Partial Differential Equations

Timothy J. Sullivan
University of Warwick

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These notes are based upon several sources, notably the lectures given for MA4A2 Advanced PDEs in the spring of 2006 by Dr Valeriy Slastikov at the University of Warwick. That course drew heavily on Evans' monograph [Ev]. When that course ran as a reading course two years later, with Dr Florian Theil leading the course and with me as his assistant, we altered the syllabus slightly, skipping over the introductory classical theory so that the Lax–Milgram theory for parabolic problems and/or non-cylindrical domains could be explored some more; this material is treated in Showalter's book [Sh] and is not contained in these notes.

I would like to thank Eriatarka and Clare Jones for pointing out typographical errors in these notes. There are no doubt many more lurking undetected, for which I take the blame. I would welcome corrections at t.j.sullivan@warwick.ac.uk.

T.J.S.

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References

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0 Some Basic Facts about PDEs

Definition 0.1. A *partial differential equation* (PDE) is an equation involving an unknown function and its partial derivatives.

In general, we need further information in order to solve a PDE: for example, consider the Poisson[1] equation

$$\Delta u(x) = f(x),$$

for $x \in \Omega \subseteq \mathbb{R}^2$, say. We also specify boundary conditions (BCs), for instance of Dirichlet type

$$u|_{\partial\Omega} = g$$

or Neumann type

$$\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = h,$$

where $\frac{\partial u}{\partial \nu}$ means $\nabla u \cdot \nu$, and ν is the unit outward normal field to $\partial\Omega$.

Example 0.2. For $u : \Omega \to \mathbb{R}$ and $x \in \Omega \subseteq \mathbb{R}^n$, we specify $F : \mathbb{R}^{n^2} \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ and ask that

$$F(\mathrm{D}^2 u, \mathrm{D}u, u, x) = 0 \quad \forall x \in \Omega. \tag{0.1}$$

This is the general form of the second order PDE. If u satisfies (0.1) then we call u a *solution* of the PDE.

The choice of BCs can affect the solution a great deal. For example, if we try to solve Laplace's equation

$$\Delta u = 0$$

on the half-plane $\Omega := \{(x, y) \in \mathbb{R}^2 | x \geq 0\}$ subject to the single BC $u|_{\{x=0\}} = 0$, then we have two solutions:

$$u(x, y) = 0 \text{ and } u(x, y) = x.$$

If we impose a second BC that $u(x) \to 0$ as $x \to \infty$, we get the first of these two solutions; the BC $u(x) - x \to 0$ as $x \to \infty$ gives us the second.

In general we wish to find the solution u to our PDE explicitly. However, we are very lucky if we can do this. Otherwise, we must settle merely for proving the existence of solutions and their uniqueness.

Solutions to PDEs may not be unique. For example, if we seek to solve the nonlinear equation

$$\Delta u = u(u^2 - 1)$$

on a region $\Omega \subseteq \mathbb{R}^2$ that is symmetric about both coordinate axes and not convex, then whenever u is a solution, so is $-u$. If we cannot prove uniqueness, then we must specify or choose a solution by other means.

The regularity (continuity, differentiability, & c.) of solutions is also of concern.

[1] Siméon-Denis Poisson (1781–1840).

We make frequent use of multi-index notation: for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, we set

$$|\alpha| := \alpha_1 + \cdots + \alpha_n,$$

and

$$\mathrm{D}^\alpha u := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} : \Omega \to \mathbb{R}.$$

We have the *gradient*:

$$\mathrm{D}u \equiv \nabla u := \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n} \right) : \Omega \to \mathbb{R}^n;$$

and *Hessian*:

$$\mathrm{D}^2 u := \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)_{1 \le i,j \le n} : \Omega \to \mathbb{R}^{n \times n}.$$

Definition 0.3. A PDE is *linear* if it is of the form

$$\sum_{|\alpha| \le k} a_\alpha(x) \mathrm{D}^\alpha u = f(x). \tag{0.2}$$

For linear PDEs, we have *superposition* of solutions: if u_1, u_2 are solutions then $\beta_1 u_1 + \beta_2 u_2$ is a solution for all $\beta_1, \beta_2 \in \mathbb{R}$.

Definition 0.4. A PDE is *semilinear* if the highest-order terms are linear:

$$\sum_{|\alpha| = k} a_\alpha(x) \mathrm{D}^\alpha u + a_0(\mathrm{D}^{k-1} u, \dots, \mathrm{D}u, u, x) = 0. \tag{0.3}$$

For example, $a(x)\Delta u = f$.

Definition 0.5. A PDE is *quasilinear* if it is of the form

$$\sum_{|\alpha| = k} a_\alpha(\mathrm{D}^{k-1} u, \dots, \mathrm{D}u, u, x) \mathrm{D}^\alpha u + a_0(\mathrm{D}^{k-1} u, \dots, \mathrm{D}u, u, x) = 0. \tag{0.4}$$

For example, $xu\Delta u = f$.

Definition 0.6. A PDE is *nonlinear* if the highest-order derivative enters non-linearly; i.e., it is neither linear, semilinear nor quasilinear.

For example, $F(\mathrm{D}^2 u) = 0$ is nonlinear if F is nonlinear.

Examples 0.7. Some classical examples of PDEs:

1. Laplace: $\Delta u = 0$.
2. Poisson: $\Delta u = f(x)$. (Nonlinear Poisson: $\Delta u = f(u, x)$.)
3. Heat: $u_t - \Delta u = 0$.
4. Wave: $u_{tt} - \Delta u = 0$.
5. Eikonal: $|\mathrm{D}u| = 1$.
6. Hamilton-Jacobi: $u_t + H(\mathrm{D}u, x) = 0$, H convex.
7. Reaction-Diffusion: $u_t - \Delta u = f(x)$.

1 Laplace's Equation

1.1 Laplace's Equation

We seek solutions $u\colon \Omega \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}^n$, to Laplace's equation [2]

$$\Delta u = 0, \tag{1.1}$$

where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

We typically think of Ω as an open set, and so use $\bar{\Omega}$ for a closed set. We also assume that Ω is connected.

Definition 1.1. $u\colon \Omega \to \mathbb{R}$ is *harmonic* if $\Delta u = 0$ on Ω.

For $\Omega \subset \mathbb{R}^n$ with measure $|\Omega|$, let $\fint$ denote the mean value integral:

$$\fint_\Omega := \frac{1}{|\Omega|} \int_\Omega .$$

In particular, we use this notation for the spherical mean:

$$\fint_{\partial \mathbb{B}_r(x)} = \frac{1}{\omega_n r^{n-1}} \int_{\partial \mathbb{B}_r(x)},$$

and

$$\fint_{\mathbb{B}_r(x)} = \frac{n}{\omega_n r^n} \int_{\mathbb{B}_r(x)},$$

where ω_n is the surface area of the unit sphere $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$.

Definition 1.2. $u\colon \Omega \to \mathbb{R}$ satisfies the *mean value property* (MVP) if either

(i) $u(x) = \fint_{\partial \mathbb{B}_r(x)} u(y)\, \mathrm{d}S(y)$ for all $\mathbb{B}_r(x) \subseteq \Omega$; or

(ii) $u(x) = \fint_{\mathbb{B}_r(x)} u(y)\, \mathrm{d}y$ for all $\mathbb{B}_r(x) \subseteq \Omega$.

Exercise 1.3. Show that versions (i) and (ii) of the MVP are equivalent.

Theorem 1.4. *If $u \in C^2(\Omega)$ is harmonic, then u satisfies the* MVP.

Proof. Fix $x \in \Omega$ and define

$$\phi(r) := \fint_{\partial \mathbb{B}_r(x)} u(y)\, \mathrm{d}S(y).$$

We want to show that $\phi'(r) = 0$. $\phi'(r) = \frac{\mathrm{d}}{\mathrm{d}r} \fint_{\partial \mathbb{B}_r(x)} u(y)\, \mathrm{d}S(y)$. Setting $z := \frac{y-x}{r}$,

$$\fint_{\partial \mathbb{B}_r(x)} u(y)\, \mathrm{d}S(y) = \fint_{\partial \mathbb{B}_1(0)} u(x + rz)\, \mathrm{d}S(z).$$

[2]Pierre-Simon, Marquis de Laplace (1749–1827).

$$
\begin{aligned}
\phi'(r) &= \fint_{\partial\mathbb{B}_1(0)} \mathrm{D}u(x+rz)\cdot z\,\mathrm{d}S(z) \\
&= \fint_{\partial\mathbb{B}_r(x)} \mathrm{D}u(y)\cdot \frac{y-x}{r}\,\mathrm{d}S(y) \\
&= \fint_{\partial\mathbb{B}_r(x)} \frac{\partial u}{\partial \nu}(y)\,\mathrm{d}S(y) \text{ since } \frac{y-x}{r} \perp \partial\mathbb{B}_r(x) \\
&= \fint_{\mathbb{B}_r(x)} \Delta u(y)\,\mathrm{d}y \\
&= 0 \text{ since } u \text{ is harmonic.}
\end{aligned}
$$

So $\fint_{\partial\mathbb{B}_r(x)} u(y)\,\mathrm{d}S(y) = \text{constant} = u(x)$ since $\fint_{\partial\mathbb{B}_r(x)} u(y)\,\mathrm{d}S(y) \to u(x)$ as $r \to 0$. Hence u satisfies the MVP. ☐

Theorem 1.5. *If $u \in C^2(\Omega)$ has the* MVP*, then u is harmonic.*

Proof. Suppose not, i.e. that there exists $x_0 \in \Omega$ with $\Delta u(x_0) \neq 0$. Without loss of generality, assume $\Delta u(x_0) > 0$. So there is a ball $\mathbb{B}_r(x_0) \subseteq \Omega$ such that $\Delta u > 0$ on $\mathbb{B}_r(x_0)$.

$$
\phi'(r) = \frac{\mathrm{d}}{\mathrm{d}r} \fint_{\partial\mathbb{B}_r(x_0)} u(y)\,\mathrm{d}S(y) = \fint_{\mathbb{B}_r(x_0)} \Delta u(y)\,\mathrm{d}y > 0.
$$

So ϕ is not constant, so u is not harmonic. ☐

Proposition 1.6. *If $u \in C(\bar{\Omega})$ satisfies the* MVP*, then the minimum and maximum of u are attained on $\partial\Omega$ or else u is constant.*

Proof. We prove the claim for maximum; the same proof applies for minimum. Set

$$
\Omega_M := \left\{ x \in \Omega \,\middle|\, u(x) = M := \max_{y\in\bar{\Omega}} u(y) \right\}.
$$

Obviously, Ω_M is relatively closed with respect to Ω. We want to show that Ω_M is also relatively open. Then, since Ω is assumed to be connected, we have $\Omega_M = \Omega$, and result follows. Take $x_0 \in \Omega_M$, $\mathbb{B}_r(x_0) \subseteq \Omega$.

$$
M \geq \fint_{\mathbb{B}_r(x_0)} u(y)\,\mathrm{d}y = u(x_0) = M.
$$

We have equality if, and only if, $u(y) = M$ for all $y \in \mathbb{B}_r(x_0)$, so $\mathbb{B}_r(x_0) \subseteq \Omega_M$. So, for all $x_0 \in \Omega_M$, there exists a ball $\mathbb{B}_r(x_0) \subseteq \Omega_M$, so Ω_M is (relatively) open, as required. ☐

Corollary 1.7. *Given $f \in C(\Omega)$, and $g \in C(\partial\Omega)$, there exists at most one $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying Poisson's equation*

$$
\begin{cases} \Delta u = f \text{ in } \Omega, \\ u = g \text{ on } \partial\Omega. \end{cases}
$$

Proof. Suppose that u_1, u_2 both satisfy the above. Consider $w := u_1 - u_2$. Then $\Delta w = 0$ in Ω and $w = 0$ on $\partial\Omega$. w satisfies the MVP, and so attains its maximum and minimum on $\partial\Omega$, where it is zero. So w is identically zero, and $u_1 = u_2$. ☐

There are also questions of regularity concerning solutions of Laplace's equation.

Theorem 1.8. *If $u \in C(\Omega)$ is harmonic (or satisfies the MVP), then $u \in C^\infty(\Omega)$. (In fact, u is analytic, $u \in C^\omega(\Omega)$.)*

Proof. Let

$$\eta(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2-1}\right), & |x| < 1; \\ 0, & |x| \geq 1; \end{cases} \tag{1.2}$$

with C chosen so that $\int_{\mathbb{R}^n} \eta(x)\,\mathrm{d}x = 1$. Define $\eta_\varepsilon(x) := \varepsilon^{-n}\eta(x/\varepsilon)$. Take

$$x \in \Omega_\varepsilon := \{x \in \Omega | d(x, \partial\Omega) > \varepsilon\}. \tag{1.3}$$

For such an x define

$$u_\varepsilon(x) := (\eta_\varepsilon \star u)(x) := \int_\Omega \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y.$$

Obviously, $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$. We wish to show that $u(x) = u_\varepsilon(x)$ for $x \in \Omega_\varepsilon$.

$$\begin{aligned}
\int_\Omega \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y &= \frac{1}{\varepsilon^n}\int_\Omega \eta\left(\frac{|x-y|}{\varepsilon}\right)u(y)\,\mathrm{d}y \\
&= \frac{1}{\varepsilon^n}\int_{\mathbb{B}_\varepsilon(x)} \eta\left(\frac{|x-y|}{\varepsilon}\right)u(y)\,\mathrm{d}y \\
&= \frac{1}{\varepsilon^n}\int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right)\int_{\partial\mathbb{B}_r(x)} u(y)\,\mathrm{d}S(y)\mathrm{d}r \\
&= \frac{1}{\varepsilon^n}u(x)\int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right)\omega_n r^{n-1}\,\mathrm{d}r \\
&= u(x)\int_{\mathbb{B}_1(0)} \eta(r)\,\mathrm{d}r \\
&= u(x).
\end{aligned}$$

So for all $x \in \Omega_\varepsilon$, $u(x) = u_\varepsilon(x)$. So $u \in C^\infty(\Omega_\varepsilon)$ for all $\varepsilon > 0$, so $u \in C^\infty(\Omega)$. □

Theorem 1.8 extends our notion of "harmonic" to functions that are not twice differentiable.

Theorem 1.9 (*A priori* derivative estimates)**.** *Fix x_0 and write $B_R := \mathbb{B}_R(x_0)$. Let $u \in C(\overline{B_R})$ be harmonic. Then for $1 \leq i \leq n$,*

$$|\mathrm{D}_i u(x_0)| \leq \frac{n}{R}\max_{\overline{B_R}}|u|,$$

where $\mathrm{D}_i := \frac{\partial}{\partial x_i}$.

Proof. To simplify the proof, we assume that $u \in C^1(\overline{B_R})$. $\Delta(\mathrm{D}_i u) = 0 \Rightarrow \mathrm{D}_i u$ is harmonic.

$$\begin{aligned}
\mathrm{D}_i u(x_0) &= ⨍_{B_R} \mathrm{D}_i u(y)\,\mathrm{d}y \\
&= \frac{n}{\omega_n R^n}\int_{B_R} \mathrm{D}_i u(y)\,\mathrm{d}y \\
&= \frac{n}{\omega_n R^n}\int_{\partial B_R} u(y)\nu_i(y)\,\mathrm{d}S(y),
\end{aligned}$$

where ν_i is the i^{th} component of the unit normal ν. Hence,

$$|\mathrm{D}_i u(x_0)| \leq \frac{n}{\omega_n R^n}\omega_n R^{n-1} \max_{\partial B_R} |u| \leq \frac{n}{R} \max_{\overline{B_R}} |u|. \qquad \square$$

Corollary 1.10 (Liouville's theorem.[3])**.** *If u is harmonic on $\mathbb{R}^n$ and bounded, then u is constant.*

Proof. For any $R > 0$ and $x_0 \in \mathbb{R}^n$,

$$\begin{aligned}|\mathrm{D}_i u(x_0)| &\leq \frac{n}{R} \max_{\overline{\mathbb{B}_R(x_0)}} |u| \\ &\leq \frac{n}{R} \cdot \text{constant}.\end{aligned}$$

So $|\mathrm{D}_i u(x_0)| = 0$ for all i and x_0. So u is constant. $\square$

Theorem 1.11 (Theorem 1.9 improved.)**.** *If $u \in C(\overline{B_R})$ is harmonic, then for $|\alpha| = m \in \mathbb{N}$,*

$$|\mathrm{D}^\alpha u(x_0)| \leq \frac{n^m e^{m-1} m!}{R^m} \max_{\overline{B_R}} |u|.$$

Proof. For simplicity, we treat only unmixed derivatives and write D_i^m for $\frac{\partial^m}{\partial x_i^m}$. We apply proof by induction. The case $m = 1$ is true as shown above. Assume true for m. Take $r = (1-\theta)R$, $0 < \theta < 1$.

$$|\mathrm{D}_i^{m+1} u(x_0)| \leq \frac{n}{r} \max_{\overline{B_r}} |\mathrm{D}_i^m u|.$$

However, the estimate $|\mathrm{D}_i^m u(x_0)| \leq \frac{n^m e^{m-1} m!}{R^m} \max_{\overline{B_R}} |u|$ is not good enough. We need to estimate $|\mathrm{D}_i^m u(y)|$ for $y \in \mathbb{B}_r(x_0) = B_r$. Take $\mathbb{B}_\rho(y) \subseteq \mathbb{B}_R(x_0)$ for all $y \in \mathbb{B}_r(x_0)$; i.e., take $\rho = R - r$. We use the inequality for m in the ball $\mathbb{B}_{R-r}(y)$ for all $y \in B_r$.

$$\begin{aligned}\max_{\overline{B_r}} |\mathrm{D}_i^m u(y)| &\leq \frac{n^m e^{m-1} m!}{(R-r)^m} \max_{\overline{\mathbb{B}_{R-r}(y)}} |u| \\ &\leq \frac{n^m e^{m-1} m!}{(R-r)^m} \max_{\overline{B_R}} |u|.\end{aligned}$$

So

$$\begin{aligned}|\mathrm{D}_i^{m+1} u(x_0)| &\leq \frac{n}{r} \frac{n^m e^{m-1} m!}{(R-r)^m} \max_{\overline{B_R}} |u| \\ &= \frac{n}{(1-\frac{m}{m+1})R} \frac{n^m e^{m-1} m!}{(\frac{m}{m+1})^m R^m} \max_{\overline{B_R}} |u| \text{ with } \theta = \frac{m}{m+1} \\ &= \frac{n^{m+1}}{R^{m+1}} e^{m-1} m! \left(\frac{m+1}{m}\right)^m (m+1) \max_{\overline{B_R}} |u| \\ &= \frac{n^{m+1}}{R^{m+1}} (m+1)! e^{m-1} \left(1 + \frac{1}{m}\right)^m \max_{\overline{B_R}} |u| \\ &\leq \frac{n^{m+1} e^m (m+1)!}{R^{m+1}} \max_{\overline{B_R}} |u|\end{aligned}$$

since $(1 + \frac{1}{m})^m < e$. $\square$

[3] Joseph Liouville (1809–1882).

Theorem 1.12. *If u is harmonic on Ω, then u is analytic.*

Proof. **Exercise.** Prove this using the above estimates of $|\mathrm{D}^\alpha u(x_0)|$. □

1.2 The Fundamental Solution

We seek to solve $\Delta u = 0$ on $\mathbb{R}^n$, assuming that u is spherically symmetric, so $u = u(r)$. In spherical polar coordinates

$$\frac{\partial u}{\partial x_i} = u'(r)\frac{x_i}{r}$$

and

$$\frac{\partial^2 u}{\partial x_i^2} = u''(r)\frac{x_i 2}{r^2} = u'(r)\left(\frac{1}{r} - \frac{x_i^3}{r^3}\right).$$

Thus,

$$\Delta u = u''(r) + \frac{n-1}{r}u'(r).$$

So, we solve the ODE $u''(r) + \frac{n-1}{r}u'(r) = 0$, writing $v := u'$:

$$\begin{aligned} v'(r) &= -\frac{n-1}{r}v(r) \\ \Rightarrow \quad \frac{\mathrm{d}v}{v(r)} &= -(n-1)\frac{\mathrm{d}r}{r} \\ \Rightarrow \quad \log v(r) &= -(n-1)\log r + \log C \\ \Rightarrow \quad v(r) &= C/r^{n-1} \end{aligned}$$

Thus,

$$\begin{aligned} u(r) &= C_1 \int \frac{\mathrm{d}r}{r^{n-1}} + C_2 \\ &= \begin{cases} C_1 \log r + C_2, & n = 2; \\ C_3/r^{n-2} + C_4, & n \geq 3. \end{cases} \end{aligned}$$

By a choice of constants we have:

Definition 1.13. The *fundamental solution* of Laplace's equation on $\mathbb{R}^n$ is

$$\Phi(x) := \begin{cases} -\frac{1}{2\pi}\log|x|, & n = 2; \\ \frac{1}{\omega_n(n-2)}\frac{1}{|x|^{n-2}}, & n \geq 3. \end{cases} \tag{1.4}$$

We now use this to solve the Poisson equation $-\Delta u = f$ on $\mathbb{R}^n$:

Theorem 1.14. *If $f \in C^2_{\mathrm{c}}(\mathbb{R}^n) := \left\{ f \in C^2(\mathbb{R}^n) \,\middle|\, \operatorname{supp} f := \overline{\{x | f(x) \neq 0\}} \text{ is compact} \right\}$, then*

$$u(x) := \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, \mathrm{d}y$$

solves $-\Delta u = f$.

Proof. Note that

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} \Phi(y) f(x-y)\,\mathrm{d}y.$$

So

$$\begin{aligned}
\Delta u(x) &= \Delta \int_{\mathbb{R}^n} \Phi(y) f(x-y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} \Phi(y) \Delta f(x-y)\,\mathrm{d}y \\
&= \int_{\mathbb{B}_\varepsilon(0)} \Phi(y) \Delta f(x-y)\,\mathrm{d}y + \int_{\mathbb{B}_\varepsilon(0)^\complement} \Phi(y) \Delta f(x-y)\,\mathrm{d}y.
\end{aligned}$$

We estimate:

$$\begin{aligned}
\int_{\mathbb{B}_\varepsilon(0)} \Phi(y) \Delta f(x-y)\,\mathrm{d}y &\le \sup |\Delta f| \int_{\mathbb{B}_\varepsilon(0)} |\Phi(y)|\,\mathrm{d}y \\
&= \begin{cases} C\varepsilon^2 |\log \varepsilon|, & n = 2; \\ C\varepsilon^2, & n \ge 3. \end{cases}
\end{aligned}$$

Also, using the facts that $\Delta_x \Phi(x-y) = \Delta_y \Phi(x-y)$ and $\partial\left(\mathbb{B}_\varepsilon(0)^\complement\right) = \partial \mathbb{B}_\varepsilon(0)$,

$$\int_{\mathbb{B}_\varepsilon(0)^\complement} \Phi(y) \Delta f(x-y)\,\mathrm{d}y = \int_{\partial \mathbb{B}_\varepsilon(0)} \Phi(y) \frac{\partial f}{\partial \nu}(x-y)\,\mathrm{d}S(y) - \int_{\mathbb{B}_\varepsilon(0)^\complement} \nabla \Phi(y) \cdot \nabla f(x-y)\,\mathrm{d}y.$$

We estimate:

$$\begin{aligned}
\int_{\partial \mathbb{B}_\varepsilon(0)} \Phi(y) \frac{\partial f}{\partial \nu}(x-y)\,\mathrm{d}S(y) &\le C \int_{\partial \mathbb{B}_\varepsilon(0)} |\Phi(y)|\,\mathrm{d}S(y) \\
&= \begin{cases} C\varepsilon |\log \varepsilon|, & n = 2; \\ C\varepsilon, & n \ge 3. \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
-\int_{\mathbb{B}_\varepsilon(0)^\complement} \nabla \Phi(y) \cdot \nabla f(x-y)\,\mathrm{d}y &= \int_{\mathbb{B}_\varepsilon(0)^\complement} \Delta \Phi(y) f(x-y)\,\mathrm{d}y - \int_{\partial \mathbb{B}_\varepsilon(0)} \frac{\partial \Phi}{\partial \nu}(y) f(x-y)\,\mathrm{d}S(y) \\
&= -\int_{\partial \mathbb{B}_\varepsilon(0)} \frac{\partial f}{\partial \nu}(y) f(x-y)\,\mathrm{d}S(y)
\end{aligned}$$

since the normal $\nu = -\frac{y}{|y|}$, and $\mathrm{D}\Phi(y) = \frac{-1}{\omega_n} \frac{y}{|y|^{n-1}}$, $\frac{\partial \Phi}{\partial \nu} = \mathrm{D}\Phi \cdot \nu = \frac{1}{\omega_n |y|^{n-1}} = \frac{1}{\omega_n \varepsilon^{n-1}}$,

$$\begin{aligned}
&= \frac{-1}{\omega_n \varepsilon^{n-1}} \int_{\partial \mathbb{B}_\varepsilon(0)} f(x-y)\,\mathrm{d}S(y) \\
&= \frac{-1}{\omega_n \varepsilon^{n-1}} \int_{\partial \mathbb{B}_\varepsilon(x)} f(y)\,\mathrm{d}S(y) \\
&= -\!\!\!\!\!\!\int_{\partial \mathbb{B}_\varepsilon(x)} f(y)\,\mathrm{d}S(y) \\
&\xrightarrow[\varepsilon \to 0]{} -f(x)
\end{aligned}$$

So $\Delta u(x) = O(\varepsilon) - f(x)$. Take the limit as $\varepsilon \to 0$. $\square$

Having solved Poisson's equation on $\mathbb{R}^n$, we now seek to find a solution $u \in C^2(\bar{\Omega})$ to

$$\begin{cases} -\Delta u = f, & \text{on } \Omega; \\ u = g, & \text{on } \partial\Omega. \end{cases}$$

Our idea is to use Green's identity on $V_\varepsilon := \Omega \setminus \mathbb{B}_\varepsilon(x)$:

$$\begin{aligned} &\int_{V_\varepsilon} \Delta\Phi(y-x)u(y)\,\mathrm{d}y - \int_{V_\varepsilon} \Phi(y-x)\Delta u(y)\,\mathrm{d}y \\ &\quad = \int_{\partial V_\varepsilon} \frac{\partial \Phi}{\partial \nu}(y-x)u(y)\,\mathrm{d}S(y) - \int_{\partial V_\varepsilon} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,\mathrm{d}S(y) \end{aligned}$$

$$\int_{V_\varepsilon} \Phi(y-x)\Delta u(y)\,\mathrm{d}y \underset{\varepsilon\to 0}{\longrightarrow} \int_{\Omega} \Phi(y-x)\Delta u(y)\,\mathrm{d}y.$$

$$\int_{\partial V_\varepsilon} \frac{\partial \Phi}{\partial \nu}(y-x)u(y)\,\mathrm{d}S(y) = \int_{\partial\Omega} \frac{\partial \Phi}{\partial \nu}(y-x)u(y)\,\mathrm{d}S(y) + \underbrace{\int_{\partial \mathbb{B}_\varepsilon(x)} \frac{\partial \Phi}{\partial \nu}(y-x)u(y)\,\mathrm{d}S(y)}_{\underset{\varepsilon\to 0}{\longrightarrow} u(x)}$$

$$\int_{\partial V_\varepsilon} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,\mathrm{d}S(y) = \int_{\partial\Omega} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,\mathrm{d}S(y) + \underbrace{\int_{\partial \mathbb{B}_\varepsilon(0)} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,\mathrm{d}S(y)}_{\leq C \int_{\mathbb{B}_\varepsilon(0)} |\Phi(y)|\,\mathrm{d}y \to 0}$$

$$-\int_{\Omega} \Phi(y-x)\Delta u(y)\,\mathrm{d}y = \int_{\partial\Omega} \frac{\partial \Phi}{\partial \nu}(y-x)u(y)\,\mathrm{d}S(y) + u(x) - \int_{\partial\Omega} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,\mathrm{d}S(y)$$

Hence,

$$\begin{aligned} u(x) &= \int_{\partial\Omega} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,\mathrm{d}S(y) - \int_{\partial\Omega} \underbrace{u(y)}_{=g}\frac{\partial \Phi}{\partial \nu}(y-x)\,\mathrm{d}S(y) \\ &\quad - \int_{\Omega} \Phi(y-x)\underbrace{\Delta u(y)}_{=f}\,\mathrm{d}y. \end{aligned}$$

We now define Γ^x by

$$\begin{cases} \Delta\Gamma^x = 0 \text{ in } \Omega \\ \Gamma^x(y) = \Phi(y-x) \text{ on } \partial\Omega \end{cases}$$

and $G(x,y) := \Phi(y-x) - \Gamma^x(y)$. Then

$$\begin{aligned} u(x) &= -\int_{\partial\Omega} u(y)\frac{\partial G}{\partial \nu}(x,y)\,\mathrm{d}S(y) - \int_{\Omega} \Delta u(y)G(x,y)\,\mathrm{d}y \\ &= -\int_{\partial\Omega} g(y)\frac{\partial G}{\partial \nu}(x,y)\,\mathrm{d}S(y) + \int_{\Omega} f(y)G(x,y)\,\mathrm{d}y. \end{aligned}$$

Exercises 1.15. (i) Check that the above definitions make sense.

(ii) Show that G is symmetric: $G(x,y) \equiv G(y,x)$.

2 Function Spaces

2.1 Weak Derivatives and Sobolev Spaces

In the sequel, much use will be made of *smooth functions with compact support*, which will also be referred to as *test functions*:

$$C_{\mathrm{c}}^{\infty}(\Omega) := \left\{ \phi \colon \Omega \to \mathbb{R} \,\middle|\, \begin{array}{c} \phi \text{ is infinitely differentiable and} \\ \operatorname{supp}\phi := \overline{\{x \in \Omega | \phi(x) \neq 0\}} \text{ is compact} \end{array} \right\}.$$

Recall that if $u \in C^1(\Omega)$, then for all $\phi \in C_{\mathrm{c}}^{\infty}(\Omega)$,

$$\int_{\Omega} \frac{\partial u}{\partial x_i}(x)\phi(x)\,\mathrm{d}x = -\int_{\Omega} u(x)\frac{\partial \phi}{\partial x_i}(x)\,\mathrm{d}x.$$

Note, however, that the RHS makes sense even if u is not differentiable, but only in $L^1(\Omega)$.

Definition 2.1. If $u, v \in L^1(\Omega)$ and, for all $\phi \in C_{\mathrm{c}}^{\infty}(\Omega)$,

$$\int_{\Omega} u(x)\frac{\partial \phi}{\partial x_i}(x)\,\mathrm{d}x = -\int_{\Omega} v(x)\phi(x)\,\mathrm{d}x,$$

we say $\frac{\partial u}{\partial x_i} := v$ in a *weak sense.*

Definition 2.2. If $u, v \in L^1(\Omega)$ and, for all $\phi \in C_{\mathrm{c}}^{\infty}(\Omega)$,

$$\int_{\Omega} u(x)\mathrm{D}^{\alpha}\phi(x)\,\mathrm{d}x = (-1)^{|\alpha|}\int_{\Omega} v(x)\phi(x)\,\mathrm{d}x,$$

we say $\mathrm{D}^{\alpha}u := v$ in a *weak sense.*

Example 2.3. Let $u(x) = x$ for $0 \leq x \leq 1$ and $= 1$ for $1 \leq x \leq 2$. The classical ("strong") derivative $u'(x) = 1$ for $0 \leq x < 1$ and $= 0$ for $1 < x \leq 2$.

$$\begin{aligned} \int_0^2 u(x)\phi'(x)\,\mathrm{d}x &= \int_0^1 x\phi'(x)\,\mathrm{d}x + \int_1^2 \phi'(x)\,\mathrm{d}x \\ &= -\int_0^1 \phi(x)\,\mathrm{d}x + \phi(1) - \phi(1) \\ &= -\int_0^2 v(x)\phi(x)\,\mathrm{d}x \end{aligned}$$

where

$$v(x) := \begin{cases} 1, & 0 \leq x \leq 1; \\ 0, & 1 < x \leq 2. \end{cases}$$

Note that we can change the value of $v(1)$ to be 1 and still retain the desired equality, so weak derivatives are defined only up to sets of measure zero.

Example 2.4. Not all functions are weakly differentiable. Consider

$$u(x) := \begin{cases} x, & 0 \le x \le 1; \\ 2, & 1 < x \le 2. \end{cases}$$

We have $\int_0^2 u(x)\phi'(x)\,\mathrm{d}x = -\int_0^1 \phi(x)\,\mathrm{d}x - \phi(1)$. Suppose there is a $v \in L^1((0,2))$ such that

$$\int_0^2 \phi(x)\,\mathrm{d}x + \phi(1) = \int_0^2 v(x)\phi(x)\,\mathrm{d}x$$

for all $\phi \in C_{\mathrm{c}}^\infty((0,2))$. Take a sequence ϕ_m in $C_{\mathrm{c}}^\infty((0,2))$ with $\phi_m(1) = 1$ for all m and $\phi_m(x) \underset{m\to\infty}{\longrightarrow} \delta_{1,x}$. Then for each m,

$$\int_0^2 \phi_m(x)\,\mathrm{d}x + \phi_m(1) = \int_0^2 v(x)\phi_m(x)\,\mathrm{d}x.$$

Take the limit as $m \to \infty$. $\int_0^1 \phi_m(x)\,\mathrm{d}x \to 0$ since $\phi_m(x) \to 0$ almost everywhere and $|\phi_m| \le 1$. Similarly, $\int_0^1 v(x)\phi_m(x)\,\mathrm{d}x \to 0$. Hence, $1 = 0$, a contradiction. So u has no weak derivative.

Lemma 2.5. *A weak derivative, if it exists, is unique (up to sets of measure zero).*

Proof. Suppose that u has two weak derivatives v_1 and v_2. Then, for all $\phi \in C_{\mathrm{c}}^\infty(\Omega)$,

$$(-1)^{|\alpha|}\int_\Omega v_1(x)\phi(x)\,\mathrm{d}x = \int_{OM} u(x)\mathrm{D}^\alpha\phi(x)\,\mathrm{d}x = (-1)^{|\alpha|}\int_\Omega v_2(x)\phi(x)\,\mathrm{d}x.$$

So $\int_\Omega (v_1(x) - v_2(x))\phi(x)\,\mathrm{d}x = 0$, and so $v_1 = v_2$ almost everywhere. □

Corollary 2.6. *If a function has a strong derivative, then it has a weak derivative and the two are equal.*

Definition 2.7. Given $k \in \mathbb{N}_0$ and $1 \le p \le \infty$, define the *Sobolev space*[4] $W^{k,p}(\Omega)$ to be

$$W^{k,p}(\Omega) := \{u : \Omega \to \mathbb{R} | u \in L^p(\Omega) \text{ and for all } |\alpha| \le k, \mathrm{D}^\alpha u \in L^p(\Omega)\}.$$

We often write $H^k(\Omega)$ for $W^{k,2}(\Omega)$.

Lemma 2.8. (Product rule.) *If $u \in W^{1,p}(\Omega)$, $\xi \in C_{\mathrm{c}}^\infty(\Omega)$, then*

$$(\mathrm{D}_{x_i}(u\xi))(x) = (\mathrm{D}_{x_i}u(x))\xi(x) + u(x)(\mathrm{D}_{x_i}\xi(x)).$$

Proof.

$$\begin{aligned}\int_\Omega u(x)\xi(x)\frac{\partial\phi}{\partial x_i}(x)\,\mathrm{d}x &= \int_\Omega u(x)\frac{\partial\phi\xi}{\partial x_i}(x)\,\mathrm{d}x - \int_\Omega u(x)\frac{\partial\xi}{\partial x_i}(x)\phi(x)\,\mathrm{d}x \\ &= \int_\Omega (\mathrm{D}_{x_i}u(x))\phi(x)\xi(x)\,\mathrm{d}x - \int_\Omega u(x)\frac{\partial\xi}{\partial x_i}(x)\phi(x)\,\mathrm{d}x \\ &= -\int_\Omega \phi(x)\left((\mathrm{D}_{x_i}u(x))\xi(x) + u(x)\frac{\partial\xi}{\partial x_i}(x)\right)\,\mathrm{d}x\end{aligned}$$

So the weak derivative exists and is as claimed. □

[4] Sergei L'vovich Sobolev (1908–1989).

Exercises 2.9. 1. Show that $\mathrm{D}^\alpha \mathrm{D}^\beta u = \mathrm{D}^{\alpha+\beta} u = \mathrm{D}^\beta \mathrm{D}^\alpha u$.

2. Show that $\mathrm{D}u = 0 \Rightarrow u = \text{constant}$.

Definition 2.10. For $u \in W^{k,p}(\Omega)$, define

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha| \le k} \int_\Omega |\mathrm{D}^\alpha u(x)|^p \, \mathrm{d}x \right)^{1/p}, & 1 \le p < \infty; \\ \sum_{|\alpha| \le k} \operatorname{ess\,sup}_{x \in \Omega} |\mathrm{D}^\alpha u(x)|, & p = \infty. \end{cases} \tag{2.1}$$

Lemma 2.11. *$\| \cdot \|_{W^{k,p}(\Omega)}$ is a norm on $W^{k,p}(\Omega)$.*

Proof. It is easy to see that $\|u\|_{W^{k,p}(\Omega)} \ge 0$ for all u, and that $\|u\|_{W^{k,p}(\Omega)} = 0$ if, and only if, $u = 0$ almost everywhere. It is also clear that $\|\lambda u\|_{W^{k,p}(\Omega)} = |\lambda| \|u\|_{W^{k,p}(\Omega)}$. So it remains only to check the triangle inequality. For finite p:

$$\begin{aligned} \|u_1 + u_2\|_{W^{k,p}(\Omega)} &= \left(\sum_{|\alpha| \le k} \int_\Omega |\mathrm{D}^\alpha (u_1 + u_2)|^p \right)^{1/p} \\ &= \left(\sum_{|\alpha| \le k} \int_\Omega |\mathrm{D}^\alpha u_1 + \mathrm{D}^\alpha u_2|^p \right)^{1/p} \\ &= \left(\sum_{|\alpha| \le k} \|\mathrm{D}^\alpha u_1 + \mathrm{D}^\alpha u_2\|_{L^p(\Omega)}^p \right)^{1/p} \\ &\le \left(\sum_{|\alpha| \le k} \left(\|\mathrm{D}^\alpha u_1\|_{L^p(\Omega)} + \|\mathrm{D}^\alpha u_2\|_{L^p(\Omega)} \right)^p \right)^{1/p} \\ &\le \left(\sum_{|\alpha| \le k} \|\mathrm{D}^\alpha u_1\|_{L^p(\Omega)}^p \right)^{1/p} + \left(\sum_{|\alpha| \le k} \|\mathrm{D}^\alpha u_2\|_{L^p(\Omega)}^p \right)^{1/p} \quad \text{by Minkowski's inequality} \\ &= \|u_1\|_{W^{k,p}(\Omega)} + \|u_2\|_{W^{k,p}(\Omega)}. \end{aligned}$$

The check for $p = \infty$ is similar. □

Theorem 2.12. *$W^{k,p}(\Omega)$ is a Banach space for $1 \le p \le \infty$. For $1 \le p < \infty$, it is also separable.*

Proof. (Proved only for $1 \le p < \infty$.) $W^{k,p}(\Omega)$ is linear and normed with norm $\|\cdot\|_{W^{k,p}(\Omega)}$ as above. To show that $W^{k,p}(\Omega)$ is complete, let (u_n) be a Cauchy sequence in $W^{k,p}(\Omega)$. We have to show that $u_n \underset{n\ to\infty}{\longrightarrow} u$ in $W^{k,p}(\Omega)$ for some $u \in W^{k,p}(\Omega)$.

For $|\alpha| \le k$, $(\mathrm{D}^\alpha u_n)$ is Cauchy in $L^p(\Omega)$, and u_n is Cauchy in $L^p(\Omega)$. Therefore, by the completeness of $L^p(\Omega)$, $u_n \to u$ in $L^p(\Omega)$ and $\mathrm{D}^\alpha u_n \to u_\alpha$ in $L^p(\Omega)$. We want to show that $u \in W^{k,p}(\Omega)$, or, equivalently, that $u_\alpha = \mathrm{D}^\alpha U$. For $|\alpha| \le k$,

$$\begin{array}{ccc} \int_\Omega u_m \mathrm{D}^\alpha \phi & = & (-1)^{|\alpha|} \int_\Omega (\mathrm{D}^\alpha u_m)\phi \\ {\scriptstyle m\to\infty}\downarrow & & \downarrow{\scriptstyle m\to\infty} \\ \int_\Omega u \mathrm{D}^\alpha \phi & = & (-1)^{|\alpha|} \int_\Omega u_\alpha \phi \end{array}$$

So u has α^{th} weak derivative and $\mathrm{D}^\alpha u = u_\alpha$. So $u \in W^{k,p}(\Omega)$ and $u_n \to u$ in $W^{k,p}(\Omega)$.

For $p < \infty$, $L^p(\Omega)$ is separable, so $\prod_{|\alpha|\le k} L^p(\Omega)$ is separable. Since any subspace of a separable metric space is itself separable, $W^{k,p}(\Omega)$ is separable. □

Corollary 2.13. *$H^k(\Omega)$ is a separable Hilbert space with inner product*

$$\begin{aligned} (f,g)_{H^k(\Omega)} &:= \sum_{|\alpha|\le k} \int_\Omega \mathrm{D}^\alpha f(x) \mathrm{D}^\alpha g(x)\,\mathrm{d}x \\ &= \int_\Omega f(x)g(x) + \nabla f(x)\cdot \nabla g(x) + \nabla^2 f(x) : \nabla^2 g(x) + \ldots\,\mathrm{d}x. \end{aligned}$$

2.2 Approximation of Weakly Differentiable Functions

Definition 2.14. We say V is *compactly embedded* in Ω, and write $V \subset\subset \Omega$, if $V \subseteq \overline{V} \subseteq \operatorname{Int}\Omega$, with $\overline{V}$ compact.

Definitions 2.15. The *local L^p space* $L^p_{\text{loc}}(\Omega)$ is defined by $u \in L^p_{\text{loc}}(\Omega) \Leftrightarrow u \in L^p(V)$ for all $V \subset\subset \Omega$. The *local Sobolev spaces* $W^{k,p}_{\text{loc}}(\Omega)$ are defined in the obvious way.

Theorem 2.16. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $u \in W^{k,p}(\Omega)$. As before, set $\Omega_\varepsilon := \{x \in \Omega | d(x, \partial\Omega) > \varepsilon\}$, set*

$$\eta(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2-1}\right), & |x| < 1; \\ 0, & |x| \ge 1. \end{cases}$$

such that $\int_{\mathbb{R}^n} \eta(x)\,\mathrm{d}x = 1$, and set $\eta_\varepsilon(x) = \varepsilon^{-n}\eta(x/\varepsilon)$. Define

$$u_\varepsilon(x) := (\eta_\varepsilon \star u)(x) := \int_\Omega \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y.$$

Then $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$ and $u_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u$ in $W^{k,p}_{\text{loc}}(\Omega)$.

Proof. $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$ follows immediately from the definition of u_ε and the standard smoothness properties of a convolution.

We now show that $\mathrm{D}^\alpha u_\varepsilon = \eta_\varepsilon \star \mathrm{D}^\alpha u$.

$$\begin{aligned} \mathrm{D}^\alpha \int_\Omega \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y &= \int_\Omega \mathrm{D}^\alpha_x \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y \\ &= (-1)^{|\alpha|} \int_\Omega \mathrm{D}^\alpha_y \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y \\ &= \int_\Omega \eta_\varepsilon(x-y)\mathrm{D}^\alpha_y u(y)\,\mathrm{d}y. \end{aligned}$$

For each $x \in \Omega_\varepsilon$, $\eta_\varepsilon(x-y)$ has compact support $\overline{\mathbb{B}_\varepsilon(x)}$ in Ω, and is smooth, and so lies in $C_c^\infty(\Omega)$. So

$$\mathrm{D}^\alpha u_\varepsilon(x) = \int_\Omega \eta_\varepsilon(x-y)\mathrm{D}^\alpha u(y)\,\mathrm{d}y = (\eta_\varepsilon \star \mathrm{D}^\alpha u)(x).$$

We now want to show that $u_\varepsilon \to u$ in $L^p_{\mathrm{loc}}(\Omega)$. Take $x \in V \subset\subset \Omega$ with $V \subseteq \Omega_\varepsilon$ for small ε.

$$\begin{aligned}
|u_\varepsilon(x) - u(x)| &= \left|\int_\Omega \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y - u(x)\right| \\
&= \left|\int_\Omega \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y - \int_\Omega \eta_\varepsilon(x-y)u(x)\,\mathrm{d}y\right| \text{ since } \int_\Omega \eta_\varepsilon(x-y)\,\mathrm{d}y = 1 \\
&\leq \int_\Omega \eta_\varepsilon(x-y)|u(x)-u(y)|\,\mathrm{d}y \\
&= \int_\Omega |\eta_\varepsilon(x-y)|^{1/q}|\eta_\varepsilon(x-y)|^{1/p}|u(x)-u(y)|\,\mathrm{d}y \text{ where } 1/p+1/q=1 \\
&\leq \underbrace{\left|\int_\Omega \eta_\varepsilon(x-y)\,\mathrm{d}y\right|^{1/q}}_{=1} \left|\int_\Omega \eta_\varepsilon(x-y)|u(x)-u(y)|^p\,\mathrm{d}y\right|^{1/p} \text{ by Hölder's inequality.}
\end{aligned}$$

Also,

$$\begin{aligned}
\int_V |u_\varepsilon(x)-u(x)|^p\,\mathrm{d}x &\leq \int_V \int_\Omega \eta_\varepsilon(x-y)|u(x)-u(y)|^p\,\mathrm{d}y\mathrm{d}x \\
&= \int_W \int_{\mathbb{B}_\varepsilon(0)} \eta_\varepsilon(z)|u(x-z)-u(x)|^p\,\mathrm{d}z\mathrm{d}x \text{ with } z := x-y, W \subseteq \Omega \\
&\leq \underbrace{\int_{\mathbb{B}_\varepsilon(0)} \eta_\varepsilon(z)\,\mathrm{d}z}_{=1} \sup_{|z|<\varepsilon} \int_W |u(x-z)-u(x)|^p\,\mathrm{d}x.
\end{aligned}$$

By Lebesgue's dominated convergence theorem,

$$\sup_{|z|<\varepsilon} \int_W |u(x-z)-u(x)|^p\,\mathrm{d}x \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

But $\int_V |u_\varepsilon(x)-u(x)|^p\,\mathrm{d}x \underset{\varepsilon\to 0}{\longrightarrow} 0$, since $u_\varepsilon \to u$ in $L^p(V)$ if, and only if, $u_\varepsilon \to u$ in $L^p_{\mathrm{loc}}(\Omega)$; $\mathrm{D}^\alpha u_\varepsilon = \eta_\varepsilon \star \mathrm{D}^\alpha u$, so $\mathrm{D}^\alpha u_\varepsilon \to \mathrm{D}^\alpha u$ in $L^p(V)$, equivalently, in $L^p_{\mathrm{loc}}(\Omega)$. □

Theorem 2.17. *If $u \in W^{k,p}(\Omega)$, then there is a sequence of functions $u_m \in C^\infty(\bar{\Omega}) \cap W^{k,p}(\Omega)$ such that $u_m \to u$ in $W^{k,p}(\Omega)$ (not just in $W^{k,p}_{\mathrm{loc}}(\Omega)$).*

Proof. Omitted. □

Thus, we have a useful alternative definition of weak derivative:

Definition 2.18. Let $u, v \in L^1(\Omega)$. If $u_m \to u$ in $L^1(\Omega)$ and $\mathrm{D}^\alpha u_m \to v$ in $L^1(\Omega)$, then u has α^{th} *weak derivative* $\mathrm{D}^\alpha u := v$.

Lemma 2.19. *The norm*

$$\|u\| := \sum_{|\alpha|\le k} \|\mathrm{D}^\alpha u\|_{L^p(\Omega)},$$

is equivalent to the usual norm $\|\cdot\|_{W^{k,p}(\Omega)}$ *on* $W^{k,p}(\Omega)$.

Proof. Recall that $W^{k,p}(\Omega)$ is a Banach space with norm

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha|\le k} \|\mathrm{D}^\alpha u\|^p_{L^p(\Omega)}\right)^{1/p}.$$

It is enough to show that for all $n, p \ge 1$ and $a_i \ge 0$,

$$\frac{1}{n}\sum_{i=1}^n a_i \le \left(\sum_{i=1}^n a_i^p\right)^{1/p} \le \sum_{i=1}^n a_i$$

It is easy to see that $\sum_{i=1}^n a_i^p \le (\sum_{i=1}^n a_i)^p$ for $p \ge 1$. Note also that

$$\left(\sum_{i=1}^n a_i\right)^p \le \left(n \max_i a_i\right)^p = n^p \max_i a_i^p \le n^p \sum_{i=1}^n a_i^p,$$

and the result follows. ☐

Definition 2.20. The closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$ is denoted by $W_0^{k,p}(\Omega)$. (Some authors denote this space by $\overset{\circ}{W}{}^k_p(\Omega)$.)

Note that $W_0^{k,p}(\Omega) \subset W^{k,p}(\Omega)$ is a Banach space with norm $\|\cdot\|_{W^{k,p}(\Omega)}$.

Lemma 2.21. *If* $u \in W_0^{k,p}(\Omega)$*, then* u *extends to*

$$\tilde{u}(x) := \begin{cases} u(x), & x \in \Omega; \\ 0, & x \notin \Omega; \end{cases}$$

and $\tilde{u} \in W_0^{k,p}(\Omega_1)$ *for all* $\Omega_1 \supseteq \Omega$.

Proof. $u \in W_0^{k,p}(\Omega) \Leftrightarrow$ there exist $u_m \in C_c^\infty(\Omega)$ such that $u_m \to u$ in $W^{k,p}(\Omega)$. Define $\tilde{u}_m := \widetilde{u_m}$ as above. Clearly $\tilde{u}_m \in C_c^\infty(\Omega_1)$ for all $\Omega_1 \supseteq \Omega$. Then

$$\|\tilde{u}_m - \tilde{u}\|_{W^{k,p}(\Omega_1)} = \|u_m - u\|_{W^{k,p}(\Omega)} \underset{m\to\infty}{\longrightarrow} 0,$$

so $\tilde{u}_m \to \tilde{u}$ in $W^{k,p}(\Omega_1)$, and so $\tilde{u} \in W_0^{k,p}(\Omega_1)$. ☐

Lemma 2.22. *If* $u \in W_0^{k,p}(\Omega)$*, then* $\widetilde{u}_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u$ *in* $W^{k,p}(\Omega)$.

Proof. **Exercise.** ☐

Theorem 2.23 (Friedrichs' inequality[5]). *If $u \in W_0^{k,p}(\Omega)$ and Ω is bounded, then*

$$\|u\|_{L^p(\Omega)} \leq (\operatorname{diam}(\Omega))^k \left(\sum_{|\alpha|=k} \|\mathrm{D}^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p}.$$

Proof. Set $d := \operatorname{diam}(\Omega)$ and, without loss of generality, suppose that

$$\Omega \subseteq Q := \{x \in \mathbb{R}^n | 0 < x_i < d \text{ for } 1 \leq i \leq n\}.$$

Extend u to be zero outside Ω and obtain $\tilde{u} \in W_0^{k,p}(Q)$. If we write $x = (x_1, \ldots, x_{n-1}, x_n) = (x', x_n)$, then, by the fundamental theorem of calculus,

$$\tilde{u}(x) = \int_0^{x_n} \frac{\partial \tilde{u}}{\partial x_n}(x', y) \, \mathrm{d}y.$$

By Hölder's inequality, with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned} |\tilde{u}(x)| &= \left| \int_0^{x_n} \frac{\partial \tilde{u}}{\partial x_n} \, \mathrm{d}y \right| \\ &\leq \left(\int_0^{x_n} \left| \frac{\partial \tilde{u}}{\partial x_n} \right|^p \mathrm{d}y \right)^{1/p} \left(\int_0^{x_n} 1^q \, \mathrm{d}y \right)^{1/q} \\ &= d^{1/q} \left(\int_0^{x_n} \left| \frac{\partial \tilde{u}}{\partial x_n} \right|^p \mathrm{d}y \right)^{1/p}. \end{aligned}$$

Hence,

$$\begin{aligned} \int_\Omega |u(x)|^p \, \mathrm{d}x &= \int_Q |\tilde{u}(x)|^p \, \mathrm{d}x \\ &= d^{p/q} \int_Q \int_0^{x_n} \left| \frac{\partial \tilde{u}}{\partial x_n} \right|^p \mathrm{d}y \mathrm{d}x \\ &\leq d^{p/q} d \int_Q \left| \frac{\partial \tilde{u}}{\partial x_n} \right|^p \mathrm{d}x \\ &= d^{(p+q)/q} \int_Q \left| \frac{\partial \tilde{u}}{\partial x_n} \right|^p \mathrm{d}x. \end{aligned}$$

And so

$$\|u\|_{L^p(\Omega)} \leq d \left\| \frac{\partial u}{\partial x_n} \right\|_{L^p(\Omega)} \leq d^2 \left\| \frac{\partial^2 u}{\partial x_n^2} \right\|_{L^p(\Omega)} \leq \cdots \leq d^k \left\| \frac{\partial^k u}{\partial x_n^k} \right\|_{L^p(\Omega)},$$

and this is at most $d^k \left(\sum_{|\alpha|=k} \|\mathrm{D}^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p}$; it is also at most $d^k \sum_{|\alpha|=k} \|\mathrm{D}^\alpha u\|_{L^p(\Omega)}^p$. □

[5]Kurt O. Friedrichs (1901–1982)

Note that Friedrichs' inequality fails on the space $W^{k,p}(\Omega)$: simply take u to be any non-zero constant function. There are variants of Friedrichs' inequality that allow for non-zero boundary values at the cost of including a term of the form $\|u\|_{L^p(\partial\Omega)}$. However, it is not immediately clear how u can be restricted to the boundary of Ω (a set of Lebesgue measure zero) in a well-defined sense: see the discussion of traces later for an answer to this problem.

In $W_0^{k,p}(\Omega)$, therefore, the following four norms are all equivalent:

- $\left(\sum_{|\alpha|\leq k} \|\mathrm{D}^\alpha u\|_{L^p(\Omega)}^p\right)^{1/p}$;
- $\sum_{|\alpha|\leq k} \|\mathrm{D}^\alpha u\|_{L^p(\Omega)}$;
- $\left(\sum_{|\alpha|= k} \|\mathrm{D}^\alpha u\|_{L^p(\Omega)}^p\right)^{1/p}$;
- $\sum_{|\alpha|= k} \|\mathrm{D}^\alpha u\|_{L^p(\Omega)}$.

So, for example, in $H_0^1(\Omega) = W_0^{1,2}(\Omega)$,

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x\right)^{1/2}$$

is equivalent to the usual norm

$$\|u\|_{H_0^1(\Omega)} := \left(\int_\Omega |u(x)|^2 \,\mathrm{d}x + \int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x\right)^{1/2}.$$

2.3 Weak Solutions to the Poisson Equation

We now wish to find $u \in H_0^1(\Omega) = W_0^{1,2}(\Omega)$ such that

$$\begin{cases} -\Delta u = f & \text{on } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

To do this we solve the variational problem of finding

$$\inf_{u\in H_0^1(\Omega)} \frac{1}{2}\int_\Omega |\mathrm{D}u(x)|^2 \,\mathrm{d}x - \int_\Omega f(x)u(x)\,\mathrm{d}x.$$

We claim that these two problems are equivalent:

Proof. ($\Rightarrow$) Write

$$I(u) := \frac{1}{2}\int_\Omega |\mathrm{D}u(x)|^2 \,\mathrm{d}x - \int_\Omega f(x)u(x)\,\mathrm{d}x$$

and suppose that $u_0 \in H_0^1(\Omega)$ is a minimizer for I. Let $f(t) := I(u_0 + t\phi)$ for $\phi \in H_0^1(\Omega)$ and $t \in \mathbb{R}$; then $f(0)$ delivers the minimum (0 is a minimizer). f is smooth in t, so $f'(0) = 0$. Thus,

$$I(u_0 + t\phi) = \frac{1}{2}\int_\Omega |\mathrm{D}(u_0(x) + t\phi(x))|^2 \,\mathrm{d}x - \int_\Omega f(x)(u_0(x) + t\phi(x))\,\mathrm{d}x$$

and

$$\frac{\partial}{\partial t} I(u_0 + t\phi)\bigg|_{t=0} = \int_\Omega \mathrm{D}u_0(x) \cdot \mathrm{D}\phi(x)\,\mathrm{d}x - \int_\Omega f(x)\phi(x)\,\mathrm{d}x = 0.$$

So $\int_\Omega \mathrm{D}u_0(x) \cdot \mathrm{D}\phi(x)\,\mathrm{d}x = \int_\Omega f(x)\phi(x)\,\mathrm{d}x$ for all $\phi \in H_0^1(\Omega)$, so

$$-\int_\Omega \Delta u_0(x)\phi(x)\,\mathrm{d}x = \int_\Omega f(x)\phi(x)\,\mathrm{d}x,$$

and so

$$\begin{cases} -\Delta u_0 = f & \text{on } \Omega; \\ u_0 = 0 & \text{on } \partial\Omega; \end{cases}$$

since $u_0 \in H_0^1(\Omega)$.

($\Leftarrow$) Suppose that $u \in H_0^1(\Omega)$ solves

$$\begin{cases} -\Delta u = f & \text{on } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, for all $\phi \in H_0^1(\Omega)$,

$$\int_\Omega \mathrm{D}u(x) \cdot \mathrm{D}\phi(x) - f(x)\phi(x)\,\mathrm{d}x = \int_\Omega (\Delta u(x) - f(x))\phi(x)\,\mathrm{d}x = 0.$$

So, if we write $\phi = u - w$, where $u, w \in H_0^1(\Omega)$,

$$\int_\Omega \mathrm{D}u(x) \cdot (\mathrm{D}u(x) - \mathrm{D}w(x)) - f(x)(u(x) - w(x))\,\mathrm{d}x = 0.$$

Hence,

$$\begin{aligned} \int_\Omega |\mathrm{D}u(x)|^2\,\mathrm{d}x - \int_\Omega f(x)u(x)\,\mathrm{d}x &= \int_\Omega \mathrm{D}u(x) \cdot \mathrm{D}w(x)\,\mathrm{d}x - \int_\Omega f(x)w(x)\,\mathrm{d}x \\ &\le \frac{1}{2}\int_\Omega |\mathrm{D}u(x)|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\mathrm{D}w(x)|^2\,\mathrm{d}x - \int_\Omega f(x)w(x)\,\mathrm{d}x, \end{aligned}$$

where the inequality is the Cauchy–Schwarz inequality in the form

$$\int_\Omega \mathrm{D}u(x) \cdot \mathrm{D}w(x)\,\mathrm{d}x \le \int_\Omega \frac{1}{2}|\mathrm{D}u(x)|^2 + \frac{1}{2}|\mathrm{D}w(x)|^2\,\mathrm{d}x.$$

Thus, $I(u) \le I(w)$ for all $w \in H_0^1(\Omega)$. □

Proposition 2.24. *If Ω is bounded, then there exists a unique element of $H_0^1(\Omega)$ that attains*

$$\inf_{u \in H_0^1(\Omega)} \frac{1}{2}\int_\Omega |\mathrm{D}u(x)|^2\,\mathrm{d}x - \int_\Omega f(x)u(x)\,\mathrm{d}x.$$

Proof. Let $I(u)$ be as before and set $d := \mathrm{diam}(\Omega)$.

Step 1. We show that I is bounded below.

$$\frac{1}{2}\int_\Omega |\mathrm{D}u(x)|^2\,\mathrm{d}x \ge \frac{d}{2}\int_\Omega |u(x)|^2\,\mathrm{d}x.$$

$$\int_\Omega f(x)u(x)\,\mathrm{d}x \le \frac{d}{2}\int_\Omega |u(x)|^2\,\mathrm{d}x + \frac{1}{2d}\int_\Omega f(x)^2\,\mathrm{d}x$$

Hence,

$$\begin{aligned} I(u) &= \frac{1}{2}\int_\Omega |\mathrm{D}u(x)|^2\,\mathrm{d}x - \int_\Omega f(x)u(x)\,\mathrm{d}x \\ &\ge \frac{d}{2}\int_\Omega |u(x)|^2\,\mathrm{d}x - \frac{d}{2}\int_\Omega |u(x)|^2\,\mathrm{d}x + \frac{1}{2d}\int_\Omega f(x)^2\,\mathrm{d}x \\ &= -\frac{1}{2d}\int_\Omega f(x)^2\,\mathrm{d}x \\ &= -\text{constant}, \end{aligned}$$

and so I is bounded below.

Step 2. Now let (u_n) be a minimizing sequence, i.e.

$$I(u_n) \underset{n\to\infty}{\longrightarrow} \inf_{u\in H_0^1(\Omega)} I(u) = a \ne \pm\infty.$$

We wish to show compactness in some sense: $I(u_n) \le$ some constant for all n should imply that there is a u with $I(u_n) \to I(u) < 0$. There are two equivalent ways to approach this: either show that $I(u_n) \le 0$ for large n, or show that $I(u_n) \le C$ for all n.

We will then apply the following weak compactness result from functional analysis:

Theorem 2.25. (Banach–Alaoglu.) *Let X be a separable Banach space, and X^* its dual space. If $(x_n^*) \subset X^*$ is a bounded sequence, then there is a subsequence $(x_{n_k}^*)$ that converges in a weak topology: $\langle x_{n_k}^*, x\rangle \to \langle x^*, x\rangle$ for all $x \in X$.*

So, if we can prove that (u_n) is bounded in $H_0^1(\Omega)$, (u_n) will be weakly compact.

$$\begin{aligned} \frac{d}{2}\int_\Omega |u_n(x)|^2\,\mathrm{d}x &\le \frac{1}{2}\int_\Omega |\mathrm{D}u_n(x)|^2 \text{ by Friedrichs' inequality} \\ &\le \int_\Omega f(x)u_n(x)\,\mathrm{d}x + C \\ &\le \frac{1}{2}\left(\frac{d}{2}\int_\Omega |u_n(x)|^2\,\mathrm{d}x + \frac{2}{d}\int_\Omega f(x)^2\,\mathrm{d}x\right) + C \\ &= \frac{d}{4}\int_\Omega |u_n(x)|^2\,\mathrm{d}x + \frac{1}{d}\int_\Omega f(x)^2\,\mathrm{d}x + C. \end{aligned}$$

So $\frac{d}{4}\int_\Omega |u_n(x)|^2\,\mathrm{d}x \le \frac{1}{d}\int_\Omega f(x)^2\,\mathrm{d}x + C \le$ some constant, so (u_n) is bounded in $L^2(\Omega)$. Since

$$\begin{aligned} \frac{1}{2}\int_\Omega |\mathrm{D}u_n(x)|^2\,\mathrm{d}x &\le \int_\Omega f(x)u_n(x)\,\mathrm{d}x + C \\ &\le \left(\int_\Omega f(x)^2\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega |u_n(x)|^2\,\mathrm{d}x\right)^{1/2} + C \\ &\le \text{ constant} \end{aligned}$$

So $(\mathrm{D}u_n)$ is bounded in $L^2(\Omega)$. So $\|u_n\|_{H_0^1(\Omega)}$ is bounded, and so there exists a subsequence $(u_{n_k}) \subset (u_n)$ such that $u_{n_k} \rightharpoonup u \in H_0^1(\Omega)$.

Now, by Cauchy–Schwarz,

$$\begin{aligned}\inf_{v \in H_0^1(\Omega)} I(v) &= \liminf_{k\to\infty} \frac{1}{2}\int_\Omega \|\mathrm{D}u_{n_k}(x)\|^2 \,\mathrm{d}x - \int_\Omega f(x)u_{n_k}(x)\,\mathrm{d}x \\ &\geq \frac{1}{2}\int_\Omega |\mathrm{D}u(x)|^2\,\mathrm{d}x - \int_\Omega f(x)u(x)\,\mathrm{d}x,\end{aligned}$$

so u is indeed a minimizer of I on $H_0^1(\Omega)$.

Step 3. To show uniqueness, suppose that there are two minimizers u_1 and u_2, and set $u := \alpha u_1 + (1-\alpha)u_2$. Then, by the strict convexity of I,

$$\begin{aligned}I(\alpha u_1 + (1-\alpha)u_2) &< \alpha I(u_1) + (1-\alpha)I(u_2) \\ &= \alpha \min I + (1-\alpha)\min I \\ &= \min I,\end{aligned}$$

a contradiction. So there is a unique minimizer. □

2.4 Approximation up to the Boundary

We have approximations of elements of $W^{k,p}_{\mathrm{loc}}(\Omega)$ by sequences in $C^\infty(\Omega)$, and elements of $W^{k,p}(\Omega)$ by sequences in $C^\infty(\Omega)\cap W^{k,p}(\Omega)$. We now prove another approximation theorem:

Theorem 2.26. *Suppose that Ω is bounded and that $\partial\Omega$ is C^1. If $u \in W^{k,p}(\Omega)$, then there exists a sequence $(u_m) \subset C^\infty(\bar{\Omega})$ such that $u_m \to u$ in $W^{k,p}(\Omega)$.*

Proof. We will prove this only for star-shaped[6] domains.

Let $u \in W^{k,p}(\Omega)$ and set $\Omega_m := \left\{x \,\middle|\, \frac{m-1}{m}x \in \Omega\right\}$. Observe that

$$\Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_m \supset \cdots \supset \Omega$$

and $\Omega \subset\subset \Omega_m$ for all m. Since Ω is star-shaped (about 0, by translation), we can set $u_m(x) := u\left(\frac{m-1}{m}x\right)$ for $x \in \Omega_m$. Note that $u_m \in W^{k,p}(\Omega_m)$. We want u_m to approximate u in $W^{k,p}(\Omega)$.

$$\begin{aligned}\|u_m - u\|^p_{L^p(\Omega)} &\leq \int_\Omega \left| u\left(\frac{m-1}{m}x\right) - u(x)\right|^p \mathrm{d}x \\ &= \int_\Omega \left| u\left(x - \frac{x}{m}\right) - u(x)\right|^p \mathrm{d}x \\ &\leq \sup_{|z|\leq d/m} \int_\Omega |u(x+z) - u(x)|^p\,\mathrm{d}x \\ &\underset{m\to\infty}{\longrightarrow} 0,\end{aligned}$$

[6]Ω is *star-shaped* with respect to $x_0 \in \Omega$ if any ray (half line) based at x_0 intersects $\partial\Omega$ only once.

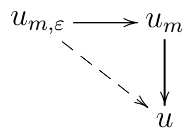
so $u_m \to u$ in $L^p(\Omega)$.

Similarly,

$$
\begin{aligned}
\|D^\alpha u_m - \mathrm{D}^\alpha u\|^p_{L^p(\Omega)} &= \int_\Omega |\mathrm{D}^\alpha u_m(x) - \mathrm{D}^\alpha u(x)|^p \,\mathrm{d}x \\
&\leq \int_\Omega \left| \mathrm{D}^\alpha u \left(\frac{m-1}{m} x \right) - \mathrm{D}^\alpha u(x) \right|^p \,\mathrm{d}x \\
&= \int_\Omega \left| \left(\frac{m-1}{m} \right)^{|\alpha|} \left(\mathrm{D}^\alpha_{\frac{m-1}{m}x} u \left(\frac{m-1}{m} x \right) \right) - \mathrm{D}^\alpha u(x) \right|^p \,\mathrm{d}x \\
&= \int_\Omega \left| \left(\left(\frac{m-1}{m} \right)^{|\alpha|} - 1 + 1 \right) \mathrm{D}^\alpha u(x) - \mathrm{D}^\alpha u(x) \right|^p \,\mathrm{d}x \\
&\leq C_1 \left| 1 - \left(\frac{m-1}{m} \right)^{|\alpha|} \right|^p \int_\Omega \left| \mathrm{D}^\alpha u \left(\frac{m-1}{m} x \right) \right|^p \,\mathrm{d}x \\
&\quad + C_2 \int_\Omega \left| \mathrm{D}^\alpha u \left(x - \frac{x}{m} \right) - \mathrm{D}^\alpha u(x) \right|^p \,\mathrm{d}x \\
&\xrightarrow[m\to\infty]{} 0,
\end{aligned}
$$

and so $\mathrm{D}^\alpha u_m \to \mathrm{D}^\alpha u$ in $L^p(\Omega)$.

We now have $u_m \in W^{k,p}(\Omega_m)$, $\Omega \subset\subset \Omega_m$, such that $\|u_m - u\|_{W^{k,p}(\Omega)} \to 0$. Now approximate each u_m by smooth functions $u_{m,\varepsilon} \in C^\infty(\Omega_m)$ such that $u_{m,\varepsilon} \xrightarrow[\varepsilon\to 0]{0} u_m$ in both $W^{k,p}_{\mathrm{loc}}(\Omega_m)$ and $W^{k,p}(\Omega)$, since $\Omega \subset\subset \Omega_m$. Also, $u_{m,\varepsilon} \in C^\infty(\bar{\Omega})$, since $\Omega \subseteq \bar{\Omega} \subseteq \Omega_m$.



So we can choose $\varepsilon(m) \to 0$ such that $u_{m,\varepsilon(m)} \in C^\infty(\bar{\Omega})$ and $u_{m,\varepsilon(m)} \to u$ in $W^{k,p}(\Omega)$. □

2.5 Extensions of Weakly Differentiable Functions

Theorem 2.27 (Extension theorem)**.** *Let Ω be bounded and let $\partial\Omega$ be C^1. Let $\Omega \subset\subset V$ and $u \in W^{k,p}(\Omega)$. Then there exists a bounded linear operator $E\colon W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)$ such that*

(i) $Eu(x) = u(x)$ *for all* $x \in \Omega$*;*

(ii) $\|Eu\|_{W^{k,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{k,p}(\Omega)}$*, where C depends only on k, p, Ω, V;*

(iii) $\operatorname{supp} Eu \subseteq V$*, so, in effect,* $Eu \in W^{k,p}(V)$*.*

Exercise 2.28. Use the extension theorem to prove the previous result about approximation up to the boundary.

Proof of theorem 2.27. We prove the result for $W^{1,p}(\Omega)$.

Step 1. Make a transformation of coordinates so that $\partial\Omega$ is the flat hyperplane $\{x_n = 0\}$ near $x_0 \in \partial\Omega$. Write $B := \mathbb{B}_r(x_0)$ as $B_+ \uplus B_-$, where B_+ has $x_n \geq 0$ (inside Ω) and B_- has $x_n < 0$ (outside Ω). Suppose that $u \in C^\infty(\bar{\Omega})$. Extend $u : B_+ \to \mathbb{R}$ to $\bar{u} : B \to \mathbb{R}$ by

$$\bar{u}(x) = \begin{cases} u(x), & x \in B_+; \\ -3u(x', -x_n) + 4u(x', -x_n/2), & x \in B_-. \end{cases}$$

We check continuity of $\bar{u}$:

$$\bar{u}|_{\partial\Omega}(x) = u(x', 0) = -3u(x', 0) + 4u(x', 0).$$

Clearly $\frac{\partial \bar{u}}{\partial x_i}$ is continuous for $1 \leq i < n$. Also,

$$\left.\frac{\partial \bar{u}}{\partial x_n}\right|_{\partial\Omega} = \frac{\partial \bar{u}}{\partial x_n}(x', 0) = 3\frac{\partial \bar{u}}{\partial x_n}(x', 0) - 2\frac{\partial \bar{u}}{\partial x_n}(x', 0).$$

So $\bar{u} \in C^1(B)$. We now check that $\|\bar{u}\|_{W^{1,p}(B)} \leq C\|u\|_{W^{1,p}(B_+)}$. From the definition of $\bar{u}$, it is clear that

$$\int_B |\bar{u}(x)|^p \,\mathrm{d}x \leq C \int_{B_+} |u(x)|^p \,\mathrm{d}x.$$

Similarly, for $1 \leq i < n$,

$$\int_B \left|\frac{\partial \bar{u}}{\partial x_i}(x)\right|^p \,\mathrm{d}x \leq C \int_{B_+} \left|\frac{\partial u}{\partial x_i}(x)\right|^p \,\mathrm{d}x.$$

Since

$$\frac{\partial \bar{u}}{\partial x_n}(x) = \begin{cases} \frac{\partial \bar{u}}{\partial x_n}(x), & x \in B_+; \\ 3\frac{\partial \bar{u}}{\partial x_n}(x', -x_n) - 2\frac{\partial \bar{u}}{\partial x_n}\left(x', -\frac{x_n}{2}\right), & x \in B_-, \end{cases}$$

we also have

$$\int_B \left|\frac{\partial \bar{u}}{\partial x_n}(x)\right|^p \,\mathrm{d}x \leq C \int_{B_+} \left|\frac{\partial u}{\partial x_n}(x)\right|^p \,\mathrm{d}x.$$

Hence, $\|\bar{u}\|_{W^{1,p}(B)} \leq C\|u\|_{W^{1,p}(B_+)}$.

Step 2. Cover $\partial\Omega$ with balls B_i as in Step 1. Since $\partial\Omega$ is compact, we can take a finite subcover and a partition of unity subordinate to that finite subcover. The sum of the $\bar{u}_i$'s, weighted by that partition of unity, then gives our extension Eu, defined for $u \in C^\infty(\bar{\Omega})$. Now extend E from the dense subspace $C^\infty(\bar{\Omega})$ to all of $W^{k,p}(\Omega)$. □

2.6 Trace

Suppose that we try to solve the Poisson equation

$$\begin{cases} -\Delta u(x) = f(x), & x \in \Omega; \\ u(x) = g(x), & x \in \partial\Omega; \end{cases}$$

where f, g are both L^2. The previous results suggest that the solution u should lie in $W^{1,2}(\Omega)$. However, such a u is only defined up to sets of measure zero, and $\partial\Omega$

has zero measure in Ω, so what does the BC $u = g$ on $\partial\Omega$ actually mean? Given $u \in W^{1,2}(\Omega)$, if we change its value on a null set, we do not change $\int_\Omega |u(x)|^2 \, \mathrm{d}x$. However, it turns out that if we control both u and $\mathrm{D}u$ on Ω, then we automatically control u on $\partial\Omega$.

Theorem 2.29 (Trace theorem)**.** *Let Ω be bounded and let $\partial\Omega$ be C^1 Then there exists a bounded linear operator $T\colon W^{1,p}(\Omega) \to L^p(\partial\Omega)$, called the trace operator, such that*

(i) $Tu = u|_{\partial\Omega}$ if $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$, i.e. T extends the notion of "restriction to the boundary";

(ii) $\|Tu\|_{L^p(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$ for all $u \in W^{1,p}(\Omega)$, where C depends only on Ω and p.

Proof. Step 1. As before, take $\partial\Omega$ to be flat near $x_0 \in \partial\Omega$. Take $u \in C^\infty(\bar{\Omega})$ (although it is only necessary that $u \in C^1(\bar{\Omega})$). Choose a non-negative function $\xi \in C_c^\infty(\mathbb{B}_r(x_0))$ such that $\xi(x) = 1$ for $x \in \mathbb{B}_{r/2}(x_0)$ (and so ξ decays smoothly to 0 at $\partial\mathbb{B}_r(x_0)$. Let $\Gamma := \mathbb{B}_{r/2}(x_0) \cap \partial\Omega$. Then

$$\int_\Gamma |u(x')|^p \, \mathrm{d}x' \leq \int_{\partial\Omega \cap \mathbb{B}_r(x_0)} \xi(x')|u(x')|^p \, \mathrm{d}x'$$

and

$$\begin{aligned}\int_{B_+} \frac{\partial}{\partial x_n} \left(\xi(x)|u(x)|^p\right) \, \mathrm{d}x &= -\int_{\partial B_+} \xi(x)|u(x)|^p \nu_n(x) \, \mathrm{d}x \\ &= -\int_{\partial\Omega \cap \mathbb{B}_r(x_0)} \xi(x')|u(x')|^p \, \mathrm{d}x'\end{aligned}$$

since $\xi = 0$ on $\partial\mathbb{B}_r(x_0)$.

$$\begin{aligned}\int_\Gamma |u(x')|^p \, \mathrm{d}x' &\leq \left| \int_{\partial\Omega \cap \mathbb{B}_r(x_0)} \xi(x')|u(x')|^p \, \mathrm{d}x' \right| \\ &= \left| \int_{B_+} \frac{\partial}{\partial x_n} \left(\xi(x)|u(x)|^p\right) \, \mathrm{d}x \right| \\ &= \left| \int_{B_+} \frac{\partial \xi}{\partial x_n}(x)|u(x)|^p + \xi(x)p|u(x)|^{p-1} \operatorname{sgn} u(x) \frac{\partial u}{\partial x_n}(x) \, \mathrm{d}x \right| \\ &\leq \int_{B_+} \left|\frac{\partial \xi}{\partial x_n}(x)\right| |u(x)|^p \, \mathrm{d}x + \int_{B_+} |\xi(x)|p|u(x)|^{p-1} \left|\frac{\partial u}{\partial x_n}(x)\right| \, \mathrm{d}x \\ &\leq C_1 \int_{B_+} |u(x)|^p \, \mathrm{d}x + C_2 \int_{B_+} |u(x)|^{p-1} \left|\frac{\partial u}{\partial x_n}(x)\right| \, \mathrm{d}x \\ &\leq C \left(\int_{B_+} |u(x)|^p \, \mathrm{d}x + \int_{B_+} |\mathrm{D}u(x)|^p \, \mathrm{d}x \right),\end{aligned}$$

where the last line follows by Young's inequality $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ in the form

$$|u(x)|^{p-1} \left|\frac{\partial u}{\partial x_n}(x)\right| \leq C \left(\left|\frac{\partial u}{\partial x_n}(x)\right|^p + |u(x)|^{(p-1)q} \right) = C \left(\left|\frac{\partial u}{\partial x_n}(x)\right|^p + |u(x)|^p \right).$$

Thus, $\|u\|_{L^p(\Gamma)} \le C\|u\|_{W^{1,p}(\Omega)}$.

Now cover $\partial\Omega$ by a finite number of such sets Γ_i to give

$$\|Tu\|_{L^p(\partial\Omega)} = \|u\|_{L^p(\partial\Omega)} \le \sum_i \|u\|_{L^p(\Gamma_i)} \le C\|u\|_{W^{1,p}(\Omega)}.$$

Step 2. We extend the above to functions $u \in W^{1,p}(\Omega)$. Take a sequence $(u_m) \subset C^\infty(\bar{\Omega})$ such that $u_m \to u$ in $W^{1,p}(\Omega)$. The operator T is clearly linear by its definition as "restriction to the boundary".

$$\begin{aligned} \|Tu_m - Tu_n\|_{L^p(\partial\Omega)} &= \|u_m - u_n\|_{L^p(\partial\Omega)} \\ &\le C\|u_m - u_n\|_{W^{1,p}(\Omega)} \\ &\underset{m,n\to\infty}{\longrightarrow} 0 \end{aligned}$$

since (u_m) is Cauchy, so (Tu_m) is Cauchy in $L^p(\partial\Omega)$. Since $L^p(\partial\Omega)$ is complete, $Tu_m \to v =: Tu$. For each m,

$$\|Tu_m\|_{L^p(\partial\Omega)} \le C\|u_m\|_{W^{1,p}(\Omega)},$$

and so, taking the limit as $m \to \infty$, $\|Tu\|_{L^p(\partial\Omega)} \le C\|u\|_{W^{1,p}(\Omega)}$. Thus, Tu is well-defined and depends only on u, not the choice of sequence (u_m).

Step 3. If $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$, then $Tu = u|_{\partial\Omega}$. This is clear, since $Tu_m = u_m \to u$ in $C(\partial\Omega)$ and $Tu_m \to Tu$ in $L^p(\partial\Omega)$ (convergence almost everywhere). Hence, $Tu = u|_{\partial\Omega}$. ☐

Theorem 2.30. *Let Ω be bounded and let $\partial\Omega$ be C^1. Then $u \in W_0^{1,p}(\Omega)$ if, and only if, $u \in W^{1,p}(\Omega)$ and $Tu = 0$.*

This result extends to higher-order derivatives. Thus, we have an equivalent definition for $W_0^{k,p}(\Omega)$:

$$W_0^{k,p}(\Omega) := \left\{ u \in W^{k,p}(\Omega) \middle| \forall |\alpha| \le k-1, T\mathrm{D}^\alpha u = 0 \right\}.$$

Theorem 2.31 (Green's theorem)**.** *Suppose that $u \in W^{1,p}(\Omega)$ and $v \in W^{1,q}(\Omega)$, with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\int_\Omega \frac{\partial u}{\partial x_i}(x)v(x)\,\mathrm{d}x = -\int_\Omega \frac{\partial v}{\partial x_i}(x)u(x)\,dx + \int_{\partial\Omega} u(x)v(x)\nu_i(x)\,\mathrm{d}S(x).$$

Proof. **Exercise.** *Hint:* Take sequences $(u_m), (v_m) \subset C^\infty(\bar{\Omega})$ converging to u and v respectively. Use $Tu_m \to T_u$, $Tv_m \to Tv$. ☐

2.7 Review of Sobolev Spaces

Suppose we wish to solve

$$\begin{cases} -\Delta u = f \in L^p(\Omega); \\ u|_{\partial\Omega} = 0. \end{cases}$$

1. Weak derivatives. Even if $u \notin C^1$, we can still define a derivative, so the solution u to the above doesn't have to lie in $C^2(\bar{\Omega})$. We define the weak derivative $\frac{\partial u}{\partial x_i}$ by

$$\int_\Omega u(x)\frac{\partial \phi}{\partial x_i}(x)\,\mathrm{d}x = -\int_\Omega \frac{\partial u}{\partial x_i}(x)\phi(x)\,\mathrm{d}x \quad \forall \phi \in C_\mathrm{c}^\infty(\Omega).$$

   For this derivative to be well-defined we need that $u \in L^1(\Omega)$.

2. Sobolev spaces.

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) | \mathrm{D}^\alpha u \in L^p(\Omega), |\alpha| \le k\}.$$

   (a) $W^{k,p}(\Omega)$, $1 \le p < \infty$, is a Banach space.
   (b) $H^k(\Omega) = W^{k,2}(\Omega)$ is a Hilbert space.
   (c) $W^{k,p}(\Omega)$, $1 < p < \infty$, is reflexive ($X^{**} = X$).
   (d) $W^{k,p}(\Omega)$, $1 \le p < \infty$, is separable.

   Theorem 2.32. (Banach–Alaoglu.) *If X is a reflexive Banach space and (u_m) is a bounded sequence in X, then there is a subsequence (u_{m_j}) such that $u_{m_j} \rightharpoonup u \in X$.*

   So if $\|u_m\|_{W^{k,p}(\Omega)} \le C$, $1 < p < \infty$, then there is a subsequence (u_{m_j}) such that

$$\int_\Omega u_{m_j}(x)\phi(x)\,\mathrm{d}x \to \int_\Omega u(x)\phi(x)\,\mathrm{d}x \quad \forall \phi \in L^q(\Omega),$$

$$\int_\Omega \mathrm{D}^\alpha u_{m_j}(x)\phi(x)\,\mathrm{d}x \to \int_\Omega D^\alpha u(x)\phi(x)\,\mathrm{d}x \quad \forall \phi \in L^q(\Omega), |\alpha| \le k.$$

3. $C^\infty(\bar{\Omega})$ is dense in $W^{k,p}(\Omega)$. If we prove some property for $u \in C^\infty(\bar{\Omega})$ and this property respects limits in the $W^{k,p}(\Omega)$ norm, then that property holds for $u \in W^{k,p}(\Omega)$ as well.

4. $W_0^{k,p}(\Omega) :=$ closure of $C_\mathrm{c}^\infty(\Omega)$ in $W^{k,p}(\Omega)$. For $W_0^{k,p}(\Omega)$ we have better approximation theorems (not just on $V \subset\subset \Omega$). $W_0^{k,p}(\Omega) = W^{k,p}(\mathbb{R}^n)$ if we extend by 0 outside Ω. We have an explicit formula for a sequence of approximations $u_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} u$ in $W_0^{k,p}(\Omega) = W^{k,p}(\mathbb{R}^n)$:

$$u_\varepsilon(x) := \int_{\mathbb{R}^n} \eta\left(\frac{x-y}{\varepsilon}\right) u(y)\,\mathrm{d}y,$$

   where

$$\eta(x) = \begin{cases} C\exp\left(\frac{1}{|x|^2-1}\right), & |x| < 1; \\ 0, & |x| \ge 1; \end{cases}$$

   and $\int_{\mathbb{R}^n} \eta(x)\,\mathrm{d}x = 1$.

5. Extension of $W^{k,p}(\Omega)$ to $W^{k,p}(V)$, $\Omega \subset\subset V$.

$$\|u\|_{W^{k,p}(\Omega)} \leq \|Eu\|_{W_0^{k,p}(V)} \leq C\|u\|_{W^{k,p}(\Omega)}$$

where $Eu = u$ on Ω. $C_c^\infty(\Omega)$ is dense in $W_0^{k,p}(\Omega)$.

6. Trace. Does the condition $u|_{\partial\Omega} = 0$ make sense, given that $\partial\Omega$ has zero measure? There is a linear map $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that

   (a) $Tu = u|_{\partial\Omega}$ for $u \in C(\bar{\Omega}) \cap W^{1,p}(\Omega)$;

   (b) $\|Tu\|_{L^p(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$.

   Thus, we have an an equivalent definition for $W_0^{k,p}(\Omega)$:

$$W_0^{k,p}(\Omega) := \left\{ u \in W^{k,p}(\Omega) \middle| \forall |\alpha| \leq k-1, T\mathrm{D}^\alpha u = 0 \right\}.$$

7. A bound $\int_\Omega |\mathrm{D}u_m(x)|^2 \,\mathrm{d}x \leq C$ is enough for weak compactness, $\mathrm{D}u_{m_j} \rightharpoonup v$. If $u \in W^{1,p}(\Omega)$, then $\|u\|_{L^p(\Omega)} = \mathrm{diam}(\Omega)\|\mathrm{D}u\|_{L^p(\Omega)}$, and so $v = \mathrm{D}u$.

If we wish to solve

$$\begin{cases} -\Delta u = f \in L^p(\Omega); \\ u|_{\partial\Omega} = g; \end{cases}$$

then we find w smooth enough with $w|_{\partial\Omega} = g$ and solve

$$\begin{cases} -\Delta(u-w) = f + \Delta w; \\ (u-w)|_{\partial\Omega} = 0. \end{cases}$$

2.8 Sobolev Inclusions

If $u \in W^{k,p}(\Omega)$, does it belong to any other function space, such as $L^q(\Omega)$, $C(\Omega)$, $C^1(\Omega)$?

Theorem 2.33 (Gagliardo–Nirenberg inequality[7] for $\mathbb{R}^n$)**.** *Let $\Omega = \mathbb{R}^n$, $1 \leq p < n$, and $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. Then there is a constant C, depending only on n and p, such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|\mathrm{D}u\|_{L^p(\mathbb{R}^n)} \quad \forall u \in W^{1,p}(\mathbb{R}^n).$$

Equivalently, $W^{1,p}(\mathbb{R}^n) \subseteq L^{p^*}(\mathbb{R}^n)$ and the inclusion $i : W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$ is continuous.

Definition 2.34. $p^* = \frac{np}{n-p}$ is sometimes known as the *Sobolev conjugate* of p.

Proof of theorem 2.33. First consider the case $u \in C_c^\infty(\mathbb{R}^n)$, $p = 1$, $p^* = \frac{n}{n-1}$.

$$\begin{aligned} |u(x)| &= \left| \int_{-\infty}^{x_i} \frac{\partial u}{\partial x_i}(x_1, \dots, y_i, \dots, x_n) \,\mathrm{d}y_i \right| \\ &\leq \int_{\mathbb{R}} |\mathrm{D}u(y)| \,\mathrm{d}y_i \end{aligned}$$

[7] Emilio Gagliardo, Louis Nirenberg (1925–)

for each $i = 1, \ldots, n$. Hence, $|u(x)|^n \leq \prod_{i=1}^n \int_{\mathbb{R}} |\mathrm{D}u(y)|\, \mathrm{d}y_i$, and so

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{\mathbb{R}} |\mathrm{D}u(y)|\, \mathrm{d}y_i \right)^{\frac{1}{n-1}}.$$

So

$$\begin{aligned}
\int_{\mathbb{R}} |u(x)|^{\frac{n}{n-1}}\, \mathrm{d}x_1 &= \int_{\mathbb{R}} \prod_{i=1}^{n} \left(\int_{\mathbb{R}} |\mathrm{D}u(y)|\, \mathrm{d}y_i \right)^{\frac{1}{n-1}} \mathrm{d}x_1 \\
&= \left(\int_{\mathbb{R}} |\mathrm{D}u(y)|\, \mathrm{d}y_1 \right)^{\frac{1}{n-1}} \int_{\mathbb{R}} \prod_{i=2}^{n} \left(\int_{\mathbb{R}} |\mathrm{D}u(y)| \mathrm{d}y_i \right)^{\frac{1}{n-1}} \mathrm{d}x_1
\end{aligned}$$

The usual Hölder inequality can be generalized to

$$\int f_1 \ldots f_n \leq \left(\int f_1^{p_1} \right)^{\frac{1}{p_1}} \cdots \left(\int f_n^{p_n} \right)^{\frac{1}{p_n}}$$

for $\frac{1}{p_1} + \cdots + \frac{1}{p_n} = 1$, $p_i > 1$. Thus,

$$\int_{\mathbb{R}} \prod_{i=2}^{n} \left(\int_{\mathbb{R}} |\mathrm{D}u(y)| \mathrm{d}y_i \right)^{\frac{1}{n-1}} \mathrm{d}x_1 \leq \prod_{i=2}^{n} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |\mathrm{D}u(y)|\, \mathrm{d}x_1 \mathrm{d}y_i \right)^{\frac{1}{n-1}}$$

$$\int_{\mathbb{R}} |u(x)|^{\frac{n}{n-1}}\, \mathrm{d}x_1 \leq \left(\int_{\mathbb{R}} |\mathrm{D}u(y)|\, \mathrm{d}y_1 \right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |\mathrm{D}u(y)|\, \mathrm{d}x_1 \mathrm{d}y_i \right)^{\frac{1}{n-1}}$$

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}} |u(x)|^{\frac{n}{n-1}}\, \mathrm{d}x_1 \mathrm{d}x_2 \\
&\leq \int_{\mathbb{R}} \left[\left(\int_{\mathbb{R}} |\mathrm{D}u(y)|\, \mathrm{d}y_1 \right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |\mathrm{D}u(y)|\, \mathrm{d}x_1 \mathrm{d}y_i \right)^{\frac{1}{n-1}} \right] \mathrm{d}x_2 \\
&= \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |\mathrm{D}u(y)|\, \mathrm{d}x_1 \mathrm{d}y_2 \right)^{\frac{1}{n-1}} \int_{\mathbb{R}} \left[\left(\int_{\mathbb{R}} |\mathrm{D}u(y)|\, \mathrm{d}y_1 \right)^{\frac{1}{n-1}} \prod_{i=3}^{n} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |\mathrm{D}u(y)|\, \mathrm{d}x_1 \mathrm{d}y_i \right)^{\frac{1}{n-1}} \right] \mathrm{d}x_2 \\
&\leq \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |\mathrm{D}u(y)|\, \mathrm{d}x_1 \mathrm{d}y_2 \right)^{\frac{1}{n-1}} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |\mathrm{D}u(y)|\, \mathrm{d}y_1 \mathrm{d}x_2 \right)^{\frac{1}{n-1}} \prod_{i=3}^{n} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} |\mathrm{D}u(y)|\, \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}y_i \right)^{\frac{1}{n-1}}
\end{aligned}$$

Thus, repeating this procedure a total of n times:

$$\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}}\, \mathrm{d}x \leq \prod_{i=1}^{n} \left(\int_{\mathbb{R}^n} |\mathrm{D}u(x)|\, \mathrm{d}x \right)^{\frac{1}{n-1}} = \left(\int_{\mathbb{R}^n} |\mathrm{D}u(x)|\, \mathrm{d}x \right)^{\frac{n}{n-1}}.$$

This proves the claim for $p = 1$ with constant $C = 1$.

For $p > 1$, set $v = |u|^b$, $b > 1$:

$$\int_{\mathbb{R}^n} |v(x)|^{\frac{n}{n-1}}\, \mathrm{d}x \leq \left(\int_{\mathbb{R}^n} |\mathrm{D}v(x)|\, \mathrm{d}x \right)^{\frac{n}{n-1}}.$$

I.e.,

$$\int_{\mathbb{R}^n} |u(x)|^{\frac{bn}{n-1}}\,\mathrm{d}x \le \left(\int_{\mathbb{R}^n} b|\mathrm{D}u(x)||u(x)|^{b-1}\,\mathrm{d}x\right)^{\frac{n}{n-1}}$$

$$\int_{\mathbb{R}^n} |\mathrm{D}u(x)||u(x)|^{b-1}\,\mathrm{d}x \le \left(\int_{\mathbb{R}^n} |\mathrm{D}u(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |u(x)|^{(b-1)\frac{p}{p-1}}\,\mathrm{d}x\right)^{\frac{p-1}{p}}$$

So

$$\int_{\mathbb{R}^n} |u(x)|^{\frac{bn}{n-1}}\,\mathrm{d}x \le b\left(\int_{\mathbb{R}^n} |\mathrm{D}u(x)|^p\,\mathrm{d}x\right)^{\frac{n}{p(n-1)}} \left(\int_{\mathbb{R}^n} |u(x)|^{(b-1)\frac{p}{p-1}}\,\mathrm{d}x\right)^{\frac{p-1}{p}\frac{n}{n-1}}.$$

Solving $\frac{bn}{n-1} = (b-1)\frac{p}{p-1}$ for b gives $b = \frac{p(n-1)}{n-p}$, so $\frac{bn}{n-1} = \frac{np}{n-p} = p^*$. Therefore,

$$\left(\int_{\mathbb{R}^n} |u(x)|^{p^*}\,dx\right)^{\frac{1}{p^*}} \le C\left(\int_{\mathbb{R}^n} |\mathrm{D}u(x)|^p\,dx\right)^{\frac{1}{p}}.$$

where C depends only on n and p.

Finally, to obtain the result for $u \in W^{1,p}(\mathbb{R}^n)$, note that if $u_m \to u$, then

$$\left(\int_{\mathbb{R}^n} |u_m(x) - u_n(x)|^{p^*}\,dx\right)^{\frac{1}{p^*}} \le C\left(\int_{\mathbb{R}^n} |\mathrm{D}u_m(x) - \mathrm{D}u_n(x)|^p\,dx\right)^{\frac{1}{p}},$$

and so the theorem holds by the usual approximation results. □

Theorem 2.35 (Gagliardo–Nirenberg inequality for $\Omega \subset \mathbb{R}^n$)**.** *If $\Omega \subset \mathbb{R}^n$ is bounded, $\partial\Omega$ is C^1, and $1 \le p < n$, then $W^{1,p}(\Omega) \subseteq L^{p^*}(\Omega)$ and*

$$\|u\|_{L^{p^*}(\Omega)} \le C\|u\|_{W^{1,p}(\Omega)}$$

for all $u \in W^{1,p}(\Omega)$, where C depends only on Ω and p. (Hence, also, $W^{1,p}(\Omega) \subseteq L^q(\Omega)$ for $q \le p^*$.)

Proof. This follows from the extension theorem (theorem 2.27) and the inequalities

$$\|u\|_{L^{p^*}(\Omega)} \le \|Eu\|_{L^{p^*}(\mathbb{R}^n)} \le C\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\Omega)}.$$ □

For $W_0^{1,p}(\Omega)$, we have $\|u\|_{L^{p^*}(\Omega)} \le C\|\mathrm{D}u\|_{L^p(\Omega)}$.

Theorem 2.36. *Let $\Omega \subset \mathbb{R}^n$ be bounded with $\partial\Omega$ C^1. Then $W^{k,p}(\Omega) \subseteq L^q(\Omega)$, where $\frac{1}{q} = \frac{1}{p} + \frac{k}{n}$, $p \ge 1$, $kp < n$, and $\|u\|_{L^q(\Omega)} \le C\|u\|_{W^{k,p}(\Omega)}$ for all $u \in W^{k,p}(\Omega)$, where C depends only on Ω, k and p.*

Proof. First take $u \in W^{k,p}(\mathbb{R}^n)$. We know that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C\|\mathrm{D}u\|_{L^p(\mathbb{R}^n)},$$

and that for $|\alpha| = k$,

$$\|\mathrm{D}^{\alpha-1}u\|_{L^{p^*}(\mathbb{R}^n)} \le C\|\mathrm{D}^\alpha u\|_{L^p(\mathbb{R}^n)}.$$

Take $v := \mathrm{D}^{\alpha-1}u$ and apply the Gagliardo–Nirenberg inequality:

$$\|\mathrm{D}^{\alpha-2}u\|_{L^{p^{**}}(\mathbb{R}^n)} \leq C\|\mathrm{D}^{\alpha-1}u\|_{L^p(\mathbb{R}^n)},$$

where $\frac{1}{p^{**}} = \frac{1}{p} - \frac{2}{n}$. Applying this a total of k times yields

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\|u\|_{W^{k,p}(\mathbb{R}^n)},$$

or, by equivalence, for $|\alpha| = k$,

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\|\mathrm{D}^\alpha u\|_{L^p(\mathbb{R}^n)}.$$

Now, for $u \in W^{k,p}(\Omega)$, $Eu \in W^{k,p}(\mathbb{R}^n)$ and

$$\|u\|_{L^q(\Omega)} \leq \|Eu\|_{L^q(\mathbb{R}^n)} \leq C\|Eu\|_{W^{k,p}(\mathbb{R}^n)} \leq C'\|u\|_{W^{k,p}(\Omega)}. \qquad \square$$

What about the case $p = n$? Intuitively, for $W^{1,n}(\Omega)$, $q = \frac{np}{n-p} = \infty$. However, $W^{1,n}(\Omega) \not\subseteq L^\infty(\Omega)$, although $W^{1,n}(\Omega) \subseteq L^q(\Omega)$ for all finite q: take $u(x) := \log\log(1 - |x|^{-1})$ on $\Omega := \mathbb{B}_1(0) \subset \mathbb{R}^2$. $u \in W^{1,n}(\Omega)$ but $u(x) \to \infty$ as $|x| \to 0$.

Theorem 2.37 (Poincaré inequality[8]). *Let $\Omega \subset \mathbb{R}^n$ be bounded and connected with $\partial\Omega$ C^1. Then, for all $u \in W^{1,p}(\Omega)$,*

$$\int_\Omega \left| u(x) - ⨍_\Omega u \right|^p \mathrm{d}x \leq C \int_\Omega |\mathrm{D}u(x)|^p \,\mathrm{d}x,$$

where, as in our treatment of the Laplace equation, $⨍_\Omega u := \frac{1}{|\Omega|}\int_\Omega u(x)\,\mathrm{d}x$ denotes the mean value of u on Ω, and C depends only on Ω and p.

Proof. First consider the case $n = 1$, $u \in C^\infty([0,1])$. By the mean value theorem, there exists an $a \in (0,1)$ such that $u(a) = ⨍_\Omega u$.

$$\begin{aligned} \left| u(x) - ⨍_\Omega u \right|^p &= \left| \int_a^x u'(x)\,\mathrm{d}x \right| \\ &\leq \int_0^1 |u'(x)|\,\mathrm{d}x \\ &\leq C \left(\int_0^1 |u'(x)|^p \,\mathrm{d}x \right)^{\frac{1}{p}}. \end{aligned}$$

Hence,

$$\int_0^1 \left| u(x) - ⨍_\Omega u \right|^p \mathrm{d}x \leq C \int_0^1 \int_0^1 |u'(y)|\,\mathrm{d}y\mathrm{d}x = C \int_0^1 |u'(x)|^p \,\mathrm{d}x.$$

To treat the other cases, we first need the following definition and compactness theorem:

Definition 2.38. Suppose that X and Y are Banach spaces with $X \subseteq Y$. We say that X is *compactly embedded* in Y, and write $X \subset\subset Y$, if

[8] (Jules) Henri Poincaré (1854–1912)

(i) there is a C such that $\|x\|_Y \leq \|x\|_X$ for all $x \in X$;

(ii) any bounded set in X is precompact in Y (i.e., every sequence in such a bounded set has a subsequence that is Cauchy in Y).

Theorem 2.39 (Rellich–Kondrachov compactness theorem)**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is open, bounded (and connected), and that $\partial\Omega$ is C^1. If $1 \leq p < n$ and $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$, then $W^{1,p}(\Omega) \subseteq L^{p^*}(\Omega)$ and $W^{1,p}(\Omega) \subset\subset L^q(\Omega)$ for $1 \leq q < p^*$.*

Now consider $n \geq 1$. Suppose that the inequality is false. Then there is a sequence $(u_k) \subset W^{1,p}(\Omega)$ such that

$$\int_\Omega \left| u_k(x) - \fint_\Omega u_k \right|^p \mathrm{d}x \geq k^p \int_\Omega |\mathrm{D}u_k(x)|^p \,\mathrm{d}x,$$

i.e., $\left\| u_k - \fint_\Omega u_k \right\|_{L^p(\Omega)} \geq k \|\mathrm{D}u_k\|_{L^p(\Omega)}$. Let

$$v_k(x) := \frac{u_k(x) - \fint_\Omega u_k}{\left\| u_k - \fint_\Omega u_k \right\|_{L^p(\Omega)}}.$$

Thus, $\fint_\Omega v_k = 0$, $\|v_k\|_{L^p(\Omega)} = 1$, and $\mathrm{D}v_k = \frac{\mathrm{D}u_k}{\left\| u_k - \fint_\Omega u_k \right\|_{L^p(\Omega)}}$, so $\|\mathrm{D}v_k\|_{L^p(\Omega)} \leq \frac{1}{k}$. In particular, v_k is bounded in $W^{1,p}(\Omega)$. So, by the Rellich–Kondrachov theorem, $v_k \to v$ in $L^p(\Omega)$. So $\fint_\Omega v = 0$, $\|v\|_{L^p(\Omega)} = 1$, and

$$\int_\Omega v(x) \frac{\partial \phi}{\partial x_i}(x) \,\mathrm{d}x = \lim_{k \to \infty} \int_\Omega v_k(x) \frac{\partial \phi}{\partial x_i}(x) \,\mathrm{d}x = -\lim_{k \to \infty} \int_\Omega \frac{\partial v_k}{\partial x_i}(x) \phi(x) \,\mathrm{d}x = 0.$$

Hence, $\int_\Omega v(x) \mathrm{D}\phi(x) \,\mathrm{d}x = 0$, so $\mathrm{D}v = 0$, and so, since Ω is connected, v is constant with value 0, since $\fint_\Omega v = 0$. This contradicts $\|v\|_{L^p(\Omega)} = 1$. □

We now prove the Rellich–Kondrachov theorem, as used above:

Proof of theorem 2.39. By the Gagliardo–Nirenberg inequality(theorem 2.35), $W^{1,p}(\Omega) \subseteq L^q(\Omega)$ for $1 \leq q \leq p^*$, with

$$\|u\|_{L^{p^*}(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}.$$

We have to show that $\|u\|_{L^q(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}$:

$$\begin{aligned} \int_\Omega |u(x)|^q \,\mathrm{d}x &= \int_\Omega 1 |u(y)|^q \,\mathrm{d}y \\ &\leq C \left(\int_\Omega |u(x)|^{p^*} \,\mathrm{d}x \right)^{\frac{q}{p^*}}. \end{aligned}$$

Hence, $\|u\|_{L^q(\Omega)} \leq \|u\|_{L^{p^*}(\Omega)} \leq C' \|u\|_{W^{1,p}(\Omega)}$.

We now have to show that if $(u_m) \subset W^{1,p}(\Omega)$ is a bounded sequence, then (u_m) is precompact in $L^q(\Omega)$.

Step 1. Without loss of generality, consider $u_m \in W^{1,p}(\mathbb{R}^n)$ such that $\operatorname{supp} u_m$ is compact and $\operatorname{supp} u_m \subseteq V$ for all m and some bounded set V. Take $u_m^\varepsilon := \eta_\varepsilon \star u_m \in C^\infty(\mathbb{R}^n)$ as usual. $u_m^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u_m$ uniformly in m, and $u_m^\varepsilon \to u_m$ in $W^{1,p}(\mathbb{R}^n)$.

$$
\begin{aligned}
u_m^\varepsilon(x) - u_m(x) &= \int_{\mathbb{R}^n} \eta_\varepsilon(x-y) u_m(y)\,\mathrm{d}y - u(x) \\
&= \int_{\mathbb{R}^n} \eta_\varepsilon(y) u_m(x-y)\,\mathrm{d}y - u(x) \\
&= \int_{\mathbb{R}^n} \eta(y) u_m(x-\varepsilon y)\,\mathrm{d}y - u(x) \\
&= \int_{\mathbb{R}^n} \eta(y) \left(u_m(x-\varepsilon y) - u_m(x)\right)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} \eta(y) \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} u_m(x-\varepsilon t y)\,\mathrm{d}t\mathrm{d}y.
\end{aligned}
$$

Therefore,

$$
|u_m^\varepsilon(x) - u_m(x)| \leq \int_{\mathbb{R}^n} \eta(y) \int_0^1 |\mathrm{D}u_m(x-\varepsilon t y)||\varepsilon y|\,\mathrm{d}t\mathrm{d}y,
$$

and so

$$
\begin{aligned}
\int_{\mathbb{R}^n} |u_m^\varepsilon(x) - u_m(x)|\,\mathrm{d}x &\leq \varepsilon \int_{\mathbb{B}_1(0)} \eta(y) \int_{\mathbb{R}^n} |\mathrm{D}u_m(x)|\,dx \int_0^1 1\,\mathrm{d}t\mathrm{d}y \\
&\leq \varepsilon \int_{\mathbb{R}^n} |\mathrm{D}u_m(x)|\,\mathrm{d}x,
\end{aligned}
$$

since $\operatorname{supp}\eta \subseteq \mathbb{B}_1(0)$ and $|y| \leq 1$. Hence, $\int_V |u_m^\varepsilon(x) - u_m(x)|\,\mathrm{d}x \leq \varepsilon \int_V |\mathrm{D}u_m(x)|\,\mathrm{d}x$, so

$$
\|u_m^\varepsilon - u_m\|_{L^1(V)} \leq \varepsilon C \|\mathrm{D}u_m\|_{L^1(V)} \leq \varepsilon C' \|\mathrm{D}u_m\|_{L^p(V)} \leq \varepsilon C''.
$$

We need to estimate $\|u_m^\varepsilon - u_m\|_{L^q(V)}$:

$$
\|u_m^\varepsilon - u_m\|_{L^q(V)} \leq \|u_m^\varepsilon - u_m\|_{L^1(V)}^\theta \|u_m^\varepsilon - u_m\|_{L^q(V)}^{1-\theta} \leq C \|u_m^\varepsilon - u_m\|_{L^1(V)}.
$$

This is a consequence of the following corollary of Hölder's inequality: for $0 < \theta < 1$, $\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r}$,

$$
\|u\|_{L^q} \leq \|u\|_{L^p}^\theta \|u\|_{L^r}^{1-\theta}.
$$

Step 2. u_m^ε is uniformly bounded for each $\varepsilon > 0$:

$$
|u_m^\varepsilon(x)| = \left| \int_{\mathbb{R}^n} \eta_\varepsilon(x-y) u_m(y)\,\mathrm{d}y \right| \leq \frac{C}{\varepsilon^n} \|u_m\|_{L^1(V)} \leq \frac{C'}{\varepsilon^n}.
$$

u_m^ε is uniformly equicontinuous for each $\varepsilon > 0$:

$$
|\mathrm{D}u_m^\varepsilon(x)| = \left| \int_{\mathbb{R}^n} \mathrm{D}\eta_\varepsilon(x-y) u_m(y)\,\mathrm{d}y \right| \leq \frac{C}{\varepsilon^{n+1}} \|u_m\|_{L^1(V)} \leq \frac{C'}{\varepsilon^{n+1}}.
$$

Thus, $\|u_m^\varepsilon - u_m\|_{L^q(V)} \le C\varepsilon^\theta$ and (u_m^ε) is uniformly bounded and equicontinuous for each ε.

Step 3. We want to prove that for each $\delta > 0$, there is a subsequence $(u_{m_j}) \subseteq (u_m)$ such that $\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} \le \delta$. Fix $\delta > 0$. Choose $\varepsilon > 0$ such that $\|u_m^\varepsilon - u_m\|_{L^q(V)} \le \delta/2$. By the Arzela-Ascoli theorem, there exists a subsequence $(u_{m_j}^\varepsilon) \subseteq (u_m^\varepsilon)$ such that $(u_{m_j}^\varepsilon)$ converges uniformly on compact subsets of $\mathbb{R}^n$. So

$$\sup_{x\in\overline{V}} \left| u_{m_j}^\varepsilon(x) - u_{m_k}^\varepsilon(x) \right| \underset{j,k\to\infty}{\longrightarrow} 0.$$

Thus, $\limsup_{j,k\to\infty} \|u_{m_j}^\varepsilon - u_{m_k}^\varepsilon\|_{L^q(V)} = 0$, and

$$\begin{aligned}
&\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} \\
&\quad\le \limsup_{j,k\to\infty} \Big(\|u_{m_j} - u_{m_j}^\varepsilon\|_{L^q(V)} + \|u_{m_j}^\varepsilon - u_{m_k}^\varepsilon\|_{L^q(V)} + \|u_{m_k}^\varepsilon - u_{m_k}\|_{L^q(V)} \Big) \\
&\quad\le \limsup_{j\to\infty} \|u_{m_j} - u_{m_j}^\varepsilon\|_{L^q(V)} + \limsup_{j,k\to\infty} \|u_{m_j}^\varepsilon - u_{m_k}^\varepsilon\|_{L^q(V)} + \limsup_{k\to\infty} \|u_{m_k}^\varepsilon - u_{m_k}\|_{L^q(V)} \\
&\quad\le \delta.
\end{aligned}$$

Take subsequences for $\delta = 1, \frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2^\ell}, \dots$ and take a diagonal subsequence: thus, there exists (u_{m_k}) such that $\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} = 0$. □

Thus, $W^{1,p}(\Omega) \subset\subset L^p(\Omega)$ for $1 \le p \le \infty$ — more on this later. So, if we have a weakly convergent sequence $u_m \rightharpoonup u$ in $W^{1,p}(\Omega)$, we can find a strongly convergent subsequence $u_{m_j} \to u$ in $L^p(\Omega)$.

2.9 Hölder Spaces

Definitions 2.40. Define the norm $\|\cdot\|_{C(\bar\Omega)}$ by

$$\|u\|_{C(\bar\Omega)} := \sup_{x\in\Omega} |u(x)|,$$

the seminorm $[\cdot]_{0,\gamma}$ by

$$[u]_{0,\gamma} := \sup_{x,y\in\Omega, x\neq y} \frac{|u(x) - u(y)|}{|x-y|^\gamma},$$

and the norm $\|\cdot\|_{0,\gamma}$ by

$$\|u\|_{0,\gamma} := \|u\|_{C(\bar\Omega)} + [u]_{0,\gamma}.$$

Exercise 2.41. Show that $\|\cdot\|_{0,\gamma}$ is indeed a norm.

Definition 2.42. The *Hölder space* $C^{k,\gamma}(\bar\Omega)$ is space of functions $u \in C^k(\bar\Omega)$ for which the norm

$$\|u\|_{C^{k,\gamma}(\bar\Omega)} = \sum_{|\alpha|\le k} \|\mathrm{D}^\alpha u\|_{C(\bar\Omega)} + \sum_{|\alpha|=k} [\mathrm{D}^\alpha u]_{0,\gamma}$$

is finite.

Exercise 2.43. Show that $C^{k,\gamma}(\bar{\Omega})$ is a Banach space but not a Hilbert space.

Theorem 2.44 (Morrey's inequality for $\mathbb{R}^n$)**.** *Consider $\mathbb{R}^n$, $p > n$, $\gamma := 1-\frac{n}{p}$. Then there is a constant C depending only on n and p such that, for all $u \in W^{1,p}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$,*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

Proof. Step 1. Consider $\mathbb{B}_r(x)$. We want to show that

$$\fint_{\mathbb{B}_r(x)} |u(x) - u(y)| \,\mathrm{d}y \leq C \int_{\mathbb{B}_r(x)} \frac{|\mathrm{D}u(y)|}{|x-y|^{n-1}} \,\mathrm{d}y.$$

Take $\omega \in \mathbb{S}^{n-1} = \partial\mathbb{B}_1(0)$, $0 < s < r$.

$$\begin{aligned} |u(x+s\omega) - u(x)| &= \left|\int_0^s \frac{\mathrm{d}}{\mathrm{d}t} u(x+t\omega)\,\mathrm{d}t\right| \\ &= \left|\int_0^s \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{D}u(x+t\omega)\cdot\omega\,\mathrm{d}t\right| \\ &\leq \int_0^s |\mathrm{D}u(x+t\omega)\frac{t^{n-1}}{t^{n-1}}\,\mathrm{d}t. \end{aligned}$$

$$\begin{aligned} \int_{\partial\mathbb{B}_1(0)} |u(x+s\omega) - u(x)| \,\mathrm{d}S(\omega) &\leq \int_0^s \int_{\partial\mathbb{B}_1(0)} |\mathrm{D}u(x+t\omega)|\frac{t^{n-1}}{t^{n-1}}\,\mathrm{d}S(\omega)\mathrm{d}t \\ &\leq \int_{\mathbb{B}_s(x)} \frac{|\mathrm{D}u(y)|}{|x-y|^{n-1}}\,\mathrm{d}y. \end{aligned}$$

$$\begin{aligned} s^{n-1}\int_{\partial\mathbb{B}_1(0)} |u(x+s\omega) - u(x)| \,\mathrm{d}S(\omega) &\leq s^{n-1}\int_{\mathbb{B}_s(x)} \frac{|\mathrm{D}u(y)|}{|x-y|^{n-1}}\,\mathrm{d}y \\ &\leq s^{n-1}\int_{\mathbb{B}_r(x)} \frac{|\mathrm{D}u(y)|}{|x-y|^{n-1}}\,\mathrm{d}y. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{|\mathbb{B}_r(x)|}\int_{\mathbb{B}_r(x)} |u(y) - u(x)| \,\mathrm{d}y &\leq \int_{\mathbb{B}_r(x)} \frac{|\mathrm{D}u(y)|}{|x-y|^{n-1}}\,\mathrm{d}y \int_0^r s^{n-1}\,\mathrm{d}s \\ &= \frac{1}{|\mathbb{B}_r(x)|}\frac{r^n}{n}\int_{\mathbb{B}_r(x)} \frac{|\mathrm{D}u(y)|}{|x-y|^{n-1}}\,\mathrm{d}y. \end{aligned}$$

That is,

$$\fint_{\mathbb{B}_r(x)} |u(y) - u(x)| \,\mathrm{d}y \leq C \int_{\mathbb{B}_r(x)} \frac{|\mathrm{D}u(y)|}{|x-y|^{n-1}} \,\mathrm{d}y.$$

Step 2. We now estimate $|u(x)|$ for all x: $|u(x)| \leq |u(x) - u(y)| + |u(y)|$.

$$|u(x)| = \fint_{\mathbb{B}_1(x)} |u(x)|\,\mathrm{d}y \leq \fint_{\mathbb{B}_1(x)} |u(x)-u(y)|\,\mathrm{d}y + \fint_{\mathbb{B}_1(x)} |u(y)|\,\mathrm{d}y.$$

Also,

$$\fint_{\mathbb{B}_1(x)} |u(y)|\,\mathrm{d}y \le C\|u\|_{L^p(\mathbb{B}_1(x))} \le C\|u\|_{L^p(\mathbb{R}^n)}.$$

$$\begin{aligned}\fint_{\mathbb{B}_1(x)} |u(x)-u(y)|\,\mathrm{d}y &\le C\int_{\mathbb{B}_1(x)} \frac{|\mathrm{D}u(y)|}{|x-y|^{n-1}}\,\mathrm{d}y\\ &\le C\left(\int_{\mathbb{B}_1(x)} |\mathrm{D}u(y)|^p\,\mathrm{d}y\right)^{\frac{1}{p}}\left(\int_{\mathbb{B}_1(x)} \frac{\mathrm{d}y}{|x-y|^{(n-1)\frac{p-1}{p}}}\right)^{\frac{p-1}{p}}\end{aligned}$$

Since $p > n$,

$$\left(\int_{\mathbb{B}_1(x)} \frac{\mathrm{d}y}{|x-y|^{(n-1)\frac{p-1}{p}}}\right)^{\frac{p-1}{p}} \le C\|\mathrm{D}u\|_{L^p(\mathbb{R}^n)},$$

and so

$$\sup_{x\in\mathbb{R}^n} |u(x)| \le C_1\|u\|_{L^p(\mathbb{R}^n)} + C_2\|\mathrm{D}u\|_{L^p(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

Step 3. We now estimate $[u]_{0,\gamma}$. Set $V := \mathbb{B}_r(x)\cap\mathbb{B}_r(y)$, where $r = |x-y|$.

$$\begin{aligned}|u(x)-u(y)| &= \fint_V |u(x)-u(y)|\,\mathrm{d}z\\ &\le \fint_V |u(x)-u(z)|\,\mathrm{d}z + \fint_V |u(y)-u(z)|\,\mathrm{d}z\end{aligned}$$

$$\begin{aligned}\frac{1}{|V|}\int_V |u(x)-u(z)|\,\mathrm{d}z &\le \frac{|\mathbb{B}_r(x)|}{|V|}\fint_{\mathbb{B}_r(x)} |u(x)-u(z)|\,\mathrm{d}z\\ &\le 2\fint_{\mathbb{B}_r(x)} |u(x)-u(z)|\,\mathrm{d}z\end{aligned}$$

$$|u(x)-u(y)| \le C_1\fint_{\mathbb{B}_r(x)} |u(x)-u(z)|\ \ dz + C_2\fint_{\mathbb{B}_r(y)} |u(y)-u(z)|\,\mathrm{d}z$$

Estimating each of these terms:

$$\begin{aligned}\fint_{\mathbb{B}_r(x)} |u(x)-u(z)|\,\mathrm{d}z &\le C\int_{\mathbb{B}_r(x)} \frac{|\mathrm{D}u(z)|}{|x-z|^{n-1}}\,\mathrm{d}z\\ &\le C\|\mathrm{D}u\|_{L^p(\mathbb{R}^n)}\left(\int_{\mathbb{B}_r(x)} \frac{\mathrm{d}z}{|x-z|^{(n-1)\frac{p-1}{p}}}\right)^{\frac{p-1}{p}},\end{aligned}$$

and the last factor on the RHS is

$$\le C\left(\int_0^r x^{n-1(n-1)\frac{p}{p-1}}\,\mathrm{d}x\right)^{\frac{p-1}{p}} \le C\left(r^{n-(n-1)\frac{p}{p-1}}\right)^{\frac{p-1}{p}} = Cr^{\gamma}.$$

Thus,

$$\fint_{\mathbb{B}_r(x)} |u(x)-u(z)|\,\mathrm{d}z \le Cr^{\gamma}\|\mathrm{D}u\|_{L^p(\mathbb{R}^n)},$$

so $|u(x)-u(y)| \leq Cr^\gamma \|\mathrm{D}u\|_{L^p(\mathbb{R}^n)}$. Hence,

$$\sup_{x,y\in\mathbb{R}^n, x\neq y} \frac{|u(x)-u(y)|}{|x-y|} \leq C\|\mathrm{D}u\|_{L^p(\mathbb{R}^n)},$$

and so $\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$. □

Theorem 2.45 (Morrey's inequality for $\Omega \subset \mathbb{R}^n$)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, with $\partial\Omega$ C^1. Let $p > n$ and $\gamma := 1 - \frac{n}{p}$. Then there is a constant C such that*

$$\|u\|_{C^{0,\gamma}(\bar{\Omega})} \leq C\|u\|_{W^{1,p}(\Omega)}.$$

Proof. Apply Morrey's inequality for $\mathbb{R}^n$ and the extension theorem:

$$\|u\|_{C^{0,\gamma}(\bar{\Omega})} \leq \|Eu\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C'\|u\|_{W^{1,p}(\Omega)}.$$ □

Theorem 2.46. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, with $\partial\Omega$ C^1, and let $kp > n$. Then $W^{k,p}(\Omega) \subseteq C^{\ell,\gamma}(\bar{\Omega})$, where $\ell := k - \lfloor \frac{n}{p} \rfloor - 1$ and*

$$\gamma = \begin{cases} \lfloor \frac{n}{p} \rfloor + 1 - \frac{n}{p}, & \frac{n}{p} \notin \mathbb{Z}; \\ \text{any number } \in [0,1), & \frac{n}{p} \in \mathbb{Z}. \end{cases}$$

Also, there is a constant C, depending only on Ω, k and p, so that, for all $u \in W^{k,p}(\Omega)$,

$$\|u\|_{C^{\ell,\gamma}(\bar{\Omega})} \leq C\|u\|_{W^{k,p}(\Omega)}.$$

Thus, we now have information on $W^{k,p}(\Omega)$ for $kp < n$ and $kp > n$. For instance:

- $1 \leq p < n \Rightarrow W^{1,p}(\Omega) \subseteq L^{p^*}(\Omega)$, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$;
- $n < p < \infty \Rightarrow W^{1,p}(\Omega) \subseteq C^{0,\gamma}(\bar{\Omega})$, $\gamma = 1 - \frac{n}{p}$.

What about the boundary case $kp = n$?

Theorem 2.47. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, with $\partial\Omega$ C^1. If $u \in W_0^{1,p}(\Omega)$, $p = n$, then*

$$\int_\Omega \exp\left[\left(\frac{|u(x)|}{C_1\|\mathrm{D}u\|_{L^p(\Omega)}}\right)^{\frac{p}{p-1}}\right] \mathrm{d}x \leq C_2|\Omega|.$$

Proof. The proof relies on inequalities with Riesz potentials. See [GT], Chapter 7. □

Definition 2.48. Given $\varphi : [0,\infty) \to \mathbb{R}$ such that $\varphi(0) = 0$ and φ is increasing and convex, the *Orlicz*[9] *space* $L^\varphi(\Omega)$ is the space of all integrable $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L^\varphi(\Omega)} := \inf\left\{\lambda > 0 \,\middle|\, \int_\Omega \varphi\left(\frac{|u(x)|}{\lambda}\right) \mathrm{d}x \leq 1\right\} < \infty.$$

[9]Władysław Orlicz (1903–1990).

For example, if $\varphi(t) = |t|^p$, then $\|u\|_{L^\varphi(\Omega)} = \|u\|_{L^p(\Omega)}$. In fact, the Orlicz space $L^\varphi(\Omega)$ is a Banach space. With this notation, theorem 2.47 is the statement that $W_0^{1,p}(\Omega) \subseteq L^\varphi(\Omega)$ for

$$\varphi(t) = \exp\left(|t|^{\frac{p}{p-1}}\right) - 1.$$

Theorem 2.49. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, with $\partial\Omega$ C^1. If $u \in W_0^{k,p}(\Omega)$, $kp = n$, then*

$$\int_\Omega \exp\left[\left(\frac{|u(x)|}{C_1 \|\mathrm{D}^k u\|_{L^p(\Omega)}}\right)^{\frac{p}{p-1}}\right] \mathrm{d}x \leq C_2 |\Omega|.$$

The space of theorem 2.49 is known as the *Orlicz-Sobolev space.*

In the discussion so far, we have not treated the case $p = \infty$.

Theorem 2.50. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, with $\partial\Omega$ C^1. Then $u \in W^{1,\infty}(\Omega)$ if, and only if, u is Lipschitz (i.e., is equivalent to a Lipschitz function).*

Proof. See [Ev], Chapter 5. □

2.10 Logarithmic Sobolev Inequalities

The following theorem is used in quantum mechanics. Its main feature and utility is its independence of dimension, n.

Theorem 2.51. *Define the measure $\mathrm{d}\mu := e^{-\pi|x|^2}\,\mathrm{d}x$. Then, for all $u \in C_\mathrm{c}^\infty(\Omega)$,*

$$\frac{1}{\pi}\int_{\mathbb{R}^n} |\mathrm{D}u(x)|^2 \,\mathrm{d}\mu \geq \int_{\mathbb{R}^n} |u(x)|^2 \log\left(\frac{|u(x)|^2}{\|u\|^2_{L^2(\mathbb{R}^n)}}\right) \mathrm{d}\mu.$$

2.11 $H^{-1}(\Omega)$ and Duality

Definitions 2.52. Given $\Omega \subseteq \mathbb{R}^n$, $H^{-k}(\Omega)$ is defined to be the dual space of $H_0^k(\Omega) := W_0^{k,2}(\Omega)$, i.e. the space of bounded linear functionals on $H_0^k(\Omega)$:

$$H^{-k}(\Omega) := \left\{ f : H_0^k \to \mathbb{R} \,\middle|\, f \text{ is linear and } \sup \frac{|\langle f, u\rangle|}{\|u\|_{H_0^k(\Omega)}} < \infty \right\}.$$

The norm on $H^{-k}(\Omega)$ is

$$\|f\|_{H^{-k}(\Omega)} := \sup\left\{ |\langle f, u\rangle| \,\middle|\, u \in H_0^k(\Omega), \|u\|_{H_0^k(\Omega)} \leq 1 \right\}.$$

In what follows, we confine our attention to the space $H^{-1}(\Omega)$.

Theorem 2.53. *Let $\Omega \subseteq \mathbb{R}^n$ and $f \in H^{-1}(\Omega)$. Then*

(i) there exist functions $f^0, f^1, \ldots, f^n \in L^2(\Omega)$ such that, for all $v \in H_0^1(\Omega)$,

$$\langle f, v\rangle = \int_\Omega f^0(x)v(x) + \sum_{i=1}^n f^i(x)\frac{\partial v}{\partial x_i}(x)\,\mathrm{d}x; \tag{2.2}$$

(ii) the norm of f is also given by

$$\|f\|_{H^{-1}(\Omega)} = \inf \left\{ \left(\int_\Omega \sum_{i=0}^n |f^i(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \middle| \, f \textit{ satisfies } (2.2) \textit{ and, for all } i, \; f^i \in L^2(\Omega) \right\}.$$

Proof. (i) Recall that $H_0^1(\Omega)$ is a Hilbert space with inner product

$$(u, v)_{H_0^1(\Omega)} := \int_\Omega u(x)v(x) + \mathrm{D}u(x) \cdot \mathrm{D}v(x) \, \mathrm{d}x.$$

Recall:

Theorem 2.54. (Riesz representation theorem.) *Let $\mathcal{H}$ be a real Hilbert space with dual $\mathcal{H}^*$. Then $\mathcal{H}^*$ can be canonically identified with $\mathcal{H}$: for all $f \in \mathcal{H}^*$, there is a unique $f^\sharp \in \mathcal{H}$ such that $(f^\sharp, v)_\mathcal{H} = \langle f, v \rangle$ for all $v \in \mathcal{H}$, and the map $f \mapsto f^\sharp$ is a linear isomorphism of $\mathcal{H}^*$ onto $\mathcal{H}$.*

So, given $f \in H^{-1}(\Omega)$, choose the unique $u := f^\sharp \in H_0^1(\Omega)$ such that, for all $v \in H_0^1(\Omega)$,

$$\langle f, v \rangle = (u, v)_{H_0^1(\Omega)} = \int_\Omega u(x)v(x) + \mathrm{D}u(x) \cdot \mathrm{D}v(x) \, \mathrm{d}x.$$

We want

$$\langle f, v \rangle = \int_\Omega f^0(x)v(x) + \sum_{i=1}^n f^i(x) \frac{\partial v}{\partial x_i}(x) \, \mathrm{d}x,$$

so simply take $f^0 := u$ and $f^i := \frac{\partial u}{\partial x_i} \in L^2(\Omega)$. Clearly, f^0, f^i satisfy (2.2).

(ii) Suppose that we also write the action of $f \in H^{-1}(\Omega)$ as

$$\langle f, v \rangle = \int_\Omega g^0(x)v(x) + \sum_{i=1}^n g^i(x) \frac{\partial v}{\partial x_i}(x) \, \mathrm{d}x.$$

Take $v = u = f^\sharp$ as in part (i).

$$\begin{aligned} \int_\Omega |\mathrm{D}u(x)|^2 + |u(x)|^2 \, \mathrm{d}x &= \int_\Omega g^0(x)u(x) \sum_{i=1}^n g^i(x) \frac{\partial u}{\partial x_i}(x) \, \mathrm{d}x \\ &\leq \left(\int_\Omega \sum_{i=0}^n |g^i(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_\Omega |\mathrm{D}u(x)|^2 + |u(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}. \end{aligned}$$

Thus,

$$\left(\int_\Omega \sum_{i=0}^n |f^i(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} = \left(\int_\Omega \sum_{i=0}^n |\mathrm{D}u(x)|^2 + |u(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \leq \left(\int_\Omega \sum_{i=0}^n |g^i(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}.$$

So, for all $v \in H_0^1(\Omega)$ with $\|v\|_{H_0^1(\Omega)} \leq 1$,

$$|\langle f, v \rangle| \leq \left(\int_\Omega \sum_{i=0}^n |f^i(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}.$$

$$\begin{aligned}\|f\|_{H_0^1(\Omega)} &= \sup\left\{ |\langle f, v\rangle| \,\middle|\, \|v\|_{H_0^1(\Omega)} \leq 1 \right\} \\ &\leq \left(\int_\Omega \sum_{i=0}^n |f^i(x)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \\ &\leq \left(\int_\Omega \sum_{i=0}^n |g^i(x)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}.\end{aligned}$$

Thus,

$$\|f\|_{H^{-1}(\Omega)} \leq \inf_{\text{representations}} \left(\int_\Omega \sum_{i=0}^n |g^i(x)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}.$$

Take $v = u/\|u\|_{H_0^1(\Omega)}$. Then

$$\left\langle f, \frac{u}{\|u\|_{H_0^1(\Omega)}} \right\rangle = \|u\|_{H_0^1(\Omega)} = \left(\int_\Omega \sum_{i=0}^n |f^i(x)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}},$$

and $\|f\|_{H^{-1}(\Omega)}$ attains the infimum above. □

3 Elliptic PDEs

3.1 Weak Solutions and Fredholm Theory

We seek to solve

$$\begin{cases} Lu(x) = f(x), & x \in \Omega; \\ u(x) = 0, & x \in \partial\Omega; \end{cases} \tag{3.1}$$

where L is of the form

$$Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u,$$

known as *divergence form*, or

$$Lu = -\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j^2} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u.$$

Definition 3.1. A differential operator L of the above type is *elliptic* (or *uniformly elliptic*) if, for some $\alpha > 0$ and all $x \in \Omega$, $\xi \in \mathbb{R}^n$,

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2. \tag{3.2}$$

That is, the matrix (a_{ij}) is positive definite.

Example 3.2. If $a_{ij} = \delta_{ij}$, $b_i = c = f = 0$, then $L = -\Delta$ and (3.1) becomes Laplace's equation.

There is a bilinear form $B : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ associated to L, which is defined by

$$B(u, v) := (Lu, v)_{L^2(\Omega)}.$$

By integrating by parts, one sees that

$$B(u, v) = \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_j}(x) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i}(x)v(x) + c(x)u(x)v(x)\right) \mathrm{d}x.$$

Definition 3.3. u is a *weak solution* of (3.1) for $f \in H^{-1}(\Omega)$ if $B(u, v) = \langle f, v\rangle$ for all $v \in H_0^1(\Omega)$.

Theorem 3.4 (Lax–Milgram[10])**.** *Let $\mathcal{H}$ be a Hilbert space and let $B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bilinear form with constants $\alpha, \beta > 0$ such that for all $u, v \in \mathcal{H}$,*

$$|B(u, v)| \leq \alpha\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}}$$

and

$$B(u, u) \geq \beta\|u\|_{\mathcal{H}}^2.$$

Then, for all $f \in \mathcal{H}^*$, there exists a unique $u \in \mathcal{H}$ such that $B(u, v) = \langle f, v\rangle$ for all $v \in \mathcal{H}$.

[10] Peter Lax (1926–) and Arthur Milgram

Proof. If B is symmetric then it is an inner product on $\mathcal{H}$, so, by the Riesz representation theorem (Theorem 2.54), given $f \in \mathcal{H}^*$, there is a unique $u \in \mathcal{H}$ such that $B(u,v) = \langle f, v\rangle$ for all $v \in \mathcal{H}$.

Otherwise, for each $u \in \mathcal{H}$, $v \mapsto B(u,v)$ is a bounded linear functional on $\mathcal{H}$. So, by the Riesz representation theorem, given $u \in \mathcal{H}$, there is a unique $w \in \mathcal{H}$ such that $(w,v)_{\mathcal{H}} = B(u,v)$. Define $Au := w$. The map $A : \mathcal{H} \to \mathcal{H}$ is clearly well-defined. It is also linear: take $\alpha_1, \alpha_2 \in \mathbb{R}$ and $u_1, u_2 \in \mathcal{H}$:

$$\begin{aligned}(A(\alpha_1 u_1 + \alpha_2 u_2), v)_{\mathcal{H}} &= B(\alpha_1 u_1 + \alpha_2 u_2, v)\\ &= \alpha_1 B(u_1, v) + \alpha_2 B(u_2, v)\\ &= \alpha_1 (Au_1, v)_{\mathcal{H}} + \alpha_2 (Au_2, v)_{\mathcal{H}}\\ &= (\alpha_1 Au_1 + \alpha_2 Au_2, v)_{\mathcal{H}}.\end{aligned}$$

A is a bounded map:

$$\|Au\|_{\mathcal{H}}^2 = (Au, Au)_{\mathcal{H}} = B(u, Au) \le \alpha \|u\|_{\mathcal{H}} \|Au\|_{\mathcal{H}},$$

so $\|Au\|_{\mathcal{H}} \le \alpha \|u\|_{\mathcal{H}}$.

A is injective since

$$\|Au\|_{\mathcal{H}} \|u\|_{\mathcal{H}} \ge (Au, u)_{\mathcal{H}} = B(u,u) \ge \beta \|u\|_{\mathcal{H}}^2,$$

so $Au = 0 \Rightarrow u = 0$.

The image $\operatorname{im}(A)$ is closed: take a convergent sequence $(v_n) \subset \operatorname{im}(A)$, $v_n \to v$. Choose $u_n \in \mathcal{H}$ such that $Au_n = v_n$ for each n. (Au_n) is Cauchy, so

$$\begin{aligned}\|Au_n - Au_m\|_{\mathcal{H}} \|u_n - u_m\|_{\mathcal{H}} &\ge (Au_n - Au_m, u_n - u_m)_{\mathcal{H}}\\ &= B(u_n - u_m, u_n - u_m)\\ &\ge \beta \|u_n - u_m\|_{\mathcal{H}}^2.\end{aligned}$$

So $\beta\|u_n - u_m\|_{\mathcal{H}} \le \|v_n - v_m\|_{\mathcal{H}} \to 0$. So (u_n) is Cauchy and converges to some $u \in \mathcal{H}$. So $v_n = Au_n \to Au = v$ by the continuity (boundedness) of A, so $v \in \operatorname{im}(A)$, and so $\operatorname{im}(A)$ is closed.

Finally, A is surjective: for, if not, there is an $s \in \mathcal{H}$, $s \neq 0$, such that $s \perp \operatorname{im}(A)$. But

$$\beta \|s\|_{\mathcal{H}}^2 \le B(s,s) = (s, As)_{\mathcal{H}} = 0,$$

so $s = 0$, a contradiction.

So, take $f \in \mathcal{H}^*$. There is a unique $w \in \mathcal{H}$ such that $(w,v)_{\mathcal{H}} = \langle f, v\rangle$ for all $v \in \mathcal{H}$. $Au := w$ has a unique solution since A is invertible. So $(Au, v)_{\mathcal{H}} = \langle f, v\rangle$ for all $v \in \mathcal{H}$. But $(Au, v)_{\mathcal{H}} = B(u,v)$. So there is a unique $u \in \mathcal{H}$ such that $B(u,v) = \langle f, v\rangle$. □

Theorem 3.5 (Gårding's inequality[11])**.** *Let B be the bilinear form associated to an elliptic differential operator L with $a_{ij}, b_i, c \in L^\infty(\Omega)$. Then there are constants $\alpha, \beta, \gamma > 0$ such that*

$$|B(u,v)| \le \alpha \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)},$$
$$\gamma \|u\|_{L^2(\Omega)} + B(u,u) \ge \beta \|u\|_{H_0^1(\Omega)}^2$$

for all $u, v \in H_0^1(\Omega)$.

[11] Lars Gårding (1919–).

Proof.

$$\begin{aligned}|B(u,v)| &= \left|\int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_j}(x) + \sum_{i=1}^n b_i(x)\frac{\partial u}{\partial x_i}(x)v(x) + c(x)u(x)v(x)\right)\,\mathrm{d}x\right| \\ &\leq C_1\|\mathrm{D}u\|_{L^2(\Omega)}\|\mathrm{D}v\|_{L^2(\Omega)} + C_2\|\mathrm{D}u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + C_3\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\ &\leq C\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}.\end{aligned}$$

$$\begin{aligned}B(u,u) - \int_\Omega \sum_{i=1}^n b_i(x)\frac{\partial u}{\partial x_i}(x)u(x)\,\mathrm{d}x - \int_\Omega c(x)u(x)^2\,\mathrm{d}x &= \int_\Omega \sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_i}(x)\frac{\partial u}{\partial x_j}(x)\,\mathrm{d}x \\ &\geq \alpha\int_\Omega |\mathrm{D}u(x)|^2\,\mathrm{d}x \\ &\geq \beta\|u\|^2_{H_0^1(\Omega)}\end{aligned}$$

Also,

$$\int_\Omega \sum_{i=1}^n b_i(x)\frac{\partial u}{\partial x_i}(x)u(x)\,\mathrm{d}x \leq C_1\|\mathrm{D}u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}$$

and

$$\int_\Omega c(x)u(x)^2\,\mathrm{d}x \leq C_2\|u\|_{L^2(\Omega)}.$$

Therefore,

$$B(u,u) + C_1\|\mathrm{D}u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + C_2\|u\|^2_{L^2(\Omega)} \geq \alpha\|\mathrm{D}u\|^2_{L^2(\Omega)}.$$

But $\|\mathrm{D}u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \leq \varepsilon\|\mathrm{D}u\|^2_{L^2(\Omega)} + \frac{1}{4\varepsilon}\|u\|^2_{L^2(\Omega)}$ for all $\varepsilon > 0$. Therefore,

$$B(u,u) + \varepsilon\|\mathrm{D}u\|^2_{L^2(\Omega)} + \frac{1}{4\varepsilon}\|u\|^2_{L^2(\Omega)} + C\|u\|^2_{L^2(\Omega)} \geq \alpha\|\mathrm{D}u\|^2_{L^2(\Omega)},$$

and so, since the norm $\|\mathrm{D}u\|_{L^2(\Omega)}$ is equivalent to the norm $\|u\|_{H_0^1(\Omega)}$, for small enough ε,

$$\gamma\|u\|_{L^2(\Omega)} + B(u,u) \geq \beta\|u\|^2_{H_0^1(\Omega)}.$$ □

Suppose that we have L in divergence form,

$$Lu = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^n b_i(x)\frac{\partial u}{\partial x_i} + c(x)u,$$

with the associated bilinear form $B(u,v) = (Lu,v)_{L^2(\Omega)}$, and

$$B_\mu(u,u) = B(u,u) + \mu\|u\|^2_{L^2(\Omega)} \geq B(u,u) + \gamma\|u\|^2_{L^2(\Omega)}.$$

Then weak a solution to

$$\begin{cases} Lu + \mu u = f, & \text{on } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases}$$

is a u such that $B_\mu(u,v) = \langle f, v\rangle$.

Theorem 3.6. *There exists $\gamma \geq 0$ such that if $\mu \geq \gamma$ and $f \in L^2(\Omega)$, then there is a unique weak solution $u \in H_0^1(\Omega)$ to*

$$\begin{cases} Lu + \mu u = f, & \text{on } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Proof. Define $B_\mu(u,v) := (Lu + \mu u, v)_{L^2(\Omega)}$. Obviously,

$$|B_\mu(u,v)| \leq C\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)},$$

and

$$B_\mu(u,u) \geq C\|u\|^2_{H_0^1(\Omega)},$$

since $B_\mu(u,u) = B(u,u) + \mu\|u\|^2_{L^2(\Omega)} \geq B(u,u) + \gamma\|u\|^2_{L^2(\Omega)}$. By the Lax-Milgram Theorem, there is a unique $u \in H_0^1(\Omega)$ such that $B_\mu(u,v) = \langle f, v\rangle$. Since $f \in L^2(\Omega)$, $B_\mu(u,v) = (f,v)_{L^2(\Omega)}$. □

Definition 3.7. The *formal adjoint* of L is L^* defined by

$$L^*v = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial v}{\partial x_i}\right) - \sum_{i=1}^{n} b_i \frac{\partial v}{\partial x_i} + \left(c - \sum_{i=1}^{n}\frac{\partial b_i}{\partial x_i}\right)v.$$

Compare this with the usual definition of adjoint as $(Lu,v) = (u, L^*v)$:

$$\begin{aligned}
(Lu,v) &= \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + \sum_{i=1}^{n} b_i\frac{\partial u}{\partial x_i}v + cuv\right) \\
&= \int_\Omega \left(-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial v}{\partial x_i}\right)u - \sum_{i=1}^{n} u\frac{\partial}{\partial x_i}(b_i v) + cuv\right) \\
&= \int_\Omega \left(-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial v}{\partial x_i}\right)u - \sum_{i=1}^{n} b_i\frac{\partial v}{\partial x_i}u + \left(c - \sum_{i=1}^{n}\frac{\partial b_i}{\partial x_i}\right)vu\right) \\
&= (u, L^*v).
\end{aligned}$$

Definition 3.8. Define the bilinear form B^* by

$$B^*(v,u) := (u, L^*v)_{L^2(\Omega)} = B(u,v) \tag{3.3}$$

for all $u, v \in H_0^1(\Omega)$.

Definition 3.9. $v \in H_0^1(\Omega)$ is a *weak solution* to

$$\begin{cases} L^*v = f, & \text{on } \Omega; \\ v = 0, & \text{on } \partial\Omega; \end{cases}$$

if $B^*(v,u) = \langle f, u\rangle$ for all $u \in H_0^1(\Omega)$.

Theorem 3.10. (The Fredholm Alternative for Elliptic Operators.) *Let L be an elliptic partial differential operator.*

1. *Precisely one of the following is true: either*

   (a) *for all $f \in L^2(\Omega)$ there is a unique weak solution u of*

$$\begin{cases} Lu = f, & \text{on } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{3.4}$$

   or

   (b) *there is a weak solution $u \neq 0$ of*

$$\begin{cases} Lu = 0, & \text{on } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

2. *In the second case, the dimension of the subspace $N \subseteq H_0^1(\Omega)$ of weak solutions of* (3.5) *is finite, and equals the dimension of the subspace $N^* \subseteq H_0^1(\Omega)$ of weak solutions of*

$$\begin{cases} L^*v = 0, & \text{on } \Omega; \\ v = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

3. (3.4) *has a weak solution if, and only if, $(f, v) = 0$ for all $v \in N^*$.*

Proof. We use the following theorem of functional analysis:

Theorem 3.11 (Fredholm[12] alternative.)**.** *Let $\mathcal{H}$ be a Hilbert space and let $K : \mathcal{H} \to \mathcal{H}$ be a compact linear operator with adjoint $K^* : \mathcal{H}^* \to \mathcal{H}^*$. Then*

1. $\dim\ker(I - K) < \infty$*;*
2. $\operatorname{im}(I - K)$ *is closed;*
3. $\operatorname{im}(I - K) = \ker(I^* - K^*)^\perp$*;*
4. $\ker(I - K) = \{0\} \Leftrightarrow \operatorname{im}(I - K) = \mathcal{H}$*;*
5. $\dim\ker(I - K) = \dim\ker(I^* - K^*)$*.*

Choose $\mu = \gamma$ as in theorem 3.6 and define the bilinear form $B_\gamma(u, v) := B(u, v) + \gamma(u, v)$ corresponding to the operator $L_\gamma u := Lu + \gamma u$. Then

$$B_\gamma(u, v) \leq C\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)},$$

$$B_\gamma(u, u) \geq \alpha\|u\|^2_{H_0^1(\Omega)}.$$

Take any $g \in L^2(\Omega)$. Then the Lax–Milgram theorem implies that there is a unique $u \in H_0^1(\Omega)$ such that $B_\gamma(u, v) = (g, v)_{L^2(\Omega)}$ for all $v \in H_0^1(\Omega)$. Thus, define an operator $L_\gamma^{-1} : L^2(\Omega) \to L^2(\Omega)$ that assigns to each $g \in L^2(\Omega)$ the corresponding $u \in H_0^1(\Omega) \subset L^2(\Omega)$.

[12]Erik Ivar Fredholm (1866–1927).

It is easy to see that L_γ^{-1} is linear.

$$\begin{Bmatrix} Lu = f \text{ on } \Omega \\ u = 0 \text{ on } \partial\Omega \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} Lu + \gamma u = \gamma u + f \text{ on } \Omega \\ u = 0 \text{ on } \partial\Omega \end{Bmatrix}$$
$$\Leftrightarrow B_\gamma(u, v) = (\gamma u + f, v).$$

Hence, $u = L_\gamma^{-1}(\gamma u + f) = \gamma L_\gamma^{-1}u + L_\gamma^{-1}f$. So $u = Ku + h$, where $Ku := \gamma L_\gamma^{-1}u$ and $h := L_\gamma^{-1}f$. We claim that K is a compact linear operator:

To see this, take $g \in L^2(\Omega)$ and let $u := Kg$. Then

$$\alpha\|u\|^2_{H_0^1(\Omega)} \leq B_\gamma(u, u) = (g, u) \leq \|g\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \leq \|g\|_{L^2(\Omega)}\|u\|_{H_0^1(\Omega)},$$

and so

$$\alpha\|Kg\|_{H_0^1(\Omega)} = \alpha\|u\|_{H_0^1(\Omega)} \leq \|g\|_{L^2(\Omega)}$$

for all $g \in L^2(\Omega)$. Hence, since $H_0^1(\Omega) \subset\subset L^2(\Omega)$ by the Rellich–Kondrachov theorem (theorem 2.39), K is compact.

Now, either $\ker(I - K) = \{0\}$ or $\ker(I - K) \neq \{0\}$.

$$\ker(I - K) = \{0\} \Leftrightarrow \ker(I^* - K^*) = \{0\} \Leftrightarrow \operatorname{im}(I - K) = \mathcal{H} = L^2(\Omega),$$

and so $I - K$ is invertible, and so (3.4) has a unique weak solution. On the other hand, $\ker(I - K) \neq \{0\} \Leftrightarrow$ there exists $u \neq 0$ such that $(I - K)u = 0 \Leftrightarrow$ (3.5). That is, either

- $u - Ku = h$ has a unique solution for all $h \in L^2(\Omega)$; or
- $u - Ku = 0$ has a solution $u \neq 0$.

By the Fredholm alternative, $\dim\ker(I - K) < \infty$ and $\dim\ker(I - K) = \dim\ker(I^* - K^*)$. Elements of $\ker(I^* - K^*)$ are solutions of $(I^* - K^*)v = 0$, and $v - K^*v = 0$ if, and only if v solves (3.6).

Finally, (3.4) has a solution if, and only if, $(h, v) = 0$ for all v that solve (3.6). But, for $h \in \operatorname{im}(I - K)$ and $v \in \ker(I^* - K^*)$,

$$0 = (h, v) = (L_\gamma^{-1}f, v) = \frac{1}{\gamma}(Kf, v) = \frac{1}{\gamma}(f, K^*v) = \frac{1}{\gamma}(f, v),$$

since $\operatorname{im}(I - K) = \ker(I^* - K^*)$. So (3.4) has a solution if, and only if, $(f, v) = 0$ for all weak solutions v of (3.6). □

Theorem 3.12. 1. *There exists an at most countable set $\Sigma \subset \mathbb{R}$ such that*

$$\begin{cases} Lu = \lambda u + f, & \textit{on } \Omega; \\ u = 0, & \textit{on } \partial\Omega. \end{cases} \tag{3.7}$$

has a unique weak solution for each $f \in L^2(\Omega)$ if, and only if, $\lambda \notin \Sigma$.

2. *If Σ is infinite, then the elements of Σ form an increasing sequence $\lambda_k \underset{k\to\infty}{\longrightarrow} +\infty$.*

Definition 3.13. The set Σ is the *(real) spectrum* of the operator L.

Proof of Theorem 3.12. 1. Take $\gamma > 0$ and set $B_\gamma(u, v) := (Lu + \gamma u, v)$ as before. If $\lambda \leq -\gamma$, then $Lu - |\lambda|u = f$ has a unique solutions, since $|\lambda| \geq \gamma$, so the only interesting case is $\lambda > -\gamma$.

By the Fredholm Alternative, (3.7) has a unique solution for each $f \in L^2(\Omega)$ if, and only if, the only solution to

$$\begin{cases} Lu = \lambda u, & \text{on } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases}$$

is $u = 0$. This holds if, and only if, $u = 0$ is the only solution of

$$\begin{cases} Lu + \gamma u = (\gamma + \lambda)u, & \text{on } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

This holds precisely when

$$u = L_\gamma^{-1}((\gamma + \lambda)u) = (\gamma + \lambda)L_\gamma^{-1}u = \frac{\gamma + \lambda}{\gamma} Ku,$$

where $Ku := \gamma L_\gamma^{-1}u$ as before. So (3.7) has a unique solution for all $f \in L^2(\Omega)$ if, and only if, $Ku = \frac{\gamma}{\gamma+\lambda}u$ has only the trivial solution $u = 0$; that is, when $\frac{\gamma}{\gamma+\lambda}$ is not an eigenvalue of K. However, since K is a compact operator, the set of its eigenvalues is either finite or forms a sequence tending to 0.

2. Since $\lambda_k > -\gamma$ and $\frac{\gamma}{\gamma+\lambda_k} \to 0$, it follows that $\lambda_k \to +\infty$. □

This result tells us that if we wish to solve

$$\begin{cases} Lu = \lambda u + f, & \text{on } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases}$$

then "the chances are good" that there will be a unique solution (this occurs when $\lambda \notin \Sigma$. On the other hand, even if $\lambda \in \Sigma$, if we find v such that

$$\begin{cases} L^*v = \lambda v, & \text{on } \Omega; \\ v = 0, & \text{on } \partial\Omega; \end{cases}$$

then we can still solve for u provided that $(f, v) = 0$.

Theorem 3.14 (Boundedness of the Inverse.)**.** *If $\lambda \notin \Sigma$, then there is a constant C, depending only on λ, Ω and L, such that*

$$\|u\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

for all $f \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$ solving

$$\begin{cases} Lu = \lambda u + f, & \textit{on } \Omega; \\ u = 0, & \textit{on } \partial\Omega. \end{cases}$$

Proof. Suppose not. Then we can find sequences $(u_k) \subset H_0^1(\Omega)$ and $(f_k) \subset L^2(\Omega)$ such that

$$\begin{cases} Lu_k = \lambda u_k + f_k, & \text{on } \Omega; \\ u_k = 0, & \text{on } \partial\Omega; \end{cases}$$

and $\|u_k\|_{L^2(\Omega)} \geq k\|f_k\|_{L^2(\Omega)}$. Without loss of generality, take $\|u_k\|_{L^2(\Omega)} = 1$, so $\|f_k\|_{L^2(\Omega)} \to 0$. By the energy estimates

$$\begin{aligned} \|u_k\|^2_{H_0^1(\Omega)} - \gamma\|u_k\|^2_{L^2(\Omega)} &\leq B(u_k, u_k) \\ &= \lambda\|u_k\|^2_{L^2(\Omega)} + (f_k, u_k) \\ &\leq \lambda\|u_k\|^2_{L^2(\Omega)} + \|f_k\|^2_{L^2(\Omega)} + \|u_k\|^2_{L^2(\Omega)}, \end{aligned}$$

the sequence (u_k) is bounded in $H_0^1(\Omega)$. Then there exists a subsequence (u_{k_j}) such that $u_{k_j} \rightharpoonup u$ in $H_0^1(\Omega)$ and $u_{k_j} \to u$ in $L^2(\Omega)$. Then u is a weak solution of the limiting problem

$$\begin{cases} Lu = \lambda u, & \text{on } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

But since $\lambda \notin \Sigma$, $u = 0$, which contradicts $\|u\|_{L^2(\Omega)} = \|u_k\|_{L^2(\Omega)} = 1$. □

3.2 Eigenvalues and Eigenfunctions

Consider the problem

$$\begin{cases} Lu = \lambda u, & \text{on } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases}$$

where $Lu = -\sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right)$, with $a_{ij} = a_{ji}$ a symmetric matrix. In this case, the bilinear form B is also symmetric: $B(u, v) = B(v, u)$ for all $u, v \in H_0^1(\Omega)$.

Theorem 3.15. (Hilbert-Schmidt Theorem.)

1. *The eigenvalues of L are all real and positive.*
2. *If we repeat each eigenvalue of L according to its (finite) multiplicity, $\Sigma = \{\lambda_k\}_{k=1}^\infty$ with*
$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots,$$
and $\lambda_k \to +\infty$ as $k \to \infty$.
3. *There is an orthonormal basis $\{\omega_k\}_{k=1}^\infty$ of $L^2(\Omega)$ such that, for each k,*
$$\begin{cases} L\omega_k = \lambda_k\omega_k, & \text{on } \Omega; \\ \omega_k = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

This treatment of the Hilbert-Schmidt Theorem covers only symmetric elliptic differential operators; for the general case (L a compact, self-adjoint operator on any Hilbert space $\mathcal{H}$), see [RR], §7.5, Theorem 7.94.

Proof of Theorem 3.15. Define $S := L^{-1}$. For all $f \in L^2(\Omega)$, $Sf - u \Leftrightarrow Lu = f$. Take any $f, g \in L^2(\Omega)$. Then

$$(Sf, g)_{L^2(\Omega)} = (u, Lv)_{L^2(\Omega)} = B(v, u),$$

and

$$(Sg, f)_{L^2(\Omega)} = (v, Lu)_{L^2(\Omega)} = (Lu, v)_{L^2(\Omega)} = B(u, v).$$

So, since $B(u, v) = B(v, u)$ for all $u, v \in H_0^1(\Omega)$, $(Sf, g)_{L^2(\Omega)} = (f, Sg)_{L^2(\Omega)}$ for all $f, g \in L^2(\Omega)$. So S is a symmetric, compact, bounded, linear operator, and

$$(Sf, f)_{L^2(\Omega)} = (Lu, u)_{L^2(\Omega)} = B(u, u) > 0$$

for $u \neq 0$. Now apply the Spectral Theorem to deduce that:

1. all eigenvalues η_k of S are real and positive;
2. $\eta_k \to 0$;
3. there is an orthonormal basis $\{\omega_k\}_{k=1}^{\infty}$ of $L^2(\Omega)$ ($L^2(\Omega)$ is separable, so the basis is indeed countable) such that $S\omega_k = \eta_k \omega_k$ for each k.

Now note that

$$S\omega_k = \eta_k \omega_k \Leftrightarrow L\omega_k = \frac{1}{\eta_k}\omega_k = \lambda_k \omega_k,$$

and $\lambda_k \to \infty$. □

Definition 3.16. $\lambda_1 > 0$ is called the *principal eigenvalue* of L.

Theorem 3.17. *1. Rayleigh's formula:*

$$\lambda_1 = \min \left\{ B(u, u) \,\middle|\, u \in H_0^1(\Omega), \|u\|_{L^2(\Omega)} = 1 \right\}. \tag{3.9}$$

2. This minimum is attained by a function ω_1 that is strictly positive on Ω and solves

$$\begin{cases} L\omega_1 = \lambda_1 \omega_1, & \text{on } \Omega; \\ \omega_1 = 0, & \text{on } \partial\Omega. \end{cases}$$

3. λ_1 is a simple eigenvalue: if $u \in H_0^1(\Omega)$ is any weak solution of

$$\begin{cases} Lu = \lambda_1 u, & \text{on } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases}$$

then u is a multiple of ω_1.

Proof. Since $\{\omega_k\}_{k=1}^{\infty}$ is an orthonormal basis of $L^2(\Omega)$, for any $u \in H_0^1(\Omega)$, we can write $u = \sum_{k=1}^{\infty} d_k \omega_k$ in the L^2 sense, and $\|u\|_{L^2(\Omega)} = 1 \Rightarrow \sum_k d_k^2 = 1$. Here, $d_k := (u, \omega_k)_{L^2(\Omega)}$.

Also note that

$$B(\omega_k, \omega_\ell) = (\lambda_k \omega_k, \omega_\ell) = \lambda_k \delta_{k\ell}.$$

Thus, $\left\{\frac{\omega_k}{\sqrt{\lambda_k}}\right\}_{k=1}^{\infty}$ is an orthonormal set in $H_0^1(\Omega)$ with respect to the equivalent inner product $[u,v] := B(u,v)$, since $\left[\frac{\omega_k}{\sqrt{\lambda_k}}, \frac{\omega_\ell}{\sqrt{\lambda_\ell}}\right] = \delta_{k\ell}$. Is the set $\left\{\frac{\omega_k}{\sqrt{\lambda_k}}\right\}_{k=1}^{\infty}$ complete in $H_0^1(\Omega)$? For all $u \in H_0^1(\Omega)$, $B\left(u, \frac{\omega_k}{\sqrt{\lambda_k}}\right) = 0 \ \forall k \Rightarrow u = 0$. So, for all k,

$$0 = B\left(u, \frac{\omega_k}{\sqrt{\lambda_k}}\right) = B\left(\frac{\omega_k}{\sqrt{\lambda_k}}, u\right) = \lambda_k\left(\frac{\omega_k}{\sqrt{\lambda_k}}, u\right) = \sqrt{\lambda_k}(\omega_k, u).$$

Therefore, since $\{\omega_k\}_{k=1}^{\infty}$ is complete in $L^2(\Omega)$, $u = 0 \in H_0^1(\Omega)$. So $\left\{\frac{\omega_k}{\sqrt{\lambda_k}}\right\}_{k=1}^{\infty}$ is an orthonormal complete set in $H_0^1(\Omega)$.

So, if $u = \sum_k \beta_k \frac{\omega_k}{\sqrt{\lambda_k}}$ in $H_0^1(\Omega)$, $\beta_k := \left[u, \frac{\omega_k}{\sqrt{\lambda_k}}\right]$, it follows that $\beta_k = d_k\sqrt{\lambda_k}$ and the series $u = \sum_{k=1}^{\infty} d_k\omega_k$ converges in $H_0^1(\Omega)$ as well as in $L^2(\Omega)$. Thus, for $\|u\|_{L^2(\Omega)} = 1$,

$$\sum_k \beta_k^2 = B(u,u) = \sum_k d_k^2\lambda_k \geq \sum_k d_k^2\lambda_1 = \lambda_1,$$

and equality holds for $u = \omega_1$, thus proving Rayleigh's formula (3.9).

For the remainder of the proof, see [Ev], §6.5. □

3.3 Regularity

In this subsection, we will only consider $Lu = -\Delta u$, but the methods and results do generalize to more general differential operators.

Consider the problem

$$\begin{cases} -\Delta u = f, & \text{on } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases}$$

for $f \in L^2(\Omega)$. As we have seen this equation does not require that a solution u be C^2, or even twice-differentiable in the weak sense, since $\mathrm{D}^2u \neq \Delta u$.

Let $f \in L^2(\mathbb{R}^n)$ and assume appropriate smoothness of u. Formally:

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x)^2 \,\mathrm{d}x &= \int_{\mathbb{R}^n} (\Delta u(x))^2 \,\mathrm{d}x \\
&= \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \frac{\partial^2 u}{\partial x_i^2}\frac{\partial^2 u}{\partial x_j^2} \,\mathrm{d}x \\
&= -\sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \frac{\partial^3 u}{\partial x_i^2 \partial x_j}\frac{\partial u}{\partial x_j} \ dx \\
&= \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} \,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} |\mathrm{D}^2 u(x)|^2 \,\mathrm{d}x.
\end{aligned}$$

This suggests that

- $f \in L^2(\mathbb{R}^n) \Rightarrow u \in H^2(\mathbb{R}^n)$;

- $f \in H^1(\mathbb{R}^n) \Rightarrow \Delta u = f \Rightarrow \Delta \frac{\partial u}{\partial x_i} = \frac{\partial f}{\partial x_i} \Rightarrow \mathrm{D}u \in H^2(\mathbb{R}^n) \Rightarrow u \in H^3(\mathbb{R}^n)$;
- $f \in H^m(\mathbb{R}^n) \Rightarrow u \in H^{m+2}(\mathbb{R}^n)$.

Definition 3.18. The k^{th} *difference quotient* is

$$\mathrm{D}_k^h u(x) := \frac{u(x + he_k) = u(x)}{h}.$$

Theorem 3.19. *1. If $u \in W^{1,p}(\Omega)$ and $V \subset\subset \Omega$, then, for all h such that $0 < |h| < \frac{1}{2}d(V, \partial\Omega)$,*

$$\left\| \mathrm{D}_k^h u \right\|_{L^p(V)} \leq C \| \mathrm{D}u \|_{L^p(V)}.$$

2. If $u \in L^p(\Omega)$, $V \subset\subset \Omega$, and there exists a constant C such that for all h with $0 < |h| < \frac{1}{2}d(V, \partial\Omega)$, $\|\mathrm{D}_k^h u\|_{L^p(V)} \leq C$, then $u \in W^{1,p}(V)$ and $\|\mathrm{D}u\|_{L^p(V)} \leq C$.

Theorem 3.20 (Interior H^2 regularity)**.** *If $f \in L^2(\Omega)$, $a_{ij} \in C^1(\Omega)$, $b_i, c \in L^\infty(\Omega)$, and $u \in H^1(\Omega)$ satisfies $Lu = f$ on Ω,[13] then $u \in H^2(V)$ for all $V \subset\subset \Omega$, i.e. $u \in H^2_{\mathrm{loc}}(\Omega)$, with*

$$\|u\|_{H^2(V)} \leq C \left(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \right).$$

Proof. We consider only the case $Lu = -\Delta u$. $Lu = f \Rightarrow$ for all $v \in H^1_0(\Omega)$,

$$\sum_{i=1}^n \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, \mathrm{d}x = \int_\Omega f(x) v(x) \, \mathrm{d}x.$$

Let $U \subset\subset W \subset\subset \Omega$ and take $v = -\mathrm{D}_k^{-h}(\mathrm{D}_k^h u \cdot \xi^2)$, where $\xi \in C^\infty(\Omega)$ is 1 inside V, 0 on $\Omega \setminus W$, and $\xi(x) \in [0, 1]$ for all $x \in \Omega$. Then

$$\begin{aligned}
\sum_{i=1}^n \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, \mathrm{d}x &= -\sum_{i=1}^n \int_\Omega \frac{\partial u}{\partial x_i} \mathrm{D}_k^{-h} \left(\frac{\partial}{\partial x_i} \left(\mathrm{D}_k^h u(x) \cdot \xi(x)^2 \right) \right) \mathrm{d}x \\
&= \sum_{i=1}^n \int_\Omega \mathrm{D}_k^h \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_i} \left(\mathrm{D}_k^h u(x) \cdot \xi(x)^2 \right) \mathrm{d}x \\
&= \sum_{i=1}^n \int_\Omega \mathrm{D}_k^h \frac{\partial u}{\partial x_i} \cdot \mathrm{D}_k^h \frac{\partial u}{\partial x_i} \cdot \xi(x)^2 + \mathrm{D}_k^h \frac{\partial u}{\partial x_i} \cdot \mathrm{D}_k^h u(x) \cdot 2\xi(x) \frac{\partial \xi}{\partial x_i} \, \mathrm{d}x \\
&= \int_\Omega \left| \mathrm{D}_k^h \mathrm{D}u(x) \right|^2 \xi(x)^2 \, \mathrm{d}x + A.
\end{aligned}$$

Now estimate

$$\begin{aligned}
A &\leq C \int_\Omega \left| \mathrm{D}_k^h \mathrm{D}u(x) \right| \xi(x) \left| \mathrm{D}_k^h u(x) \right| \, \mathrm{d}x \\
&\leq \frac{1}{2} \int_\Omega \left| \mathrm{D}_k^h \mathrm{D}u(x) \right|^2 \xi(x)^2 \, \mathrm{d}x + C \int_W \left| \mathrm{D}_k^h u(x) \right|^2 \, \mathrm{d}x \\
&\leq \frac{1}{2} \int_\Omega \left| \mathrm{D}_k^h \mathrm{D}u(x) \right|^2 \xi(x)^2 \, \mathrm{d}x + C \int_\Omega \left| \mathrm{D}u(x) \right|^2 \, \mathrm{d}x.
\end{aligned}$$

[13] We do not require that $u = 0$ on $\partial\Omega$, i.e. that $u \in H^1_0(\Omega)$.

Also,

$$\begin{aligned}
\int_\Omega f(x)v(x)\,\mathrm{d}x &= -\int_\Omega f(x)\mathrm{D}_k^{-h}\left(\mathrm{D}_k^h u(x)\cdot\xi(x)^2\right)\,\mathrm{d}x \\
&\le \int_\Omega |f(x)||v(x)|\,\mathrm{d}x \\
&\le \frac{1}{\varepsilon}\int_\Omega |f(x)|^2\,\mathrm{d}x + \varepsilon\int_\Omega |v(x)|^2\,\mathrm{d}x,
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega |v(x)|^2\,\mathrm{d}x &\le C\int_\Omega \left|\mathrm{D}\left(\mathrm{D}_k^h u(x)\cdot\xi(x)^2\right)\right|^2\,\mathrm{d}x \\
&\le C\int_\Omega \left|\mathrm{D}_k^h\mathrm{D}u(x)\cdot\xi(x)^2 + 2\mathrm{D}_k^h u(x)\cdot\mathrm{D}\xi(x)\cdot\xi(x)\right|^2\,dx \\
&\le C\int_\Omega \left|\mathrm{D}_k^h\mathrm{D}u(x)\right|^2\xi(x)^2\,\mathrm{d}x + C\int_W \left|\mathrm{D}_k^h u(x)\right|^2\,\mathrm{d}x \\
&\le C\int_\Omega \left|\mathrm{D}_k^h\mathrm{D}u(x)\right|^2\xi(x)^2\,\mathrm{d}x + C\int_\Omega |\mathrm{D}u(x)|^2\,\mathrm{d}x,
\end{aligned}$$

where we have used the facts that $|a+b|^2 \le 2a^2+2b^2$, $0 \le \xi(x) \le 1 \Rightarrow \xi(x)^4 \le \xi(x)^2$, the derivative $\mathrm{D}\xi$ is bounded, and that $\xi \equiv 0$ outside W.

$$\begin{aligned}
\left|\int_\Omega f(x)v(x)\,\mathrm{d}x\right| &\le \frac{C}{\varepsilon}\int_\Omega |f(x)|^2\,\mathrm{d}x + \varepsilon\int_\Omega \left|\mathrm{D}_k^h\mathrm{D}u(x)\right|^2\xi(x)^2\,\mathrm{d}x + \varepsilon\int_\Omega |\mathrm{D}u(x)|^2\,\mathrm{d}x \\
&\le \sum_{i=1}^n \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,\mathrm{d}x \\
&= \int_\Omega f(x)v(x)\,\mathrm{d}x
\end{aligned}$$

Therefore,

$$\int_V \left|\mathrm{D}_k^h\mathrm{D}u(x)\right|^2\,dx \le \int_\Omega \left|\mathrm{D}_k^h\mathrm{D}u(x)\right|^2\xi(x)^2\,dx \le C\int_\Omega |\mathrm{D}u(x)|^2 + |f(x)|^2\,\mathrm{d}x,$$

and so $\mathrm{D}u \in H^1_{\mathrm{loc}}(\Omega)$, so $u \in H^2_{\mathrm{loc}}(\Omega)$ with

$$\|u\|_{H^2(V)} \le C\left(\|u\|_{H^1(W)} + \|f\|_{L^2(W)}\right).$$

One can show by choice of a new cut-off function ξ that is 1 on W with $\operatorname{supp}\xi \subseteq \Omega$ that

$$\|u\|_{H^1(W)} \le C\left(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\right).$$

(See [Ev] §6.3, Theorem 1.) Thus, as required,

$$\|u\|_{H^2(V)} \le C\left(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\right).$$ □

We now state without proof some further regularity results. See [Ev] for details.

Theorem 3.21 (Interior H^{m+2} regularity)**.** *If $f \in H^m(\Omega)$, $a_{ij}, b_i, c \in C^{m+1}(\Omega)$, and $u \in H^1(\Omega)$ solves $Lu = f$ on Ω, then $u \in H^{m+2}(V)$ for all $V \subset\subset \Omega$, i.e. $u \in H^{m+2}_{\text{loc}}(\Omega)$, and*

$$\|u\|_{H^{m+2}(V)} \leq C \left(\|u\|_{L^2(V)} + \|f\|_{H^m(V)} \right).$$

Theorem 3.22 (Smoothness in the interior)**.** *If $a_{ij}, b_i, c, f \in C^\infty(\Omega)$ and $u \in H^1(\Omega)$ solves $Lu = f$ on Ω, then $u \in C^\infty(\Omega)$.*

Theorem 3.23 (Boundary Regularity)**.** *If $\partial\Omega$ is C^2, $f \in L^2(\Omega)$, $a_{ij}, b_i, c \in C^1(\bar{\Omega})$ and $u \in H^1_0(\Omega)$ solves*

$$\begin{cases} Lu = f, & \text{on } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases}$$

then $u \in H^2(\Omega)$ and

$$\|u\|_{H^2(\Omega)} \leq C \left(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \right).$$

4 Parabolic PDEs

4.1 Parabolic Equations and Weak Solutions

We seek to solve the following PDE:

$$\begin{cases} \frac{\partial u}{\partial t} + Lu = f, & (x,t) \in \Omega \times (0,T]; \\ u(x,t) = g(x), & (x,t) \in \Omega \times \{t=0\}; \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times [0,T]. \end{cases} \tag{4.1}$$

Here $T > 0$ and we usually write $\Omega_T := \Omega \times (0,T]$. We take L to be of the form

$$Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u$$

with $a_{ij}, b_i, c \in L^\infty(\Omega_T)$, $f \in L^2(\Omega_T)$ and $g \in L^2(\Omega)$. We often write $u' := \frac{\partial u}{\partial t}$.

Definition 4.1. We say that $\frac{\partial}{\partial t} + L$ is *(uniformly) parabolic* if, for some $\alpha > 0$ and all $(x,t) \in \Omega_T$ and $\xi \in \mathbb{R}^n$,

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \alpha|\xi|^2. \tag{4.2}$$

Example 4.2. If $a_{ij} = \delta_{ij}$, $b_i = c = f = 0$, then $L = -\Delta$ and (4.1) becomes the heat equation.

What do we expect of weak solutions of (4.1)? Based on our earlier work, we expect something like

$$\langle u' + Lu, v\rangle = (f,v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega).$$

Here we have made use of duality: L takes two derivatives in x, so if $u \in H_0^1(\Omega)$ for each (or almost every) t, then $u' \in H^{-1}(\Omega)$ for each (almost every) t.

As before, we introduce a time-dependent bilinear form

$$\begin{aligned} B(u,v;t) &:= \langle Lu, v\rangle \\ &= \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i}v(x) + c(x,t)u(x)v(x)\right) \mathrm{d}x. \end{aligned}$$

So we require that

$$\langle u'(t), v\rangle + B(u(t),v;t) = (f(t),v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega).$$

Thus, we get a differential equation in t only:

$$\begin{cases} \langle u'(t), v\rangle + B(u(t),v;t) = (f(t),v)_{L^2(\Omega)}; \\ u(0) = g; \end{cases}$$

where this equation is required to hold for all $v \in H_0^1(\Omega)$. Here we consider u to be a function $u : [0,T] \to H_0^1(\Omega)$; similarly, $u' : [0,T] \to H^{-1}(\Omega)$ and $f : [0,T] \to L^2(\Omega)$. As before, we have some potential problems with functions being defined only up to sets of measure zero. The following definitions and result resolve these issues:

Definitions 4.3. Given a Banach space X, the *Bochner space*[14] $L^p([0,T];X)$ is the space of all measurable functions $u : [0,T] \to X$ such that

$$\|u\|_{L^p([0,T];X)} := \left(\int_0^T \|u(t)\|_X^p \, \mathrm{d}t \right)^{\frac{1}{p}} < \infty$$

for $1 \leq p < \infty$, and

$$\|u\|_{L^\infty([0,T];X)} := \operatorname*{ess\,sup}_{0 \leq t \leq T} \|u(t)\|_X < \infty.$$

The space $C([0,T];X)$ is the space of all continuous functions $u : [0,T] \to X$ such that

$$\|u\|_{C([0,T];X)} := \max_{0 \leq t \leq T} \|u(x)\|_X < \infty.$$

Theorem 4.4. *Suppose that $u \in L^2([0,T];H_0^1(\Omega))$ and $u' \in L^2([0,T];H^{-1}(\Omega))$. Then (possibly after redefinition on a null set), $u \in C([0,T];L^2(\Omega))$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{L^2(\Omega)}^2 = 2\langle u'(t), u(t) \rangle.$$

Proof. (See [Ev], §5.9.2, Theorem 3 for further details.) Fix $A > 0$ and extend u to $(-A, T+A)$. Then consider $u_\varepsilon := \eta_\varepsilon \star u$ as before, which is smooth in t. Then, for $\varepsilon, \delta > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon(t) - u_\delta(t)\|_{L^2(\Omega)}^2 = 2\langle u_\varepsilon'(t) - u_\delta'(t), u_\varepsilon(t) - u_\delta(t) \rangle$$

$u_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} u$ in $L^2([0,T];H_0^1(\Omega))$, and $u_\varepsilon' \underset{\varepsilon \to 0}{\longrightarrow} u'$ in $L^2([0,T];H^{-1}(\Omega))$. Thus,

$$\|u_\varepsilon(t) - u_\delta(t)\|_{L^2(\Omega)}^2 = \|u_\varepsilon(s) - u_\delta(s)\|_{L^2(\Omega)}^2 + 2\int_s^t \langle u_\varepsilon'(\tau) - u_\delta'(\tau), u_\varepsilon(\tau) - u_\delta(\tau) \rangle \, \mathrm{d}\tau$$

for all $0 \leq s \leq t \leq T$. Now, $u_\varepsilon(x) \underset{\varepsilon \to 0}{\longrightarrow} u(s)$ in $H_0^1(\Omega)$ for almost all s; pick any such s. Then

$$\begin{aligned}
&\limsup_{\varepsilon,\delta \to 0} \sup_{0 \leq t \leq T} \|u_\varepsilon(t) - u_\delta(t)\|_{L^2(\Omega)}^2 \\
&\quad \leq \limsup_{\varepsilon,\delta \to 0} \int_0^T \|u_\varepsilon'(\tau) - u_\delta'(\tau)\|_{H^{-1}(\Omega)}^2 + \|u_\varepsilon(\tau) - u_\delta(\tau)\|_{H_0^1(\Omega)}^2 \, \mathrm{d}\tau \\
&\quad = 0.
\end{aligned}$$

Thus, the smoothed functions u_ε converge in $C([0,T];L^2(\Omega))$ to some $v \in C([0,T];L^2(\Omega))$. Since $u_\varepsilon(t) \to u(t)$ for almost all t, we conclude that $u = v$ almost everywhere, as required. □

Having done this, we return to the weak formulation of the parabolic problem as an infinite system of ODEs, and define

[14] Salomon Bochner (1899–1982).

Definition 4.5. We say that $u \in L^2([0,T]; H_0^1(\Omega))$, with $u' \in L^2([0,T]; H^{-1}(\Omega))$, is a *weak solution* of the parabolic problem (4.1) if

$$\begin{cases} \langle u'(t), v\rangle + B(u(t), v; t) = (f(t), v)_{L^2(\Omega)}; \\ u(0) = g; \end{cases} \tag{4.3}$$

for all $v \in H_0^1(\Omega)$ and almost all $t \in [0,T]$.

Note that the "partial derivative" $u' = \frac{\partial u}{\partial t} : [0,T] \to H^{-1}(\Omega)$ is really a full derivative, since x-dependence has been removed by the use of Bochner spaces.

4.2 Galerkin Approximations

Recall that a weak solution to the parabolic problem (4.1),

$$\begin{cases} \frac{\partial u}{\partial t} + Lu = f, & (x,t) \in \Omega \times (0,T]; \\ u(x,t) = g(x), & (x,t) \in \Omega \times \{t = 0\}; \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times [0,T]; \end{cases}$$

is given by $u \in L^2([0,T]; H_0^1(\Omega))$ with $u' \in L^2([0,T]; H^{-1}(\Omega))$ such that

$$\begin{cases} \langle u'(t), v\rangle + B(u(t), v; t) = (f(t), v)_{L^2(\Omega)}; \\ u(0) = g; \end{cases}$$

for all $v \in H_0^1(\Omega)$ and almost all $t \in [0,T]$. Unfortunately, this is an infinite-dimensional ODE. Galerkin's method[15] is a scheme for solving such problems; in many ways, it generalizes the use of Fourier series, which were first used to solve the heat equation.

Let $\{\omega_k\}_{k=1}^\infty$ be an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $H_0^1(\Omega)$, where the ω_k satisfy the eigenvalue problem (3.8):

$$\begin{cases} L\omega_k = \lambda_k \omega_k, & \text{on } \Omega; \\ \omega_k = 0, & \text{on } \partial\Omega. \end{cases}$$

We seek an approximation $u_m(t) := \sum_{k=1}^m d_k^m(t)\omega_k$, $u_m : [0,T] \to H_0^1(\Omega)$, to u. First note that

$$\begin{cases} \langle u_m'(t), v\rangle + B(u_m(t), v; t) = (f(t), v), & \forall v \in \operatorname{span}\{\omega_k\}_{k=1}^m; \\ (u_m(0), v) = (g, v); \end{cases}$$

if, and only if,

$$\begin{cases} \langle u_m'(t), \omega_k\rangle + B(u_m(t), \omega_k; t) = (f(t), \omega_k), & \forall k = 1, \dots, m; \\ (u_m(0), \omega_k) = (g, \omega_k); \end{cases}$$

The approximation $u_m(t) = \sum_{k=1}^m d_k^m(t)\omega_k$ satisfies $\langle u_m'(t), \omega_k\rangle = (d_k^m)'(t)$, and

$$B\left(\sum_{\ell=1}^m d_\ell^m(t)\omega_\ell, \omega_k; t\right) = \sum_{k=1}^m d_\ell^m(t) B(\omega_\ell, \omega_k; t).$$

[15] Boris Grigoryevich Galerkin (1871–1945).

$B(t) := (B(\omega_\ell, \omega_k; t))_{\ell k}$ is an $m \times m$ matrix, whose entries are known. So, if we write $f_k(t) := (f(t), \omega_k)$ and $\bar{f}^m(t) = (f_1(t), \dots, f_m(t))$ (and similarly for g, which does not depend on t), we have

$$\begin{cases} (d_k^m)'(t) + \sum_{\ell=1}^m B(\omega_\ell, \omega_k; t) d_\ell^m(t) = f_k(t); \\ d_k^m(0) = g_k; \end{cases}$$

or, equivalently,

$$\begin{cases} (d^m)'(t) + B(t) d^m(t) = \bar{f}^m(t); \\ d^m(0) = \bar{g}^m. \end{cases} \tag{4.4}$$

By the standard existence theory for ODEs, there is a unique absolutely continuous function $d^m = (d_1^m, \dots, d_m^m)$ that solves (4.4).

Definition 4.6. The function

$$u_m(t) = \sum_{k=1}^m d_k^m(t) \omega_k$$

is called the m^{th} *Galerkin approximation* to u.

We would like to say that $u_m \underset{m \to \infty}{\longrightarrow} u$, a weak solution of (4.1). In order to do this, we first need to bound u_m uniformly in $L^2([0,T]; H_0^1(\Omega))$ and $C([0,T]; L^2(\Omega))$, and to bound u_m' uniformly in $L^2([0,T]; H^{-1}(\Omega))$.

Theorem 4.7 (Energy estimates)**.** *There is a constant $C \geq 0$, depending only on Ω, T and L, such that, for all $m \in \mathbb{N}$,*

$$\max_{0 \leq t \leq T} \|u_m(t)\|_{L^2(\Omega)} + \|u\|_{L^2([0,T]; H_0^1(\Omega))} + \|u_m'\|_{L^2([0,T]; H^{-1}(\Omega))} \leq C \left(\|f\|_{L^2([0,T]; H_0^1(\Omega))} + \|g\|_{L^2(\Omega)} \right).$$

Proof. We recall the equation

$$\langle u_m'(t), u_m(t) \rangle + B(u_m(t), \omega_k; t) = (f(t), \omega_k),$$

multiply by $d_k^m(t)$, sum over $k = 1, \dots, m$, and use the definition of u_m to obtain

$$\langle u_m'(t), u_m(t) \rangle + B(u_m(t), u_m(t); t) = (f(t), u_m(t))$$

for almost all $t \in [0,T]$. Gårding's inequality (theorem 3.5) implies that there exist constants $\beta > 0$, $\gamma \geq 0$, such that

$$B(u_m(t), u_m(t); t) \geq \beta \|u_m(t)\|_{H_0^1(\Omega)}^2 - \gamma \|u_m(t)\|_{L^2(\Omega)}^2.$$

Thus, by theorem 4.4,

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} \|u_m(t)\|_{L^2(\Omega)}^2 \right) + \beta \|u_m(t)\|_{H_0^1(\Omega)}^2 - \gamma \|u_m(t)\|_{L^2(\Omega)}^2 \leq \frac{1}{2} \|f(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|u_m(t)\|_{L^2(\Omega)}^2, \tag{4.5}$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} \|u_m(t)\|_{L^2(\Omega)}^2 \right) \leq C_1 \|u_m(t)\|_{L^2(\Omega)}^2 + C_2 \|f(t)\|_{L^2(\Omega)}^2.$$

Grönwall's Inequality[16] in the form $a'(t) \leq ca(t)+b(t) \Rightarrow a(t) \leq e^{ct}\left(a(0)+\int_0^t b(s)\,\mathrm{d}s\right)$ then yields

$$\|u_m(t)\|^2_{L^2(\Omega)} \leq C_1 e^{C_2 t}\left(\|u_m(0)\|^2_{L^2(\Omega)} + \int_0^t \|f(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s\right).$$

Since $d_k^m(0) = (g, \omega_k)$ for each k, $\|u_m(0)\|^2_{L^2(\Omega)} \leq \|g\|^2_{L^2(\Omega)}$, so

$$\max_{0\leq t\leq T}\|u_m(t)\|_{L^2(\Omega)} \leq C_1\|g\|^2_{L^2(\Omega)} + \int_0^T \|f(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t.$$

We integrate (4.5) from 0 to T to obtain

$$\begin{aligned}\|u_m\|_{L^2([0,T];H_0^1(\Omega))} &= \int_0^T \|u_m(t)\|^2_{H_0^1(\Omega)}\,\mathrm{d}t \\ &\leq C_1\int_0^T \|u_m(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t + C_2\|u_m(0)\|^2_{L^2(\Omega)} + C_3\int_0^T \|f(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \\ &\leq C\left(\|g\|^2_{L^2(\Omega)} + \|f\|^2_{L^2([0,T];L^2(\Omega))}\right).\end{aligned}$$

We now seek to bound u_m'. Pick any $v \in H_0^1(\Omega)$ with $\|v\|_{H_0^1(\Omega)} \leq 1$, and write $v = \bar{v} = v^\perp$, where $v \in \operatorname{span}\{\omega_k\}_{k=1}^m$ and $(v^\perp, \omega_k) = 0$ for $k = 1, \ldots, m$. Since the functions ω_k are orthogonal in $H_0^1(\Omega)$, $\|\bar{v}\|_{H_0^1(\Omega)} \leq \|v\|_{H_0^1(\Omega)} \leq 1$. Thus, for almost all $t \in [0,T]$,

$$\langle u_m'(t), \bar{v}\rangle + B(u_m(t), \bar{v}; t) = (f(t), \bar{v}).$$

Thus, by the definition of u_m,

$$\langle u_m'(t), v\rangle = \langle u_m'(t), \bar{v}\rangle = (f(t), \bar{v}) - B(u_m(t), \bar{v}; t).$$

Therefore, $|\langle u_m'(t), v\rangle| \leq C\left(\|f(t)\|_{L^2(\Omega)} + \|u_m(t)\|_{H_0^1(\Omega)}\right)$, since $\|\bar{v}\|_{H_0^1(\Omega)} \leq 1$. Thus,

$$\|u_m'(t)\|_{H^{-1}(\Omega)} \leq C\left(\|f(t)\|_{L^2(\Omega)} + \|u_m(t)\|_{H_0^1(\Omega)}\right),$$

and so

$$\begin{aligned}\|u_m'\|^2_{L^2([0,T];H^{-1}(\Omega))} &= \int_0^T \|u_m'(t)\|^2_{H^{-1}(\Omega)}\,\mathrm{d}t \\ &\leq C\int_0^T \|f(t)\|^2_{L^2(\Omega)} + \|u_m(t)\|^2_{H_0^1(\Omega)}\,\mathrm{d}t \\ &\leq C\left(\|g\|^2_{L^2(\Omega)} + \|f\|^2_{L^2([0,T];L^2(\Omega))}\right),\end{aligned}$$

by our estimate of $\|u_m\|_{L^2([0,T];H_0^1(\Omega))}$.

Putting together our estimates of $\max_{0\leq t\leq T}\|u_m(t)\|_{L^2(\Omega)}$, $\|u_m\|_{L^2([0,T];H_0^1(\Omega))}$ and $\|u_m'\|_{L^2([0,T];H^{-1}(\Omega))}$, we have the required result. □

[16] Thomas Hakon Grönwall (1877–1932)

4.3 Existence and Uniqueness

We now now pass to the limit $m \to \infty$ and use Galerkin approximations to build a weak solution to (4.1). The following theorem essentially amounts to a proof that the Galerkin method does indeed produce a weak solution of the PDE as $m \to \infty$.

Theorem 4.8 (Existence of weak solution)**.** *There exists a weak solution to the general parabolic problem*

$$\begin{cases} u' + Lu = f, & (x,t) \in \Omega \times (0,T]; \\ u(x,t) = g(x), & (x,t) \in \Omega \times \{t = 0\}; \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times [0,T]. \end{cases}$$

Proof. Recall that a weak solution for $f \in L^2([0,T]; H_0^1(\Omega))$, $g \in L^2(\Omega)$, satisfies

$$\begin{cases} \langle u'(t), \phi \rangle + B(u(t), \phi; t) = (f(t), \phi)_{L^2(\Omega)} \quad \forall \phi \in H_0^1(\Omega); \\ u(0) = g; \end{cases}$$

and we have Galerkin approximations u_m, $m \in \mathbb{N}$, satisfying

$$\begin{cases} \langle u_m'(t), \omega_k \rangle + B(u_m(t), \omega_k; t) = (f(t), \omega_k)_{L^2(\Omega)} \quad \forall k = 1, \dots, m; \\ u(0) = g_m := \sum_{k=1}^m (g, \omega_k)\omega_k. \end{cases}$$

First note that $g_m \underset{m\to\infty}{\longrightarrow} g$ in $L^2(\Omega)$ since $\|g_m - g\|_{L^2(\Omega)}^2 \leq \sum_{k=m+1}^\infty (g, \omega_k)^2 \underset{m\to\infty}{\longrightarrow} 0$.

Secondly, there is a constant C depending only on Ω, f and g such that, for all m,

$$\begin{aligned} \|u_m\|_{L^2([0,T];H_0^1(\Omega))} &\leq C; \\ \|u_m\|_{C([0,T];L^2(\Omega))} &\leq C; \\ \|u_m'\|_{L^2([0,T];H^{-1}(\Omega))} &\leq C. \end{aligned}$$

Hence, there is a subsequence $(u_{m_\ell}) \subset (u_m)$ such that

$$u_{m_\ell} \rightharpoonup u \text{ in } L^2([0,T]; H_0^1(\Omega)); u_{m_\ell}' \rightharpoonup u \text{ in } L^2([0,T]; H^{-1}(\Omega)). \tag{4.6}$$

Re-label u_{m_ℓ} as u_m. (In fact, by the uniqueness result, Theorem 4.9, which follows, the full sequence converges weakly.) Fix $N \in \mathbb{N}$, $N \leq m$, and choose

$$v(t) = \sum_{k=1}^N \beta_k(t)\omega_k \tag{4.7}$$

with $\beta_k \in C^1([0,T])$ (so $v \in C^1([0,T]; H_0^1(\Omega))$). For each m,

$$\int_0^T \langle u_m'(t), v(t) \rangle + B(u_m(t), v(t); t) \, \mathrm{d}t = \int_0^T (f(t), v(t))_{L^2(\Omega)} \, \mathrm{d}t,$$

and so, passing to the limit $m \to \infty$,

$$\int_0^T \langle u'(t), v(t) \rangle + B(u(t), v(t); t) \, \mathrm{d}t = \int_0^T (f(t), v(t))_{L^2(\Omega)} \, \mathrm{d}t.$$

This then holds for all $v \in L^2([0,T]; H_0^1(\Omega))$, since functions v of the form (4.7) are dense in this space as $m \to \infty$. In particular, for all $\phi \in H_0^1(\Omega)$ and almost all $t \in [0,T]$,

$$\langle u'(t), \phi\rangle + B(u(t), \phi; t) = (f(t), \phi)_{L^2(\Omega)}.$$

Thirdly, in order to prove that $u(0) = g$, we first note that for $v \in C^1([0,T]; H_0^1(\Omega))$ with $v(T) = 0$,

$$\int_0^T \langle u'(t), v(t)\rangle \,\mathrm{d}t = -\int_0^T \langle u(t), v'(t)\rangle \,\mathrm{d}t - (u(0), v(0))_{L^2(\Omega)}.$$

So, for the same class of v,

$$\int_0^T -\langle u(t), v'(t)\rangle + B(u(t), v(t); t) \,\mathrm{d}t = \int_0^T (f(t), v(t))_{L^2(\Omega)} \,\mathrm{d}t + (u(0), v(0))_{L^2(\Omega)}.$$

Similarly, for each m,

$$\int_0^T -\langle u_m(t), v'(t)\rangle + B(u_m(t), v(t); t) \,\mathrm{d}t = \int_0^T (f(t), v(t))_{L^2(\Omega)} \,\mathrm{d}t + (u_m(0), v(0))_{L^2(\Omega)}.$$

By (4.6),

$$\int_0^T -\langle u(t), v'(t)\rangle + B(u(t), v(t); t) \,\mathrm{d}t = \int_0^T (f(t), v(t))_{L^2(\Omega)} \,\mathrm{d}t + (g, v(0))_{L^2(\Omega)}$$

since $u_m(0) \to g$ in $L^2(\Omega)$. Since $v(0)$ is arbitrary, we conclude that $u(0) = g$. □

Theorem 4.9 (Uniqueness of weak solution)**.** *A weak solution to the general parabolic problem* (4.1) *is unique.*

Proof. Suppose that u_1 and u_2 are two weak solutions of (4.1) satisfying (4.3). Set $w := u_1 - u_2$. w satisfies

$$\begin{cases} \langle w'(t), \phi\rangle + B(w(t), \phi; t) = 0; \\ u(0) = 0. \end{cases}$$

Take $\phi = w(t)$. Then $\langle w', w\rangle + B(w, w; t) = 0$ and, since there exists constants β, γ such that $B(w, w; t) \geq \beta\|w\|^2_{H_0^1(\Omega)} - \gamma\|w\|^2_{L^2(\Omega)}$,

$$\begin{cases} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2_{L^2(\Omega)} - \gamma\|w(t)\|^2_{L^2(\Omega)} \leq 0; \\ \|w(0)\|^2_{L^2(\Omega)} = 0. \end{cases}$$

So, by Grönwall's Inequality, $\|w(t)\|^2_{L^2(\Omega)} \leq \|w(0)\|^2_{L^2(\Omega)} e^{2\gamma t} = 0$. Therefore, $w = 0 \in L^2([0,T]; H_0^1(\Omega))$, as required. □

4.4 Regularity

Suppose that u is a smooth solution of the heat equation

$$\begin{cases} u_t - \Delta u = f, & \text{on } \mathbb{R}^n \times (0, T]; \\ u(0) = g, & \text{on } \mathbb{R}^n \times \{t = 0\}; \end{cases}$$

and that u decays "quickly" to 0 as $|x| \to \infty$. Then

$$\begin{aligned} \int_{\mathbb{R}^n} f(x)^2 \, \mathrm{d}x &= \int_{\mathbb{R}^n} u_t(x,t) - \Delta u(x,t) \, \mathrm{d}x \\ &= \int_{\mathbb{R}^n} u_t(x,t)^2 - 2u_t(x,t)\Delta u(x,t) + (\Delta u(x,t))^2 \, \mathrm{d}x \\ &= \int_{\mathbb{R}^n} u_t(x,t)^2 + 2\mathrm{D}u_t(x,t) \cdot \mathrm{D}u(x,t) + |\mathrm{D}^2 u(x,t)|^2 \, \mathrm{d}x \\ &= \int_{\mathbb{R}^n} (u_t(x,t))^2 + \frac{\mathrm{d}}{\mathrm{d}t} |\mathrm{D}u(x,t)|^2 + |\mathrm{D}^2 u(x,t)|^2 \, \mathrm{d}x. \end{aligned}$$

Thus,

$$\int_0^t \int_{\mathbb{R}^n} f(x)^2 \, \mathrm{d}x\mathrm{d}s = \int_0^t \int_{\mathbb{R}^n} u_t(x,s)^2 + |\mathrm{D}^2 u(x,s)|^2 \, \mathrm{d}x\mathrm{d}s + \int_{\mathbb{R}^n} |\mathrm{D}u(x,t)|^2 - |\mathrm{D}u(x,0)|^2 \, \mathrm{d}x.$$

So

$$\int_0^t \int_{\mathbb{R}^n} u_t(x,s)^2 + |\mathrm{D}^2 u(x,s)|^2 \, \mathrm{d}x\mathrm{d}s + \int_{\mathbb{R}^n} |Du(x,t)|^2 \, \mathrm{d}x \le C \left(\int_0^T \int_{\mathbb{R}^n} f(x)^2 \, \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^n} |\mathrm{D}g(x)|^2 \, \mathrm{d}x \right).$$

Therefore,

$$\begin{aligned} &\max_{0 \le t \le T} \int_{\mathbb{R}^n} |\mathrm{D}u(x,t)|^2 \, \mathrm{d}x + \int_0^T \int_{\mathbb{R}^n} u_t(x,t)^2 + |\mathrm{D}^2 u(x,t)|^2 \, \mathrm{d}x\mathrm{d}t \\ &\qquad \le C \left(\int_0^T \int_{\mathbb{R}^n} f(x)^2 \, \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^n} |\mathrm{D}g(x)|^2 \, \mathrm{d}x \right). \end{aligned}$$

5 Maximum Principles

In this section we work with an elliptic operator L in the form

$$Lu = -\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j^2} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u.$$

We shall assume whatever smoothness of a_{ij}, b_i and c that we require to ensure the regularity of a solution u; usually we need that $u \in C^2(\Omega) \cap C(\bar{\Omega})$. We also assume throughout that $\Omega \subset \mathbb{R}^n$ is bounded.

Theorem 5.1. (Weak Maximum Principle for Elliptic PDEs.) *Suppose that $\Omega \subset \mathbb{R}^n$ is bounded, that $c = 0$, and that $u \in C^2(\Omega) \cap C(\bar{\Omega})$. Then*

1. *$Lu \leq 0$ on $\Omega \Rightarrow \max_{\bar{\Omega}} u = \max_{\partial\Omega} u$;*
2. *$Lu \geq 0$ on $\Omega \Rightarrow \min_{\bar{\Omega}} u = \min_{\partial\Omega} u$.*

In particular, theorem 5.1 implies that harmonic functions ($\Delta u = 0$) attain their maxima and minima on the boundary of their domain of definition. Similarly, if $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies

$$\begin{cases} Lu = 0, & \text{on } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases}$$

then it follows that $u = 0$.

Proof of Theorem 5.1. We prove the first claim only, since the second follows from the first by changing u for $-u$. Also, we consider the case $L = -\Delta$; the general case is not much more difficult.

First assume that $Lu < 0$ on Ω and there exists a local maximum $x_0 \in \Omega$ such that $u(x_0) = \max_{\bar{\Omega}} u$. Then, since x_0 is a local maximum, $\mathrm{D}u(x_0) = 0$ and $\mathrm{D}^2u(x_0) \leq 0$ in the sense that

$$\mathrm{D}^2u(x_0)\xi \cdot \xi \leq 0 \quad \forall \xi \in \mathbb{R}^n.$$

But $Lu(x) = -\Delta u(x) < 0$ for all $x \in \Omega$, so $\Delta u(x_0) > 0$, a contradiction. So, if $Lu < 0$, $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$.

Now suppose that $Lu \leq 0$ on Ω. Define $u^\varepsilon(x) := u(x) + \varepsilon e^{\lambda x_1}$.

$$Lu^\varepsilon(x) = Lu(x) + \varepsilon L e^{\lambda x_1} = -\Delta u(x) - \varepsilon\lambda^2 e^{\lambda x_1} < 0.$$

So $Lu^\varepsilon < 0$ on Ω, so $\max_{\bar{\Omega}} u^\varepsilon = \max_{\partial\Omega} u^\varepsilon$. As $\varepsilon \to 0$, $u^\varepsilon \to u$ uniformly on $\bar{\Omega}$, so $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$. ☐

Theorem 5.2 (Weak maximum principle for parabolic PDEs). *Suppose that $\Omega \subset \mathbb{R}^n$ is bounded, that $c = 0$, and that $u \in C^2(\Omega_T) \cap C(\overline{\Omega_T})$. Then*

1. *$u_t + Lu \leq 0$ on $\Omega_T \Rightarrow \max_{\overline{\Omega_T}} u = \max_{\Gamma_T} u$;*
2. *$u_t + Lu \geq 0$ on $\Omega_T \Rightarrow \min_{\overline{\Omega_T}} u = \min_{\Gamma_T} u$;*

where $\Omega_T := \Omega \times (0, T]$ *and* $\Gamma_T := \overline{\Omega_T} \setminus \Omega_T$ *is the parabolic boundary of* Ω_T.

Proof. As usual, we prove only the first statement. Assume that $u_t + Lu < 0$ and suppose that $(x_0, t_0) \in \Omega_T$ is a local maximum of u such that $u(x_0, t_0) = \max_{\overline{\Omega_T}} u$.

- $0 < t_0 < T$. $u_t(x_0, t_0) = 0$ and $D^2u(x_0, t_0) \leq 0$. As in the elliptic case, this contradicts $u_t + Lu < 0$, so $\max_{\overline{\Omega_T}} u = \max_{\Gamma_T} u$.
- $t_0 = T$. $u_t(x_0, T) \geq 0$, $Lu(x_0, T) \geq 0$, since $D^2u(x_0, T) \leq 0$. So $(u_t + Lu)(x_0, T) \geq 0$, which contradicts $u_t + Lu < 0$.

If $u_t + Lu \leq 0$, define $u^\varepsilon(x, t) := u(x, t) - \varepsilon t$. Then

$$u_t^\varepsilon + Lu^\varepsilon = u_t + Lu - \varepsilon < 0,$$

so $\max_{\overline{\Omega_T}} u^\varepsilon = \max_{\Gamma_T} u^\varepsilon$. Again, by taking the limit $\varepsilon \to 0$, $\max_{\overline{\Omega_T}} u = \max_{\Gamma_T} u$. □

We now state without proof the Strong Maximum Principle and the lemma of Hopf that is needed to prove it.

Lemma 5.3 (Hopf's lemma). *Suppose* $u \in C^2(\Omega) \cap C(\bar{\Omega})$, $c = 0$, $Lu \leq 0$ *on* Ω, *and there is a point* $x_0 \in \Omega$ *such that* $u(x_0) > u(x)$ *for all* $x \in \Omega$. *Suppose also that there is an open ball* $B \subseteq \Omega$ *with* $x_0 \in \partial B$. *Then*

1. $\frac{\partial u}{\partial \nu}(x_0) > 0$;
2. *if* $c \geq 0$ *on* Ω, *then the same is true if* $u(x_0) \geq 0$.

Proof. See [Ev], §6.4.2. □

Theorem 5.4 (Strong maximum principle for elliptic PDEs). *Suppose that* $\Omega \subset \mathbb{R}^n$ *is bounded and connected, that* $c = 0$, *and that* $u \in C^2(\Omega) \cap C(\bar{\Omega})$. *Then*

1. $Lu \leq 0$ *on* Ω *and* u *attains* $\max_{\bar{\Omega}} u$ *at some* $x_0 \in \Omega$, *then* u *is constant on* Ω;
2. $Lu \geq 0$ *on* Ω *and* u *attains* $\min_{\bar{\Omega}} u$ *at some* $x_0 \in \Omega$, *then* u *is constant on* Ω.

Proof. See [Ev], §6.4.2, Theorem 3. □

Theorem 5.5 (Harnack's inequality). *Suppose that* $u \geq 0$ *is* C^2, $Lu = 0$ *on* Ω, *and that* $V \subset\subset \Omega$ *is connected. Then there is a constant* C, *depending only on* V *and* L, *such that*

$$\sup_V u \leq C \inf_V u.$$

Proof. See [GT]. □

6 Methods Applicable to Nonlinear Problems

See also Subsection 2.3

6.1 Calculus of Variations

The methods that we have developed up to this point have been applicable to linear problems only. The calculus of variations is a way to move beyond this limitation.

Suppose that we wish to solve the *p-Laplacian equation*

$$\begin{cases} \nabla \cdot (|\mathrm{D}u(x)|^{p-2}\mathrm{D}u(x)) = f(x), & x \in \Omega; \\ u(x) = 0, & x \in \partial\Omega; \end{cases} \tag{6.1}$$

where $p > 2$ and $f \in L^q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$. This is equivalent to finding a minimizer $u \in W_0^{1,p}(\Omega)$ for the functional

$$I(u) := \frac{1}{p}\int_\Omega |\mathrm{D}u(x)|^p \,\mathrm{d}x + \int_\Omega f(x)u(x)\,\mathrm{d}x.$$

Theorem 6.1. *There is a unique minimizer of I in $W_0^{1,p}(\Omega)$.*

Proof. We first show that I is bounded below. For any $\varepsilon > 0$,

$$\int_\Omega f(x)u(x)\,\mathrm{d}x \leq \int_\Omega \frac{f(x)^q}{\varepsilon^q q}\,\mathrm{d}x + \int_\Omega \frac{\varepsilon^p u(x)^p}{p}\,\mathrm{d}x,$$

and

$$\int_\Omega |u(x)|^p \,\mathrm{d}x \leq C\int_\Omega |\mathrm{D}u(x)|^p \,\mathrm{d}x.$$

So

$$\begin{aligned} I(u) &\geq \frac{1}{p}\int_\Omega |\mathrm{D}u(x)|^p \,\mathrm{d}x - \frac{1}{\varepsilon^q q}\int_\Omega |f(x)|^q \,\mathrm{d}x - \frac{C\varepsilon^p}{p}\int_\Omega |\mathrm{D}u(x)|^p \,\mathrm{d}x \\ &\geq -\frac{1}{\varepsilon^q q}\int_\Omega |f(x)|^q \,\mathrm{d}x \end{aligned}$$

for ε so small that $C\varepsilon^p < 1$. Therefore, if (u_n) is a minimizing sequence for I, $I(u_n) \not\to -\infty$. So

$$0 \geq I(u_n) \to \inf_{v \in W_0^{1,p}(\Omega)} I(v) \geq C.$$

Therefore, take $|I(u_n)| \leq C$ for all n.

$$\begin{aligned} \frac{1}{p}\int_\Omega |\mathrm{D}u_n(x)|^p \,\mathrm{d}x &\leq \left|\int_\Omega f(x)u_n(x)\,\mathrm{d}x\right| + C' \\ &\leq \frac{1}{\varepsilon^q q}\int_\Omega |f(x)|^q \,\mathrm{d}x + \frac{C\varepsilon^p}{p}\int_\Omega |\mathrm{D}u(x)|^p \,\mathrm{d}x + C' \\ &\leq \frac{1}{\varepsilon^q q}\int_\Omega |f(x)|^q \,\mathrm{d}x + \frac{1}{2p}\int_\Omega \mathrm{D}u(x)|^p \,\mathrm{d}x + C', \end{aligned}$$

so $\int_\Omega |\mathrm{D}u(x)|^p \,\mathrm{d}x$ is bounded. Thus, we have weak convergence (of a subsequence) $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$. So

$$\liminf_{n\to\infty} \int_\Omega |\mathrm{D}u_n(x)|^p \,\mathrm{d}x \geq \int_\Omega |\mathrm{D}u(x)|^p \,\mathrm{d}x$$

and

$$\lim_{n\to\infty} \int_\Omega f(x)u_n(x) \,\mathrm{d}x = \int_\Omega f(x)u(x) \,\mathrm{d}x.$$

u is indeed a minimizer of I, since

$$\begin{aligned} I(u) &\leq \frac{1}{p} \int_\Omega |\mathrm{D}u_n(x)|^p \,\mathrm{d}x + \int_\Omega f(x)u_n(x) \,\mathrm{d}x \\ &= \liminf_{n\to\infty} I(u_n) \\ &= \inf_{v\in W_0^{1,p}(\Omega)} I(v). \end{aligned}$$

The minimizer u is unique since the functional I is strictly convex: if u_1 and u_2 are distinct minimizers, take $u = \alpha u_1 + (1-\alpha)u_2$ and observe that

$$I(\alpha u_1 + (1-\alpha)u_2) < \alpha I(u_1) + (1-\alpha)I(u_2) = I(u_1) = I(u_2),$$

a contradiction. □

Consider now the nonlinear Poisson equation

$$\begin{cases} \Delta u(x) = f(u(x)), & x \in \Omega; \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{6.2}$$

Set $F(t) := \int_0^t f(s) \,\mathrm{d}s$. The problem (6.2) is equivalent to finding a minimizer for the functional

$$I(u) := \frac{1}{2} \int_\Omega |\mathrm{D}u(x)|^2 \,\mathrm{d}x + \int_\Omega F(u(x)) \,\mathrm{d}x.$$

Theorem 6.2. *If F is bounded both above and below ($-C_1 \leq F(y) \leq C_2$), then there is a minimizer of I in $H_0^1(\Omega)$.*

Proof. As usual, I is bounded below:

$$\frac{1}{2} \int_\Omega |\mathrm{D}u(x)|^2 \,\mathrm{d}x + \int_\Omega F(u(x)) \,\mathrm{d}x \geq -C_1|\Omega|.$$

Let (u_n) be a minimizing sequence: $I(u_n) \to \inf_{v\in H_0^1(\Omega)} I(v)$. Take $|I(u_n)|$ to be bounded. Then

$$\begin{aligned} \frac{1}{2} \int_\Omega |\mathrm{D}u_n(x)|^2 \,\mathrm{d}x &\leq \int_\Omega |F(u_n(x))| \,\mathrm{d}x + C \\ &\leq C_2|\Omega| + C \\ &= C. \end{aligned}$$

Thus, we have weak convergence (of a subsequence) $u_n \rightharpoonup u$ in $H^1_0(\Omega)$. So

$$\liminf_{n\to\infty} \int_\Omega |\mathrm{D}u_n(x)|^p \,\mathrm{d}x \geq \int_\Omega |\mathrm{D}u(x)|^p \,\mathrm{d}x$$

and

$$\lim_{n\to\infty} \int_\Omega F(u_n(x)) \,\mathrm{d}x = \int_\Omega F(u(x)) \,\mathrm{d}x,$$

by Lebesgue's dominated convergence theorem, since $u_n \to u$ in $L^2(\Omega)$, so $u_n(x) \to u(x)$ for almost all $x \in \Omega$, so $F(u_n(x)) \to F(u(x))$ and $|F(u_n(x))|$ is uniformly bounded in n and x. Then

$$I(u) \leq \liminf_{n\to\infty} I(u_n) = \inf_{v\in H^1_0(\Omega)} I(v),$$

so u is indeed a minimizer of I. □

We cannot prove uniqueness as the functional I is not convex.

We now consider the Poisson equation with Neumann BCs:

$$\begin{cases} -\Delta u(x) = f(x), & x \in \Omega; \\ \frac{\partial u}{\partial \nu}(x) = 0, & x \in \partial\Omega. \end{cases} \tag{6.3}$$

The weak formulation of (6.3) is

$$\int_\Omega \mathrm{D}u(x)\cdot \mathrm{D}\phi(x)\,\mathrm{d}x - \int_\Omega f(x)\phi(x)\;\;dx = 0 \quad \forall \phi \in H^1(\Omega). \tag{6.4}$$

The weak formulation (6.4) is equivalent to finding a minimizer $u \in H^1(\Omega)$ for

$$I(u) := \frac{1}{2}\int_\Omega |\mathrm{D}u(x)|^2\,\mathrm{d}x - \int_\Omega f(x)u(x)\,\mathrm{d}x.$$

Theorem 6.3. *If $\int_\Omega f(x)\,\mathrm{d}x = 0$, then there is a unique (up to a constant) minimizer of I in $H^1(\Omega)$.*

Proof. Is I bounded below?

$$\begin{aligned} I(u) &= \frac{1}{2}\int_\Omega |\mathrm{D}u(x)|^2\,\mathrm{d}x - \int_\Omega f(x)u(x)\,\mathrm{d}x \\ &= \frac{1}{2}\int_\Omega |\mathrm{D}u(x)|^2\,\mathrm{d}x - \int_\Omega f(x)\left(u(x) - \fint_\Omega u\right)\,\mathrm{d}x. \end{aligned}$$

$$\begin{aligned} \left|\int_\Omega f(x)\left(u(x) - \fint_\Omega u\right)\,\mathrm{d}x\right| &\leq \frac{1}{2\varepsilon}\int_\Omega f(x)^2\,\mathrm{d}x + \frac{\varepsilon}{2}\int_\Omega \left(u(x) - \fint_\Omega u\right)^2\,\mathrm{d}x \\ &\leq \frac{1}{2\varepsilon}\int_\Omega f(x)^2\,\mathrm{d}x + \frac{C\varepsilon}{2}\int_\Omega |\mathrm{D}u(x)|^2\,\mathrm{d}x \end{aligned}$$

by Poincaré's Inequality. If we take $\varepsilon < \frac{1}{C}$,

$$I(u) \geq -\frac{1}{2\varepsilon}\int_\Omega |f(x)|^2\,\mathrm{d}x.$$

Let (u_n) be a minimizing sequence for I with $|I(u_n)| \leq C'$. Then

$$\begin{aligned}\frac{1}{2}\int_\Omega |\mathrm{D}u_n(x)|^2\,\mathrm{d}x &\leq \left|\int_\Omega f(x)\left(u_n(x) - \fint_\Omega u_n\right)\mathrm{d}x\right| + C' \\ &\leq \frac{1}{2\varepsilon}\int_\Omega f(x)^2\,\mathrm{d}x + \frac{C\varepsilon}{2}\int_\Omega |\mathrm{D}u_n(x)|\,\mathrm{d}x + C' \\ &\leq \frac{1}{2\varepsilon}\int_\Omega f(x)^2\,\mathrm{d}x + \frac{1}{4}\int_\Omega |\mathrm{D}u_n(x)|^2\,\mathrm{d}x + C',\end{aligned}$$

if $C\varepsilon < \frac{1}{2}$. Hence $\int_\Omega |\mathrm{D}u_n(x)|^2\,\mathrm{d}x$ is bounded uniformly in n. So

$$\int_\Omega \left|u_n(x) - \fint_\Omega u_n\right|^2 \mathrm{d}x \leq C\int_\Omega |\mathrm{D}u_n(x)|^2\,\mathrm{d}x \leq C'.$$

Set $v_n := u_n - \fint_\Omega u_n$. (v_n) is bounded in $L^2(\Omega)$, and $(\mathrm{D}v_n) = (\mathrm{D}u_n)$ is bounded in $L^2(\Omega)$ also, so $v_n \rightharpoonup v$ in $H^1(\Omega)$.

$$\begin{aligned}&\frac{1}{2}\int_\Omega \left|\mathrm{D}\left(u_n(x) - \fint_\Omega u_n\right)\right|^2 \mathrm{d}x - \int_\Omega f(x)\left(u_n(x) - \fint_\Omega u_n\right)\mathrm{d}x \\ &= I(v_n) \\ &= I(u_n) \\ &= \frac{1}{2}\int_\Omega |\mathrm{D}u_n(x)|^2\,\mathrm{d}x - \int_\Omega f(x)u_n(x)\,\mathrm{d}x.\end{aligned}$$

$$\begin{aligned}\inf_{u\in H^1(\Omega)} I(u) &= \liminf_{n\to\infty} I(u_n) \\ &= \liminf_{n\to\infty} I(v_n) \\ &\geq I(v)\end{aligned}$$

since $\liminf_{n\to\infty}\int_\Omega |\mathrm{D}v_n(x)|^2\,\mathrm{d}x \geq \int_\Omega |\mathrm{D}v(x)|^2\,\mathrm{d}x$ and $\lim_{n\to\infty}\int_\Omega f(x)v_n(x)\,\mathrm{d}x = \int_\Omega f(x)v(x)\,\mathrm{d}x$. So $v \in H^1(\Omega)$ is a minimizer of I (unique up to an additive constant since adding a constant to v does not change the energy). ☐

6.2 Monotonicity Methods

Consider the problem

$$\begin{cases} \nabla\cdot(a(\mathrm{D}u(x))) = f(x), & x \in \Omega; \\ u(x) = 0, & x \in \partial\Omega; \end{cases} \tag{6.5}$$

where $f \in L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$ is open and bounded, and $a : \mathbb{R}^n \to \mathbb{R}^n$. This can be seen as a problem of the form $Au = f$, $A : X \to X^*$. In general, we have no hope of solving this problem, but there are some accessible cases: when $a(p) = \mathrm{D}F(p)$, we can use the calculus of variations, as with the p-Laplacian (6.1).

Definition 6.4. A vector field $a : \mathbb{R}^n \to \mathbb{R}^n$ is called *monotone* if

$$(a(p) - a(q))\cdot(p - q) \geq 0 \quad \forall p, q \in \mathbb{R}^n.$$

We assume that a is monotone and that it satisfies the growth condition $|a(p)| \leq C(1+|p|)$ and the coercivity condition $a(p) \cdot p \geq \alpha |p|^2 - \beta$.

Our plan is to take a basis $\{\omega_k\}_{k=1}^\infty$ of $H_0^1(\Omega)$ consisting of (renormalized) eigenfunctions of $-\Delta$. We then project the equation to get

$$u_m = \sum_{k=1}^m d_k^m \omega_k.$$

We want to solve

$$\int_\Omega a(\mathrm{D}u_m(x)) \cdot \mathrm{D}\omega_k(x) \, \mathrm{d}x = \int_\Omega f(x)\omega_k(x) \, \mathrm{d}x \quad \forall k = 1, \ldots, m$$

to find u_m, and show that u_m converges to u in some sense. Finally, we show that u satisfies (6.5).

Theorem 6.5. *For each $m \in \mathbb{N}$, there is a u_m of the form*

$$u_m = \sum_{k=1}^m d_k^m \omega_k$$

that solves

$$\int_\Omega a(\mathrm{D}u_m(x)) \cdot \mathrm{D}\omega_k(x) \, \mathrm{d}x = \int_\Omega f(x)\omega_k(x) \, \mathrm{d}x \quad \forall k = 1, \ldots, m.$$

Proof. For $k = 1, \ldots, m$, define $v_k : \mathbb{R}^m \to \mathbb{R}$ by

$$v_k(d) := \int_\Omega a\left(\sum_{j=1}^m d_j \mathrm{D}\omega_j(x)\right) \cdot \mathrm{D}\omega_k(x) \, \mathrm{d}x - \int_\Omega f(x)\omega_k(x) \, \mathrm{d}x,$$

for $d = (d_1, \ldots, d_m) \in \mathbb{R}^m$. Set $v(d) := (v_1(d), \ldots, v_m(d))$, so $v : \mathbb{R}^m \to \mathbb{R}^m$. We want $v(d) \cdot d \geq 0$ if $|d| = r$ is large enough.

Lemma 6.6. *If $v : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous vector field and, for some $r > 0$, $v(d) \cdot d \geq 0$ for all $d \in \mathbb{B}_r(0)$, then there exists $d^0 \in \mathbb{B}_r(0)$ such that $v(d^0) = 0$.*

Now,

$$\begin{aligned}
v(d) \cdot d &= \int_\Omega a\left(\sum_{k=1}^m d_k \mathrm{D}\omega_k(x)\right) \cdot \left(\sum_{k=1}^m d_k \mathrm{D}\omega_k(x)\right) \mathrm{d}x - \int_\Omega f(x)\left(\sum_{k=1}^m d_k \mathrm{D}\omega_k(x)\right) \mathrm{d}x \\
&\geq \alpha \int_\Omega \left|\sum_{k=1}^m d_k \mathrm{D}\omega_k(x)\right|^2 - \beta - f(x)\left(\sum_{k=1}^m d_k \mathrm{D}\omega_k(x)\right) \mathrm{d}x \\
&= \alpha |d|^2 - \beta|\Omega| - \sum_{k=1}^m d_k (f, \omega_k)_{L^2(\Omega)} \\
&\geq \frac{\alpha}{2}|d|^2 - \beta|\Omega| - C\sum_{k=1}^m (f, \omega_k)_{L^2(\Omega)}^2 \\
&\geq \frac{\alpha}{2}|d|^2 - C \text{ since } \{\omega_k\} \text{ is an orthonormal set.}
\end{aligned}$$

So for $|d| = r$ large enough, $v(d) \cdot d \geq 0$. By Lemma 6.6, there exists $d^0 \in \mathbb{B}_r(0)$ with $v(d^0) = 0$:

$$\int_\Omega a\left(\sum_{k=1}^m d_k^{0,m}\omega_k(x)\right) \cdot \mathrm{D}\omega_k(x)\,\mathrm{d}x = \int_\Omega f(x)\omega_k(x)\,\mathrm{d}x \quad \forall m \in \mathbb{N}. \qquad \square$$

Now for some compactness:

$$\int_\Omega a(\mathrm{D}u_m(x)) \cdot \mathrm{D}\omega_k(x)\,\mathrm{d}x = \int_\Omega f(x)\omega_k(x)\,\mathrm{d}x \quad \forall k = 1, \ldots, m.$$

$$\begin{aligned}
\alpha \int_\Omega |\mathrm{D}u_m(x)|^2\,\mathrm{d}x - \beta|\Omega| &\leq \int_\Omega a(\mathrm{D}u_m(x)) \cdot \mathrm{D}u_m(x)\,\mathrm{d}x \\
&= \int_\Omega f(x)u_m(x)\,\mathrm{d}x \\
&\leq C\int_\Omega |f(x)|^2\,\mathrm{d}x + \frac{\alpha}{2}\int_\Omega |\mathrm{D}u_m(x)|^2\,\mathrm{d}x
\end{aligned}$$

by the Cauchy–Schwarz and Friedrichs inequalities. Therefore, we have weak convergence (of a subsequence) $u_m \rightharpoonup u$ in $H_0^1(\Omega)$ and strong convergence $u_m \to u$ in $L^2(\Omega)$. However, $a(\mathrm{D}u_m) \not\to a(\mathrm{D}u)$. We now have

Theorem 6.7. *u as constructed above is a weak solution of* (6.5).

Proof. $u_m \rightharpoonup u$ in $H_0^1(\Omega)$ and $u_m \to u$ in $L^2(\Omega)$. By the growth condition, $|a(\mathrm{D}u_m)| \leq C(1+|\mathrm{D}u_m|)$, so $\int_\Omega |a(\mathrm{D}u_m)|^2 \leq C'$, so (a subsequence of) $a(\mathrm{D}u_m)$ converges weakly to some ξ in $L^2(\Omega)^n$. For all $m \in \mathbb{N}$ and $w \in H_0^1(\Omega)$,

$$\int_\Omega (a(\mathrm{D}u_m(x)) - a(\mathrm{D}w(x))) \cdot (\mathrm{D}u_m(x) - \mathrm{D}w(x))\,\mathrm{d}x \geq 0.$$

For all $v \in H_0^1(\Omega)$,

$$\int_\Omega f(x)v(x)\,\mathrm{d}x = \int_\Omega a(\mathrm{D}u_m(x)) \cdot \mathrm{D}v(x)\,\mathrm{d}x \underset{m\to\infty}{\longrightarrow} \int_\Omega \xi(x) \cdot \mathrm{D}v(x)\,\mathrm{d}x = \int_\Omega f(x)v(x)\,\mathrm{d}x.$$

Also,

$$\int_\Omega a(\mathrm{D}u_m(x)) \cdot \mathrm{D}u_m(x)\,\mathrm{d}x = \int_\Omega f(x)u_m(x)\,\mathrm{d}x \underset{m\to\infty}{\longrightarrow} \int_\Omega f(x)u(x)\,\mathrm{d}x.$$

So

$$\underbrace{\int_\Omega a(\mathrm{D}u_m(x)) \cdot \mathrm{D}u_m(x)\,\mathrm{d}x}_{\to\int_\Omega fu} - \underbrace{\int_\Omega a(\mathrm{D}u_m(x)) \cdot \mathrm{D}w(x)\,\mathrm{d}x}_{\to\int_\Omega \xi\cdot\mathrm{D}w} - \int_\Omega a(\mathrm{D}w(x)) \cdot (\mathrm{D}u_m(x) - \mathrm{D}w(x))\,\mathrm{d}x \geq 0.$$

Take $w = u + \lambda v$, $\lambda > 0$ real:

$$\int_\Omega (\xi(x) - a(\mathrm{D}u(x) + \lambda\mathrm{D}v(x))) \cdot \mathrm{D}v(x)\,\mathrm{d}x \geq 0.$$

Taking the limit $\lambda \searrow 0$ implies that, for all $v \in H_0^1(\Omega)$,

$$\int_\Omega (\xi(x) - a(\mathrm{D}u(x))) \cdot \mathrm{D}v(x)\,\mathrm{d}x = 0.$$

Therefore,

$$\int_\Omega a(\mathrm{D}u(x)) \cdot \mathrm{D}v(x)\,\mathrm{d}x = \int_\Omega f(x)v(x)\,\mathrm{d}x \quad \forall v \in H_0^1(\Omega). \qquad \square$$

7 Review

1. Sobolev spaces.

   (a) Sobolev spaces are separable Banach spaces. They have various equivalent norms.

   (b) Rellich–Kondrachov (theorem 2.39): $W^{1,p}(\Omega) \subset\subset L^q(\Omega)$ for $1 \leq q < p^*$.

   (c) $C^\infty(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$ (theorems 2.16 and 2.17).

   (d) Trace generalizes restriction to the boundary and gives an alternative definition of $W_0^{1,p}(\Omega)$ (theorem 2.29).

   (e) Sobolev functions can be extended to $\mathbb{R}^n$ (theorem 2.27).

   (f) The relationship between $W^{1,p}(\mathbb{R}^n)$ and difference quotients.

2. Inequalities.

   (a) Friedrichs' Inequality (Theorem 2.23).

   (b) Gagliardo–Nirenberg Inequality (theorems 2.33 and 2.35): $\|u\|_{L^{p^*}(\Omega)} \leq C_{\Omega,p}\|u\|_{W^{1,p}(\Omega)}$.

   (c) Morrey's inequality (theorem 2.44): $\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C_{n,p}\|u\|_{W^{1,p}(\mathbb{R}^n)}$.

   (d) Sobolev inclusions.

   (e) Poincaré's inequality (theorem 2.37): $\int_\Omega \left|u(x) - ⨍_\Omega u\right|^p \,\mathrm{d}x \leq C_{\Omega,p} \int_\Omega |\mathrm{D}u(x)|^p \,\mathrm{d}x$.

3. Calculus of variations.

   (a) There is a correspondence between the solution of an Euler-Lagrange equation for a variational problem and the solution of a PDE.

   (b) Calculus of variations can be used to solve Poisson's equation with zero boundary conditions, by finding a minimizer for an appropriate integral functional.

   (c) If we can prove that the functional is strictly convex, then we can use this to show that the minimizer/solution is unique.

   (d) We can also solve the Neumann problem using calculus of variations.

4. Linear elliptic PDEs.

   (a) Weak solutions; the weak formulation of the PDE.

   (b) Lax–Milgram theorem (theorem 3.4).

   (c) Energy estimates (*a priori* bounds on the derivatives).

   (d) Existence and uniqueness of solutions.

   (e) Fredholm Alternative for elliptic operators.

   (f) Eigenvalues and eigenfunctions (say, of $-\Delta$ on $\mathbb{B}_1(0)$).

   (g) Regularity estimates using difference quotients.

   (h) Weak and strong maximum principles.

5. Linear parabolic PDEs.
   (a) Weak solutions lie in appropriate Bochner spaces.
   (b) Galerkin approximations can be used to find solutions.
   (c) To prove convergence of these approximations, we need energy estimates.
   (d) Existence and uniqueness of solutions.
   (e) Regularity estimates.
   (f) Weak and strong maximum principles.