INTRODUCTORY NOTES FOR PDE's

Partial Derivatives
For a function

$$u = u(x, y)$$

we define

$$\frac{\partial u}{\partial x} = \lim_{\delta x \to 0} \frac{u(x + \delta x, y) - u(x, y)}{\delta x}. \tag{1}$$

We call $\frac{\partial u}{\partial x}$ the 'partial derivative of u with respect to x.

$$\frac{\partial u}{\partial y} = \lim_{\delta x \to 0} \frac{u(x, y + \delta y) - u(x, y)}{\delta y}. \tag{2}$$

$\frac{\partial u}{\partial x}$ is sometimes also written $\partial_x u$ or u_x.

EXAMPLES
For

$$u = x/y,$$

we have

$$\frac{\partial u}{\partial x} = \lim_{\delta x \to 0} \frac{u(x + \delta x, y) - u(x, y)}{\delta x} = \lim_{\delta x \to 0} \frac{x + \delta x - x}{y \delta x} = \frac{1}{y}.$$

In fact, we can use all the norml 'rules' for differentiating (just think of the 'other' variables as constants):

$$\begin{aligned}
\frac{\partial}{\partial x} x^n f(y) &= n x^{n-1} f(y) \\
\frac{\partial}{\partial x} e^{\alpha x} f(y) &= \alpha e^{\alpha x} f(y) \\
\frac{\partial}{\partial x} \sin xy &= y \cos xy \\
\frac{\partial}{\partial x} \left(\frac{x}{y} \right) &= -\frac{x}{y^2}
\end{aligned}$$

Higher Order Derivatives
We define higher order derivatives as follows:

$$\frac{\partial^2 u}{\partial x^2} = \lim_{\delta x \to 0} \left[\frac{\partial u}{\partial x}(x + \delta x, y) - \frac{\partial u}{\partial x}(x, y) \right] / \delta x \tag{3}$$

(Here $\frac{\partial u}{\partial x}(x + \delta x, y)$ denotes $\frac{\partial u}{\partial x}$ evaluated at $(x + \delta x, y)$.) There is also a corresponding definition for $\frac{\partial^2 u}{\partial y^2}$.

The mixed second order derivative is

$$\frac{\partial^2 u}{\partial x \partial y} = \lim_{\delta x \to 0} \left[\frac{\partial u}{\partial y}(x + \delta x, y) - \frac{\partial u}{\partial y}(x, y) \right] / \delta x. \tag{4}$$

The following notations are used interchangeably

$$\frac{\partial^2 u}{\partial x^2} \quad \text{or} \quad u_{xx} \quad \text{or} \quad \partial_x^2 u.$$

We will assume that partial derivatives commute, *ie*

$$\partial_x \partial_y u = \partial_y \partial_x u. \tag{5}$$

EXAMPLE

$$\begin{aligned}
u &= x/y, \quad u_x = \frac{1}{y}, \ u_y = -\frac{x}{y^2} \\
u_{xx} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{y}\right) = 0, \\
u_{yy} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{-x}{y^2}\right) = \frac{2x}{y^3}, \\
u_{xy} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{-x}{y^2}\right) = -\frac{1}{y^2}, \\
u_{yx} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{1}{y}\right) = -\frac{1}{y^2}.
\end{aligned}$$

As expected $u_{xy} = u_{yx}$.

The Chain Rule

Suppose $u = u(x_1, x_2)$ and suppose that the variables x_i are themselves functions of other variables y_1 and y_2, so

$$x_1 = x_1(y_1, y_2) \quad x_2 = x_2(y_1, y_2),$$

ie a change of variables. An obvious example is the change of variables from cartesian to polar coordinates.

To find what are $\frac{\partial u}{\partial y_1}$ and $\frac{\partial u}{\partial y_2}$, we can use the chain rule generalized to more than one variable. For functions of two variables this states

$$\frac{\partial u}{\partial y_i} = \sum_{j=1}^{2} \frac{\partial u}{\partial x_j} \frac{\partial x_j}{\partial y_i}. \tag{6}$$

The proof of this is a fairly straightforward extension of the 1 variable case (see Analysis lectures).

It is important to beware of the problems associated with this notation. The notation assumes that it is obvious which variables are being held constant when differentiating. When we write $\frac{\partial x_1}{\partial y_2}$, we should understand that y_1 is being held constant. When we write $\frac{\partial y_2}{\partial x_1}$, we should understand that x_2 is being held constant. This means that in general we cannot assume that

$$\frac{\partial x_1}{\partial y_2} = 1 / \frac{\partial y_2}{\partial x_1}.$$

EXAMPLE

Let

$$\begin{aligned}
x_1 &= y_1 + y_2 \quad \text{so} \quad \frac{\partial x_1}{\partial y_1} = 1, \; \frac{\partial x_1}{\partial y_2} = 1 \\
x_2 &= y_1 - y_2 \quad \text{so} \quad \frac{\partial x_2}{\partial y_1} = 1, \; \frac{\partial x_2}{\partial y_2} = -1.
\end{aligned}$$

The inverse transformation is

$$\begin{aligned}
y_1 &= \frac{x_1 + x_2}{2} \quad \text{so} \quad \frac{\partial y_1}{\partial x_1} = \frac{1}{2}, \quad \frac{\partial y_1}{\partial x_2} = \frac{1}{2} \\
y_2 &= \frac{x_1 - x_2}{2} \quad \text{so} \quad \frac{\partial y_2}{\partial x_1} = \frac{1}{2}, \; \frac{\partial y_2}{\partial x_2} = -\frac{1}{2}.
\end{aligned}$$

Clearly $\frac{\partial x_1}{\partial y_2} \neq 1 / \frac{\partial y_2}{\partial x_1}$ in this case.

Using the chain rule we obtain

$$\begin{aligned}
\frac{\partial u}{\partial y_1} &= \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial y_1} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial y_1} = \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2}, \\
\frac{\partial u}{\partial y_2} &= \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial y_2} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial y_2} = \frac{\partial u}{\partial x_1} - \frac{\partial u}{\partial x_2}.
\end{aligned}$$

Taylor Series
Differentiable functions $u(x, y)$ can be approximated in the neighbourhood of a point by polynomials. (You should be familiar with this idea for functions of one variable from Analysis courses. Taylor's theorem actually bounds the error involved in using an N'th order polynomial in terms of the $N + 1$'th derivative.)

Here we give the 'recipe' for finding the 'best' coefficients of a corresponding polynomial approximation for a function of two variables. We choose the coefficients in an N'th order polynomial so that the function and all its derivatives up to N'th order (including mixed derivatives) match those of the polynomial function.

Consider the point (x_0, y_0) and let $X = x - x_0$ and $Y = y - y_0$ and define the polynomial

$$g(x, y) = a_0 + a_1 X + a_2 Y + a_{11} X^2 + a_{22} Y^2 + a_{12} XY + \dots, \tag{7}$$

where $a_0, a_1, a_2, a_{11}, \dots$ are constants.

Matching terms up to second order at (x_0, y_0) gives

$$\begin{array}{ll} a_0 = g(x_0, y_0) = u(x_0, y_0) & \\ a_1 = g_x(x_0, y_0) = u_x(x_0, y_0) & a_2 = g_y(x_0, y_0) = u_y(x_0, y_0) \\ 2a_{11} = g_{xx}(x_0, y_0) = u_{xx}(x_0, y_0) & 2a_{22} = g_{yy}(x_0, y_0) = u_{yy}(x_0, y_0) \\ a_{12} = g_{xy}(x_0, y_0) = u_{xy}(x_0, y_0) & \end{array}$$

So the 2nd order Taylor series approximation to $u(x, y)$:

$$g_2(x, y) = u(x_0, y_0) + u_x X + u_y Y + \frac{1}{2}\left[u_{xx} X^2 + u_{yy} Y^2 + 2u_{xy} XY\right]$$

Here $X = x - x_0$, $Y = y - y_0$ and the derivatives u_x, u_y, u_{xx}, u_{xy} and u_{yy} are all evaluated at the point (x_0, y_0). The subscript '2' on g just emphasises that we have only kept terms up to second order.

We can write

$$u(x, y) = g_2(x, y) + R_2(x, y).$$

The generalization of Taylor's theorem then gives bounds on the remainder R_2 and gives

$$\lim_{r \to 0} \frac{R_2(x, y)}{r^2} = 0 \qquad \text{where} \qquad r^2 = (x - x_0)^2 + (y - y_0)^2.$$

EXAMPLE
Suppose we wish to find the 2nd order Taylor series for the function

$$u(x, y) = e^{xy} \qquad \text{about} \qquad (x, y) = (c, d).$$

Then the coefficients in the polynomial are found as follows:

$$\begin{array}{rclrclcl} & & & a_0 & = & u(c, d) & = & e^{cd} \\ u_x & = & ye^{xy} & a_1 & = & u_x(c, d) & = & de^{cd} \\ u_y & = & xe^{xy} & a_2 & = & u_y(c, d) & = & ce^{cd} \\ u_{xx} & = & y^2 e^{xy} & a_{11} & = & u_{yy}(c, d) & = & d^2 ecd \\ u_{yy} & = & x^2 e^{xy} & a_{22} & = & u_{xx}(c, d) & = & c^2 e^{cd} \\ u_{xy} & = & e^{xy} + xye^{xy} & a_{12} & = & u_{xy}(c, d) & = & e^{cd}(1 + cd) \end{array}$$

This gives

$$u(x, y) \approx e^{cd}\left[1 + dX + cY + \frac{1}{2}(d^2 X^2 + 2(1 + cd)XY + c^2 Y^2)\right]$$