + [x_r y_r]$$

So $L = L_1 \oplus \ldots \oplus L_r$ as a direct sum of Lie algebras. We now see that each L_i is semisimple. Let $I \lhd L_i$ be soluble. $[IL_j] = 0$ if $i \neq j$, so $I \lhd L$. I is a soluble ideal of L, but L is semisimple, so $I = 0$, so L_i is semisimple.

We now show that H_i is a Cartan subalgebra of L_i. H is a Cartan subalgebra of L, so there is a regular element $x \in L$ such that H is the 0-(generalized) eigenspace of $\operatorname{ad} x$. Since $x \in H$ and $H = H_1 \oplus \ldots \oplus H_r$ we can write $x = x_1 + \ldots + x_r$ with $x_i \in H_i$.

$$\operatorname{ad} x_i : L_j \to \begin{cases} L_j & i = j \\ 0 & i \neq j \end{cases}$$

So the 0-eigenspace of $\operatorname{ad} x$ on L is the direct sum of the 0-eigenspaces of the $\operatorname{ad} x_i$ on the L_i. x is regular in L if and only if each x_i is regular in L_i. So each x_i is regular in L_i and the 0-eigenspace of $\operatorname{ad} x_i$ in L_i is H_i. So H_i is a Cartan subalgebra of L_i.

$$L_i = H_i \oplus \left(\oplus_{\alpha \in \Phi_i} L_\alpha \right)$$

is a Cartan decomposition of L_i. So Φ_i is the root system of L_i; Π_i is a fundamental root system of L_i; the Dynkin diagram of L_i is Δ_i. But Δ_i is connected, so L_i is simple by (a).

■

We next consider simple Lie algebras with a given indecomposable Cartan matrix A.

Existence Problem: Is there a simple Lie algebra with given Cartan matrix A?

Isomorphism Problem: Are any two such Lie algebras isomorphic?

Let L be a simple Lie algebra and H a Cartan subalgebra of L:

$$L = H \oplus \left(\oplus_{\alpha \in \Phi} L_\alpha \right)$$
$$\Phi = \Phi^+ \cup \Phi^-$$

For each $\alpha \in \Phi^+$ choose $0 \neq e_\alpha \in L_\alpha$; $L_\alpha = \mathbb{C} e_\alpha$. Choose $e_{-\alpha} \in L_{-\alpha}$ such that $[e_{-\alpha} e_\alpha] = h_\alpha$. If $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ then the h_{α_i} and e_α form a basis of L. $[L_\alpha L_\beta] \subseteq L_{\alpha+\beta}$ so $[e_\alpha e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$ if $\alpha + \beta \in \Phi$, $\alpha \neq -\beta$.

$$[h_{\alpha_i} h_{\alpha_j}] = 0$$
$$[e_\alpha h_{\alpha_i}] = \left\langle h_\alpha, h_{\alpha_i} \right\rangle e_\alpha$$
$$[e_{-\alpha} e_\alpha] = h_\alpha$$
$$[e_\alpha e_\beta] = \begin{cases} N_{\alpha,\beta} e_{\alpha+\beta} & \alpha + \beta \in \Phi \\ 0 & \text{otherwise} \end{cases}$$

The $N_{\alpha,\beta}$ are called the *structure constants*.

Proposition 12.2. *The structure constants* $N_{\alpha,\beta}$ *satisfy*
(i) $N_{\alpha,\beta} = -N_{\beta,\alpha}$*;*
(ii) if $\alpha,\beta,\gamma \in \Phi$ *have* $\alpha+\beta+\gamma=0$ *then* $N_{\alpha,\beta} = N_{\beta,\gamma} = N_{\gamma,\alpha}$*;*
(ii) if $\alpha,\beta,\gamma,\delta \in \Phi$ *have* $\alpha+\beta+\gamma+\delta=0$ *and no pair have sum zero then*

$$N_{\alpha,\beta}N_{\gamma,\delta} + N_{\beta,\gamma}N_{\alpha,\delta} + N_{\gamma,\alpha}N_{\beta,\delta} = 0$$

(if $\xi,\eta \in \Phi$*,* $\eta \neq -\xi$*,* $\eta+\xi \notin \Phi$ *take* $N_{\xi,\eta}=0$*);*
(iv) if $\alpha,\beta \in \Phi$ *have* $\alpha+\beta \in \Phi$ *then*

$$N_{\alpha,\beta}N_{-\alpha,-\beta} = -\frac{(p+1)q}{2}\langle h_\alpha, h_\alpha\rangle$$

where the α *-chain of roots through* β *is*

$$-p\alpha+\beta,\ldots,\beta,\ldots,q\alpha+\beta.$$

In particular, $N_{\alpha,\beta} \neq 0$*, so* $[L_\alpha L_\beta] = L_{\alpha+\beta}$*.*

Proof. (i) $[e_\alpha e_\beta] = -[e_\beta e_\alpha]$ so $N_{\alpha,\beta} = -N_{\beta,\alpha}$.

(ii) Suppose $\alpha+\beta+\gamma=0$.

$$\begin{aligned}
& [[e_\alpha e_\beta]e_\gamma] + [[e_\beta e_\gamma]e_\alpha] + [[e_\gamma e_\alpha]e_\beta] = 0 \\
\Rightarrow\quad & N_{\alpha,\beta}[e_{\alpha+\beta}e_\gamma] + N_{\beta,\gamma}[e_{\beta+\gamma}e_\alpha] + N_{\gamma,\alpha}[e_{\gamma+\alpha}e_\beta] = 0 \\
\Rightarrow\quad & N_{\alpha,\beta}h_\gamma + N_{\beta,\gamma}h_\alpha + N_{\gamma,\alpha}h_\beta = 0 \\
\Rightarrow\quad & \left(-N_{\alpha,\beta} + N_{\beta,\gamma}\right)h_\alpha + \left(-N_{\alpha,\beta} + N_{\gamma,\alpha}\right)h_\beta = 0
\end{aligned}$$

Since h_α and h_β are linearly independent, $N_{\alpha,\beta} = N_{\beta,\gamma} = N_{\gamma,\alpha}$.

(iii) Take $\alpha,\beta,\gamma,\delta \in \Phi$ with zero sum and no opposite pairs.

$$\begin{aligned}
& [[e_\alpha e_\beta]e_\gamma] + [[e_\beta e_\gamma]e_\alpha] + [[e_\gamma e_\alpha]e_\beta] = 0 \\
\Rightarrow\quad & N_{\alpha,\beta}[e_{\alpha+\beta}e_\gamma] + N_{\beta,\gamma}[e_{\beta+\gamma}e_\alpha] + N_{\gamma,\alpha}[e_{\gamma+\alpha}e_\beta] = 0 \\
\Rightarrow\quad & \left(N_{\alpha,\beta}N_{\alpha+\beta,\gamma} + N_{\beta,\gamma}N_{\beta+\gamma,\alpha} + N_{\gamma,\alpha}N_{\gamma+\alpha,\beta}\right)e_{\alpha+\beta+\gamma} = 0 \\
\Rightarrow\quad & N_{\alpha,\beta}N_{\gamma,\delta} + N_{\beta,\gamma}N_{\alpha,\delta} + N_{\gamma,\alpha}N_{\beta,\delta} = 0
\end{aligned}$$

(iv) Let $\alpha,\beta \in \Phi$ with $\alpha+\beta \in \Phi$.

$$
\begin{aligned}
& [[e_\alpha e_{-\alpha}]e_\beta]+[[e_{-\alpha}e_\beta]e_\alpha]+[[e_\beta e_\alpha]e_{-\alpha}]=0 \\
\Rightarrow\quad & -[h_\alpha e_\beta]+N_{-\alpha,\beta}[e_{-\alpha+\beta}e_\alpha]+N_{\beta,\alpha}[e_{\alpha+\beta}e_{-\alpha}]=0 \\
\Rightarrow\quad & \left(\beta(h_\alpha)+N_{-\alpha,\beta}N_{-\alpha+\beta,\alpha}+N_{\beta,\alpha}N_{\alpha+\beta,-\alpha}\right)e_\beta=0 \\
\Rightarrow\quad & N_{\alpha,\beta}N_{-\alpha+\beta,\alpha}+N_{\beta,\alpha}N_{\alpha+\beta,-\alpha}=-\langle h_\alpha,h_\beta\rangle \\
\Rightarrow\quad & N_{\alpha,\beta}N_{-\alpha,-\beta}-N_{\alpha,-\alpha+\beta}N_{-\alpha,\alpha-\beta}=\langle h_\alpha,h_\beta\rangle
\end{aligned}
$$

Take $M_{\alpha,\beta}=N_{\alpha,\beta}N_{-\alpha,-\beta}$, so

$$M_{\alpha,\beta}-M_{\alpha,-\alpha+\beta}=\langle h_\alpha,h_\beta\rangle$$

Let the α-chain of roots through β be

$$-p\alpha+\beta,\ldots,\beta,\ldots,q\alpha+\beta$$

So

$$
\begin{aligned}
M_{\alpha,\beta}-M_{\alpha,-\alpha+\beta}&=\langle h_\alpha,h_\beta\rangle \\
M_{\alpha,-\alpha+\beta}-M_{\alpha,-2\alpha+\beta}&=\langle h_\alpha,h_{-\alpha+\beta}\rangle \\
M_{\alpha,-2\alpha+\beta}-M_{\alpha,-3\alpha+\beta}&=\langle h_\alpha,h_{-2\alpha+\beta}\rangle \\
&\vdots \\
M_{\alpha,-p\alpha+\beta}&=\langle h_\alpha,h_{-p\alpha+\beta}\rangle
\end{aligned}
$$

So

$$
\begin{aligned}
M_{\alpha,\beta}&=(p+1)\langle h_\alpha,h_\beta\rangle-\langle h_\alpha,h_\alpha\rangle(1+2+\ldots+p) \\
&=(p+1)\langle h_\alpha,h_\beta\rangle-\tfrac{p(p+1)}{2}\langle h_\alpha,h_\alpha\rangle
\end{aligned}
$$

By, by 8.11, $2\langle h_\alpha,h_\beta\rangle/\langle h_\alpha,h_\alpha\rangle=p-q$, so

$$
\begin{aligned}
M_{\alpha,\beta}&=\langle h_\alpha,h_\alpha\rangle\left(\tfrac{(p+1)(p-q)}{2}-\tfrac{p(p+1)}{2}\right) \\
&=-\tfrac{(p+1)q}{2}\langle h_\alpha,h_\alpha\rangle
\end{aligned}
$$

So $N_{\alpha,\beta}\neq 0$; $[L_\alpha L_\beta]=L_{\alpha+\beta}$.

■

This result has certain consequences. Let $\alpha,\beta \in \Phi$, $\alpha+\beta \in \Phi$, $[e_\alpha e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$. Let $\gamma = -\alpha-\beta$. Then $\alpha+\beta+\gamma = 0$. We have the following ordered pairs of roots whose sum is a root:

$$\begin{array}{cccccc} (\alpha,\beta) & (\beta,\gamma) & (\gamma,\alpha) & (\beta,\alpha) & (\gamma,\beta) & (\alpha,\gamma) \\ (-\alpha,-\beta) & (-\beta,-\gamma) & (-\gamma,-\alpha) & (-\beta,-\alpha) & (-\gamma,-\beta) & (-\alpha,-\gamma) \end{array}$$

We have a total order $\alpha \prec \beta$. An ordered pair (α,β) such that $0 \prec \alpha \prec \beta$ is called a *special pair*.

Either one or two of α,β,γ are positive; if one is positive two of $-\alpha,-\beta,-\gamma$ are positive. Of the twelve pairs above just one is special.

$N_{\alpha,\beta}$, for any ordered pair (α,β), can be expressed in terms of $N_{\xi,\eta}$ for (ξ,η) a special pair by using 12.2(i), (ii), (iv). So consider $N_{\alpha,\beta}$ when (α,β) is special; $\alpha+\beta \in \Phi^+ \setminus \Pi$. This root may be expressible as $\alpha+\beta = \alpha'+\beta'$ where (α',β') is special and distinct from (α,β).

A special pair (α,β) is called *extra special* if for any special pair (α',β') with $\alpha+\beta = \alpha'+\beta'$ we have $\alpha \preceq \alpha'$.

The number of extra special pairs is $\left|\Phi^+ \setminus \Pi\right|$.

Now let (α',β') be special but not extra special. Then $\alpha+\beta = \alpha'+\beta'$ where (α,β) is extra special – such an extra special pair exists because the set of special and extra special pairs is finite.

$$\alpha'+\beta'+(-\alpha)+(-\beta) = 0$$
$$N_{\alpha',\beta'}N_{-\alpha,-\beta} + N_{\beta',-\alpha}N_{\alpha',-\beta} + N_{-\alpha,\alpha'}N_{\beta',-\beta} = 0$$
$$0 \prec \alpha \prec \alpha' \prec \beta' \prec \beta$$
$$N_{\alpha',\beta'}N_{-\alpha,-\beta} + N_{\beta-\alpha',\alpha'}N_{-\alpha,-(\beta'-\alpha)} + N_{\beta-\beta',\beta'}N_{-(\alpha'-\alpha),-\alpha} = 0$$

We show that $N_{\alpha',\beta'}$ is determined by $N_{\xi,\eta}$'s for extra special pairs (ξ,η). We use induction on $\alpha'+\beta'$:

$N_{\alpha',\beta'}$ is determined by $N_{\alpha,\beta}$, $N_{\beta-\alpha',\alpha'}$, $N_{\alpha,\beta'-\alpha}$, $N_{\beta-\beta',\beta'}$, $N_{\alpha'-\alpha,\alpha}$. (α,β) is extra special.

Either $(\beta-\alpha',\alpha')$ or $(\alpha',\beta-\alpha')$ is special:

$$(\beta-\alpha')+\alpha'=\beta\prec\alpha+\beta=\alpha'+\beta'$$

So $N_{\beta-\alpha',\alpha'}$ can be expressed in terms of $N_{\xi,\eta}$ for extra special pairs (ξ,η).

Either $(\alpha,\beta'-\alpha)$ or $(\beta'-\alpha,\alpha)$ is special:

$$\alpha+(\beta'-\alpha)=\beta'\prec\alpha'+\beta'$$

Either $(\beta-\beta',\beta')$ or $(\beta',\beta-\beta')$ is special:

$$(\beta-\beta')+\beta'=\beta\prec\alpha'+\beta'$$

Either $(\alpha'-\alpha,\alpha)$ or $(\alpha,\alpha'-\alpha)$ is special:

$$(\alpha'-\alpha)+\alpha=\alpha'\prec\alpha'+\beta'$$

Hence, $N_{\alpha',\beta'}$ can be expressed in terms of $N_{\xi,\eta}$'s for extra special pairs (ξ,η). So relations 12.2(i)-(iv) expresses all $N_{\alpha',\beta'}$'s in terms of $N_{\xi,\eta}$'s for extra special pairs (ξ,η).

Theorem 12.3. *There is a unique simple Lie algebra, up to isomorphism, with a given indecomposable Cartan matrix.*

$$L=H\oplus\left(\oplus_{\alpha\in\Phi}L_\alpha\right)$$
$$\dim(L)=l+|\Phi|$$

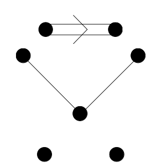
Thus, the simple Lie algebras and their dimensions are given by

$\dim(A_l)=l(l+2)$	$\dim(B_l)=l(2l+1)$	$\dim(C_l)=l(2l+1)$	$\dim(D_l)=l(2l-1)$
$(l\geq 1)$	$(l\geq 2)$	$(l\geq 3)$	$(l\geq 4)$

$\dim(E_6)=78$ $\quad\dim(E_7)=133$ $\quad\dim(E_8)=248$ $\quad\dim(F_4)=52$ $\quad\dim(G_2)=14$

Note. The following are isomorphic:

$$B_2\cong C_2$$
$$A_3\cong D_3$$
$$D_2\cong A_1\oplus A_1$$



Proof. *Uniqueness.* Let L,L' be simple Lie algebras with indecomposable Cartan matrix $A=(a_{ij})$. L has a Cartan decomposition $L=H\oplus\left(\oplus_{\alpha\in\Phi}L_\alpha\right)$. If $\Pi=\{\alpha_1,\ldots,\alpha_l\}$ then H

has basis $\{h_{\alpha_1},\dots,h_{\alpha_l}\}$; L has basis $\{h_{\alpha_1},\dots,h_{\alpha_l}\}\cup\{e_\alpha \mid \alpha\in\Phi\}$. Multiplication of basis elements:

$$[h_{\alpha_i}h_{\alpha_j}]=0$$
$$[e_\alpha h_{\alpha_i}]=\langle h_\alpha,h_{\alpha_i}\rangle e_\alpha$$
$$[e_{-\alpha}e_\alpha]=h_\alpha$$
$$[e_\alpha e_\beta]=\begin{cases} N_{\alpha,\beta}e_{\alpha+\beta} & \alpha+\beta\in\Phi \\ 0 & 0\neq\alpha+\beta\notin\Phi\end{cases}$$

All scalar products $\langle h_\alpha,h_\beta\rangle$ are determined by A. Also, all of the h_α (as linear combinations of the h_{α_i}) are determined by A.

$$s_{\alpha_i}(h_{\alpha_j})=h_{\alpha_j}-a_{ij}h_{\alpha_i}$$

So the s_{α_i} are determined by A. The Weyl group W is generated by $s_{\alpha_1},\dots,s_{\alpha_l}$. So W is determined by A. $h_\alpha=w(h_{\alpha_i})$ for some i and some $w\in W$. So the h_α are determined by A.

$$s_\alpha(h_\beta)=h_\beta-2\frac{\langle h_\alpha,h_\beta\rangle}{\langle h_\alpha,h_\alpha\rangle}h_\alpha$$

So $2\langle h_\alpha,h_\beta\rangle/\langle h_\alpha,h_\alpha\rangle$ is determined by A. But

$$\frac{1}{\langle h_\alpha,h_\alpha\rangle}=\sum_{\beta\in\Phi}\left(\frac{\langle h_\alpha,h_\beta\rangle}{\langle h_\alpha,h_\alpha\rangle}\right)^2$$

by 8.13. So $\langle h_\alpha,h_\alpha\rangle$ is determined by A. So $\langle h_\alpha,h_\beta\rangle$ is determined by A.

Suppose a basis $\{h'_{\alpha_1},\dots,h'_{\alpha_l}\}\cup\{e'_\alpha \mid \alpha\in\Phi\}$ of L' is given. We describe how to choose a basis of L. The h_{α_i} are uniquely determined. Choose $e_\alpha\neq 0$ in L_α for each $\alpha\in\Pi$. For each $\alpha\in\Phi^+\setminus\Pi$ there is a unique extra special pair (β,γ) such that $\alpha=\beta+\gamma$, $\beta,\gamma\prec\alpha$.

Assume by induction that e_β,e_γ are already chosen. Choose e_α by $e_\alpha=N_{\beta,\gamma}[e_\beta e_\gamma]$ where $N_{\beta,\gamma}=N'_{\beta,\gamma}$, the structure constant for L'. Having thus chosen e_α for $\alpha\in\Phi^+$, we choose $e_{-\alpha}$ by $[e_{-\alpha}e_\alpha]=h_\alpha$.

The $N_{\alpha,\beta}$ for arbitrary α,β are determined by the $N_{\xi,\eta}$, where (ξ,η) is extra special, by 12.2. Since $N_{\xi,\eta}=N'_{\xi,\eta}$ for all extra special (ξ,η) it follows that $N_{\alpha,\beta}=N'_{\alpha,\beta}$ for all $\alpha,\beta\in\Phi$ with $\alpha+\beta\in\Phi$.

This shows that L and L' are isomorphic.

Existence. (Sketch proof.) Begin with Cartan matrix $A=(a_{ij})$. Let H be an l-dimensional vector space over $\mathbb{C}$ with basis $h_{\alpha_1},\ldots,h_{\alpha_l}$. We define $s_{\alpha_i}:H\to H$ by $s_{\alpha_i}(h_{\alpha_j})=h_{\alpha_j}-a_{ij}h_{\alpha_i}$, a self-inverse map. Let W be the group of all non-singular linear maps $H\to H$ generated by $s_{\alpha_1},\ldots,s_{\alpha_l}$. W is finite. Correspondingly,

$$\{h_\alpha = w(h_{\alpha_i}) \mid w\in W, 1\le i\le l\}$$

is also finite. (The h_α were determined in Chapter 11.) We now define a bilinear map

$$\begin{gathered} H\times H\to\mathbb{C} \\ (x,y)\mapsto\langle x,y\rangle \end{gathered}$$

This form is uniquely determined by A. Define $\alpha\in H^*$ by $\alpha(x)=\langle h_\alpha,x\rangle$; let Φ be the set of all such α.

Let L be a vector space over $\mathbb{C}$ with $\dim(L)=\dim(H)+|\Phi|$ with basis

$$\{h_{\alpha_1},\ldots,h_{\alpha_l}\}\cup\{e_\alpha \mid \alpha\in\Phi\}$$

Define a bilinear map

$$\begin{gathered} L\times L\to L \\ (x,y)\mapsto[xy] \end{gathered}$$

We define $[\ ]$ on the basis elements by

$$\begin{gathered} [h_{\alpha_i}h_{\alpha_j}]=0 \\ [e_\alpha h_{\alpha_i}]=-[h_{\alpha_i}e_\alpha]=\langle h_\alpha,h_{\alpha_i}\rangle e_\alpha \\ [e_{-\alpha}e_\alpha]=h_\alpha \\ [e_\alpha e_\beta]=\begin{cases} N_{\alpha,\beta}e_{\alpha+\beta} & \alpha+\beta\in\Phi \\ 0 & 0\ne\alpha+\beta\notin\Phi \end{cases} \end{gathered}$$

The $N_{\alpha,\beta}$ can be chosen arbitrarily if (α,β) is extra special, e.g. $N_{\alpha,\beta}=1$. $N_{\alpha,\beta}$ is determined for all other pairs by 12.2. So multiplication of basis elements is determined by A. We make various checks:

Check $[xx]=0$ for all $x\in L$. (Easy.) Check $[[xy]z]+[[yz]x]+[[zx]y]+0$. (Most are easy, but $x=e_\alpha$, $y=e_\beta$, $z=e_\gamma$ is difficult.) Then L is a Lie algebra, $L=H\oplus\left(\oplus_{\alpha\in\Phi}L_\alpha\right)$, $L_\alpha=\mathbb{C}e_\alpha$. Check that H is a Cartan subalgebra of L. (Difficult.) Then $L=H\oplus\left(\oplus_{\alpha\in\Phi}L_\alpha\right)$ is a Cartan decomposition of L with respect to H. (Easy.) Then Φ is the set of roots of L with respect to H. $\Pi=\{\alpha_1,\ldots,\alpha_l\}$ is a fundamental system of roots inside Φ. We have

$$2\frac{\left\langle h_{\alpha_i},h_{\alpha_j}\right\rangle}{\left\langle h_{\alpha_i},h_{\alpha_i}\right\rangle}=a_{ij}$$

so A is the Cartan matrix. Finally, the argument of 12.1(a) proves that L is simple.

■

Review.

$$L\text{ semisimple}\xrightarrow[\substack{\text{choose}\\ \text{Cartan}\\ \text{subalgebra}\\ H}]{}\Phi\xrightarrow[\substack{\text{choose}\\ \text{fundamental}\\ \text{roots}\\ \Pi}]{}A=(a_{ij})$$

If we choose a different Cartan subalgebra and a different fundamental system do we get a different A?

Theorem 12.4. *(i) Let L be a Lie algebra and H_1,H_2 Cartan subalgebras. Then there exists an automorphism $\theta:L\to L$ such that $\theta(H_1)=H_2$.*
(ii) A subalgebra H of L is a Cartan subalgebra if and only if H is nilpotent and $H=\mathcal{N}(H)$.

Theorem 12.5. *Let Φ be the root system of a semisimple Lie algebra and let Π_1,Π_2 be two fundamental systems in Φ. Then there is a $w\in W$ such that $w(\Pi_1)=\Pi_2$.*

12.4 and 12.5 imply that the Cartan matrix is uniquely determined by L. So the simple Lie algebras on our list are pairwise non-isomorphic.

We have four infinite families of simple Lie algebras and five exceptional ones:

Classical				Exceptional				
A_l	B_l	C_l	D_l	E_6	E_7	E_8	F_4	G_2
$l(l+2)$	$l(2l+1)$	$l(2l+1)$	$l(2l-1)$	78	133	248	52	14
$l \geq 1$	$l \geq 2$	$l \geq 3$	$l \geq 4$					

Type A_l. We can write $\dim(A_l) = l(l+2) = (l+1)^2 - 1$. The set $\mathfrak{sl}_{l+1}(\mathbb{C})$ of all $(l+1)\times(l+1)$ matrices of trace zero forms a Lie algebra of type A_l. The diagonal subalgebra is a Cartan subalgebra.

Type B_l. The set $\mathfrak{so}_{2l+1}(\mathbb{C})$ of all $(2l+1)\times(2l+1)$ matrices X satisfying

$$X^{\mathrm{T}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{pmatrix} = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{pmatrix}X$$

forms a simple Lie algebra of type B_l. The diagonal subalgebra is a Cartan subalgebra. $\mathfrak{so}_{2l+1}(\mathbb{C})$ is isomorphic to the Lie algebra of all $(2l+1)\times(2l+1)$ skew-symmetric matrices. Elements of $\mathfrak{so}_{2l+1}(\mathbb{C})$ have the block form

$$X = \begin{pmatrix} 0 & X_{01} & X_{02} \\ -X_{02}^{\mathrm{T}} & X_{11} & X_{12} \\ -X_{01}^{\mathrm{T}} & X_{22} & -X_{11}^{\mathrm{T}} \end{pmatrix}$$

where X_{11} is an arbitrary $l \times l$ matrix, X_{12} and X_{21} are $l \times l$ symmetric matrices and X_{01} and X_{02} are arbitrary $1 \times l$ matrices (row vectors).

Type C_l. The set $\mathfrak{sp}_{2l}(\mathbb{C})$ of all $2l \times 2l$ matrices X satisfying

$$X^{\mathrm{T}}\begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix} = -\begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}X$$

forms a simple Lie algebra of type C_l. The diagonal subalgebra is a Cartan subalgebra. Elements of $\mathfrak{sp}_{2l}(\mathbb{C})$ have the block form

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & -X_{11}^{\mathrm{T}} \end{pmatrix}$$

where X_{11} is an arbitrary $l \times l$ matrix and X_{12} and X_{21} are $l \times l$ symmetric matrices.

Type D_l. The set $\mathfrak{so}_{2l}(\mathbb{C})$ of all $2l \times 2l$ matrices X such that

$$X^{\mathrm{T}}\begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix} = -\begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}X$$

forms a simple Lie algebra of type D_l. The diagonal subalgebra is a Cartan subalgebra. $\mathfrak{so}_{2l}(\mathbb{C})$ is isomorphic to the Lie algebra of all $2l \times 2l$ skew-symmetric matrices. Elements of $\mathfrak{so}_{2l}(\mathbb{C})$ have the block form

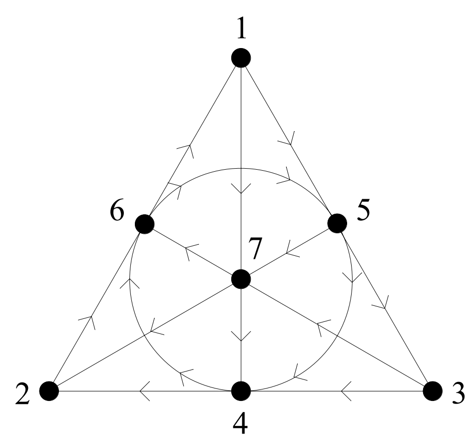
$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & -X_{11}^{\mathrm{T}} \end{pmatrix}$$

where X_{11} is an arbitrary $l \times l$ matrix and X_{12} and X_{21} are $l \times l$ skew-symmetric matrices.

$\mathfrak{sl}_m(\mathbb{C})$ is the Lie algebra of $\mathrm{SL}_m(\mathbb{C}) = \{X \in \mathrm{GL}_m(\mathbb{C}) \mid \det(X) = 1\}$; $\mathfrak{so}_m(\mathbb{C})$ is the Lie algebra of $\mathrm{SO}_m(\mathbb{C}) = \{X \in \mathrm{GL}_m(\mathbb{C}) \mid X^{\mathrm{T}}X = I_m \text{ and } \det(X) = 1\}$.

Type G_2. $\dim(G_2) = 14$. Consider the algebra of *octonians* (a.k.a. *Cayley numbers*), $\mathcal{O}$. $\dim(\mathcal{O}) = 8$. $\mathcal{O}$ has basis $1, e_1, e_2, \ldots, e_7$:

1 is the multiplicative identity;
$e_i^2 = -1$ for $1 \le i \le 7$;
$e_i e_j = \pm e_k$ for $1 \le i \ne j \le 7$.



The projective plane over the 2-field.

$e_i e_j = e_k$ if $i \to j$; $e_i e_j = -e_k$ if $i \leftarrow j$.

$\mathcal{O}$ is a non-associative algebra. The set of all derivations of $\mathcal{O}$, i.e. linear maps $D:\mathcal{O}\to\mathcal{O}$ such that $D(xy)=D(x)y+xD(y)$, forms a Lie algebra of type G_2.

Type F_4. Define the *octonian conjugate*:

$$x=a_0 1+\sum_{i=1}^{7} a_i e_i$$
$$\overline{x}=a_0 1-\sum_{i=1}^{7} a_i e_i$$
$$x=\overline{x}\Leftrightarrow x=a_0 1$$

A matrix M over $\mathcal{O}$ is called *Hermitian* if $M^{\mathrm{T}}=\overline{M}$. Let $\mathcal{J}$ be the $\mathbb{C}$-vector space of all 3×3 Hermitian matrices over $\mathcal{O}$. Such matrices have the form

$$M=\begin{pmatrix} a1 & x & y \\ \overline{x} & b1 & z \\ \overline{y} & \overline{z} & c1 \end{pmatrix}$$

where $a,b,c\in\mathbb{C}$ and $x,y,z\in\mathcal{O}$. $\dim(\mathcal{J})=27$. We define multiplication on $\mathcal{J}$ by

$$M_1\times M_2=\tfrac{1}{2}(M_1M_2+M_2M_1)$$
$$M_1\times M_2\in\mathcal{J}\ \text{ for } M_1,M_2\in\mathcal{J}$$

$\mathcal{J}$ is a commutative non-associative algebra; it is an example of a *Jordan algebra*, the axioms for which are that

$$X\times Y=Y\times X$$
$$(X^2\times Y)\times X=X^2\times(Y\times X)$$

The derivations of $\mathcal{J}$ form a simple Lie algebra of type F_4.

E_6, E_7 and E_8 can all be described in terms of $\mathcal{O}$ and $\mathcal{J}$.

There is an alternative approach to the existence theorem, which proceeds (in outline) as follows:

Let L be a simple Lie algebra with Cartan matrix $A=(a_{ij})$, $L=H\oplus(\oplus_{\alpha\in\Phi}L_\alpha)$. H has basis $h_{\alpha_1},\ldots,h_{\alpha_l}$. Let

$$h_i=\frac{2h_{\alpha_i}}{\langle h_{\alpha_i},h_{\alpha_i}\rangle}.$$

$h_1,\dots,h_l$ also form a basis of H. Choose $0 \neq e_i \in L_{\alpha_i}$ and $0 \neq f_i \in L_{-\alpha_i}$.

$$\begin{aligned}
[e_j h_i] &= \alpha_j(h_i)e_j \\
&= \left\langle h_{\alpha_j}, h_i \right\rangle e_j \\
&= \frac{2\left\langle h_{\alpha_i}, h_{\alpha_j} \right\rangle}{\left\langle h_{\alpha_i}, h_{\alpha_i} \right\rangle} \\
&= a_{ij} e_j \\
[f_j h_i] &= -a_{ij} f_j
\end{aligned}$$

Choose f_i with $[f_i e_i] = h_i$; $e_1,\dots,e_l$ generate $\oplus_{\alpha\in\Phi^+} L_\alpha$; $f_1,\dots,f_l$ generate $\oplus_{\alpha\in\Phi^-} L_\alpha$; $h_1,\dots,h_l$ generate H. So $G = \{e_i, f_i, h_i \mid 1 \le i \le l\}$ generates L. We have relations $\mathcal{R}$:

$$\begin{gathered}
[h_i h_j] = 0 \\
[e_j h_i] = a_{ij} e_j \\
[f_j h_i] = -a_{ij} f_j \\
[f_i e_i] = h_i \\
[f_j e_i] = 0 \text{ if } i \neq j \\
[e_i \dots e_i [e_i e_j]] = 0 \text{ if } i \neq j \;\; (1 - a_{ij} \;\; e_i\text{'s}) \\
[f_i \dots f_i [f_i f_j]] = 0 \text{ if } i \neq j \;\; (1 - a_{ij} \;\; f_i\text{'s})
\end{gathered}$$

(The requirements for $1 - a_{ij}$ e_i's and f_i's arise from consideration of the α_i-chain of roots through α_j.) The Lie algebra generated by G with relations $\mathcal{R}$ is a finite-dimensional Lie algebra with Cartan matrix A.

L is constructed as follows: let R be the polynomial ring $\mathbb{C}\left\langle e_1,\dots,e_l, f_1,\dots,f_l, h_1,\dots,h_l \right\rangle$ with non-commutative variables. $[R]$ is the Lie algebra obtained from R. Let M be the subalgebra generated by $e_1,\dots,e_l, f_1,\dots,f_l, h_1,\dots,h_l$. Let I be the ideal of M generated by

$$[h_i h_j],\ [e_j h_i] - a_{ij} e_j,\ [f_j h_i] + a_{ij} f_j,\ [f_j e_i] - \delta_{ij} h_i,\ [e_i \dots e_i [e_i e_j]],\ [f_i \dots f_i [f_i f_j]].$$

Then $L = M/I$. We can show that L is finite-dimensional and has Cartan matrix A.