Chapter 2
Partial Differential Equations

2.1: Equation

An equation of the form

$$G(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots) = 0$$

is a *partial differential equation*. Here x, y are the independent variables and u is the dependent variable.

The *order* of such an equation is the order of the highest-order partial derivative that appears.

The solution is usually defined only for some domain D:

$$u = u(x, y) \text{ for } (x, y) \in D$$

There is, of course, no need to insist on having merely two variables x, y; any finite number of variables greater than 1 gives a PDE.

Examples

(i) $u_{xt} - xt = 0$, which is 2$^{\text{nd}}$ order.

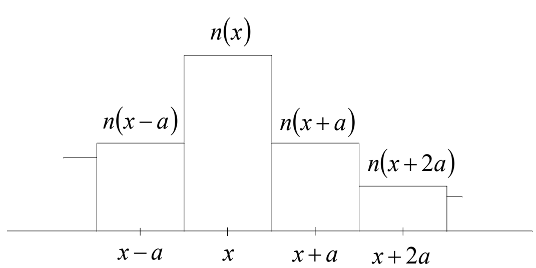
(ii) *The Diffusion Equation*

$$u_t - Du_{xx} = 0$$

In general D may be a function of x, y; we will consider it to be a constant.

The diffusion equation may be derived as a model for particles on a line.

Imagine particles on a line jumping a distance a at a rate $1/\tau$ but in a random direction.



Let $n(x,t)$ be the number of particles in $\left(x - \frac{a}{2}, x + \frac{a}{2}\right)$ at time t. At time $t + \delta t$,

$$n(x,t+\delta t) = n(x,t) - \underbrace{n(x,t)\frac{\delta t}{\tau}}_{\text{jumped out}} + \underbrace{\frac{1}{2}n(x-a,t)\frac{\delta t}{\tau}}_{\text{jumped in from right}} + \underbrace{\frac{1}{2}n(x+a,t)\frac{\delta t}{\tau}}_{\text{jumped in from left}}$$

$$\frac{n(x,t+\delta t)-n(x,t)}{\delta t} = \frac{1}{2\tau}\left(n(x+a,t)+n(x-a,t)-2n(x,t)\right)$$

We now write $n(x+a,t)$ and $n(x-a,t)$ using their Taylor series:

$$n(x+a,t) \approx n(x,t) + a\frac{\partial n}{dx}(x,t) + \frac{a^2}{2}\frac{\partial^2 n}{\partial x^2}(x,t)$$
$$n(x-a,t) \approx n(x,t) - a\frac{\partial n}{dx}(x,t) + \frac{a^2}{2}\frac{\partial^2 n}{\partial x^2}(x,t)$$

If we now substitute in and take the limit as $\delta t \to 0$ we get

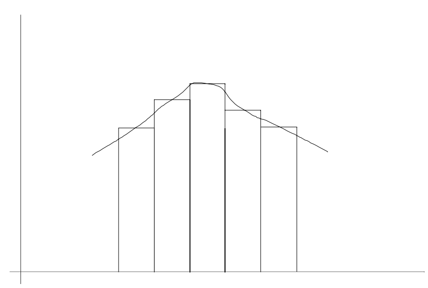
$$\frac{\partial n}{\partial t}(x,t) = \frac{a^2}{2\tau}\frac{\partial^2 n}{\partial x^2}$$

Or, using subscript notation and letting $D = a^2/2\tau$,

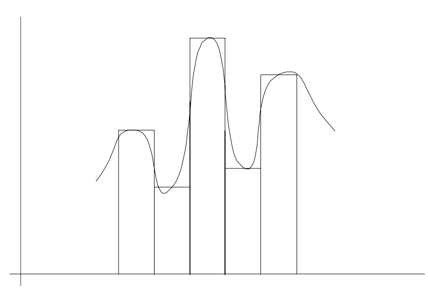
$$n_t - Dn_{xx} = 0$$

We have assumed that $n(x,t)$ varies slowly on the scale of a so that discarding third- and higher-order terms is a valid approximation.

Slow – Taylor good:



Fast – Taylor poor:



We may also define a net flow rate for the particles. The net flow from $x-a$ into x in time δt is

$$-\frac{n(x,t)}{2}\frac{\delta t}{\tau}+\frac{n(x-a,t)}{2}\frac{\delta t}{\tau}=-\frac{1}{2}\left(\frac{n(x,t)-n(x-a,t)}{a}\right)a\frac{\delta t}{\tau}=-\frac{a}{2}\frac{\partial n}{\partial x}\frac{\delta t}{\tau}$$

The flow per unit time is then

$$j(x,t)\approx -\alpha\frac{\partial n}{\partial x} \text{ where } \alpha=\frac{a}{2\tau}$$

These 'particles' we have modelled might be:

(i) 'particles of energy' – $u(x,t)$ is the energy per unit length i.e. temperature; j is the heat flow.

(ii) in 2 or more dimensions, the 'particles' might be rabid foxes / random walkers. Instead of a line we have an area, so $u=u(x,y,t)$.

(iii) *The Wave Equation*

$$u_{tt}-c^2u_{xx}=0$$

This equation models the oscillations of a string; it will be derived later.

(iv) *Laplace's Equation*

$$\nabla^2 u=u_{xx}+u_{yy}=0$$

(v) *Schrödinger's Equation*

In quantum mechanics a particle is described by a wave function, Ψ, which satisfies

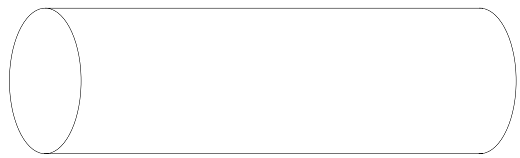
$$\Psi_t-i\Psi_{xx}=0$$

where $i^2=-1$.

2.2: Boundary Conditions

For ODEs initial / boundary values fix the arbitrary constants of integration and thus pick out a unique solution. Similarly, PDEs are often accompanied by initial / boundary conditions. (A condition given at $t=0$ in a problem involving time is usually called an 'initial' condition.)

e.g. The Diffusion Equation



If the bar has ends at $x = 0, L$ we might have

(i) constraints on u at $x = 0, L$ e.g. heat sink, bath of fixed / controlled temperature:

$$u(0,t) = T_0(t) \text{ and } u(L,t) = T_L(t)$$

(ii) a constraint on the flow $j = -\alpha u_x$:

$$j(0,t) = F_0(t) \text{ and } j(L,t) = F_L(t)$$

Insulating the bar should prevent flow into or out of the bar, i.e.

$$u_x\big|_{x=0} = u_x\big|_{x=L} = 0$$

PDEs with boundary conditions are usually called boundary value problems (BVPs).

2.3: Solutions to PDEs

Questions we might be interested in are:
(i) What are the solutions?
(ii) Are they unique?
(iii) Are they stable?

We will spend most time on (i) and (ii).

Instability: We say that a solution to a BVP

$$G(x, y, u, u_x, \ldots) = 0$$

subject to boundary conditions is 'unstable' if a small change in G or the boundary conditions changes the character of the solution. In fact, many solutions in nature are of this type, e.g. weather systems, turbulent flow.

2.4: Solving PDEs

We shall look at two methods for solving PDEs:
(i) direct integration;
(ii) separation of variables.

Direct Integration

If a PDE is in the form (or, after a change of variables, may be brought to the form)

$$\frac{\partial}{\partial y}\left(u(x,y)\right) = g(x,y)$$

then it may be solved by direct integration:

$$u(x,y) = \underbrace{\int^{y} g(x,y')dy'}_{\substack{\text{integrate keeping} \\ x \text{ constant}}} + f(x)$$

Here, f is an arbitrary function of x.

e.g.

$$u_{tx} = tx$$

Integrating with respect to t keeping x constant gives

$$\frac{\partial u}{\partial x} = \frac{1}{2}xt^2 + f(x)$$

Now integrate with respect to x:

$$u(x,t) = \frac{1}{4}x^2t^2 + g(x) + h(t)$$

There are infinitely many possible solutions to the PDE, corresponding to all the possible choices of g, h.

We expect that boundary conditions, if suitably chosen, will pick out the unique solution. However, some boundary conditions may lead to no solutions.

<u>Separation of Variables</u>

The idea is that a PDE may separate into a system of ODEs if written in the correct variables.

e.g.

$$u_x^2 + u_y^2 = 1$$

Try $u = \phi(x) + \psi(y)$:

$$(\phi'(x))^2 + (\psi'(y))^2 = 1 \Rightarrow (\phi'(x))^2 = 1 - (\psi'(y))^2$$

Here the RHS is a function only of x while the LHS is a function only of y. Since x, y are independent the only possible solution is the constant function, say α^2, so:

$$\frac{d\phi}{dx} = \alpha \Rightarrow \phi(x) = \alpha x + \beta_1$$

$$\frac{d\psi}{dy} = \sqrt{1-\alpha^2} \Rightarrow \psi(y) = y\left(1-\alpha^2\right)^{1/2} + \beta_2$$

$$u(x, y) = \alpha x + \left(1-\alpha^2\right)^{1/2} y + \beta$$

Whether or not the method of separation of variables works depends very much on the type of equation and on the boundary conditions.

We shall see that the method can cope with the diffusion equation and wave equation, which are linear equations, where we will make the substitution $u(x,t) = X(x)T(t)$.

2.5: Linearity and Homogeneity

It is usual to write PDEs in the form

$$\mathrm{L}u(x, y) = f(x, y)$$

Here L is an operator so that $\mathrm{L}u(x, y)$ contains all the terms involving u and its derivatives; $f(x, y)$ contains everything else i.e. the independent variables.

Examples

(i) $u_{xt} = xt$

$$\mathrm{L}u = \frac{\partial^2 u}{\partial x \partial t}, \; f(x,t) = xt$$

(ii) $u_t - Du_{xx} = 0$

$$\mathrm{L}u = \frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2}, \; f(x,t) = 0$$

(iii) $u_{tt} - c^2 u_{xx} = 0$

$$\mathrm{L}u = \frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2}, \; f(x,t) = 0$$

(iv) *The Korteweg - de Vries Equation*

$u_t + uu_x + ku_{xxx} = 0$, $k > 0$

$$\mathrm{L}u = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + k\frac{\partial^3 u}{\partial x^3}, \; f(x,t) = 0$$

This equation models waves propagating in shallow water.

We say the operator L is *linear* if

(i) $\mathrm{L}(u+w)=\mathrm{L}u+\mathrm{L}w$

(ii) $\mathrm{L}(cu)=c\,\mathrm{L}u$

We say that a PDE written in the above form is *homogeneous* if $f(x,y)=0$.

(i) $$\mathrm{L}(u+w)=\frac{\partial^2}{\partial x\partial t}(u+w)=\frac{\partial^2 u}{\partial x\partial t}+\frac{\partial^2 w}{\partial x\partial t}=\mathrm{L}u+\mathrm{L}w$$
$$\mathrm{L}(cu)=\frac{\partial^2}{\partial x\partial t}(cu)=c\frac{\partial^2 u}{\partial x\partial t}=c\,\mathrm{L}u$$
So L is linear and the equation is homogeneous.

(ii) $$\mathrm{L}(u+w)=\frac{\partial}{\partial t}(u+w)-D\frac{\partial^2}{\partial x^2}(u+w)=\frac{\partial u}{\partial t}+\frac{\partial w}{\partial t}-D\frac{\partial^2 u}{\partial x^2}-D\frac{\partial^2 w}{\partial x^2}=\mathrm{L}u+\mathrm{L}w$$
L is linear, and the equation is homogeneous.

(iii) As (ii) above.

(iv) $$\mathrm{L}(u+w)=(u+w)_t+(u+w)(u+w)_t+k(u+w)_{xxx}$$
$$\mathrm{L}(u+w)=u_t+uu_x+ku_{xxx}+w_t+ww_x+kw_{xxx}+uw_x+wu_x$$
$$\mathrm{L}(u+w)=\mathrm{L}u+\mathrm{L}w+uw_x+wu_x$$

So L is non-linear but the equation is homogeneous.

<u>The Principle of Superposition</u>

If the operator L is linear then we may establish the principle of superposition:

If $u_1,u_2,\ldots,u_N$ are arbitrary functions and $c_1,c_2,\ldots,c_N$ are arbitrary constants then

$$\mathrm{L}\left(\sum_{i=1}^{N}c_iu_i\right)=\sum_{i=1}^{N}c_i\,\mathrm{L}u_i$$

If a set of functions $\{u_j\}$ all satisfy the homogeneous equation

$$\mathrm{L}u_j=0$$

we may construct other solutions u by forming arbitrary linear combinations of the u_j:

$$u=\sum_j c_ju_j$$

$$\mathrm{L}u=\mathrm{L}\sum_j c_ju_j=\sum_j c_j\,\mathrm{L}u_j=\sum_j c_j0=0$$

In other words, solutions may be added.

For inhomogeneous equations

$$\mathrm{L}u = f$$

we may add to any solution u arbitrary combinations of the solutions of the corresponding homogeneous equation:

$$\mathrm{L}\left(u + \sum_j c_j u_j\right) = \mathrm{L}u + \mathrm{L}\sum_j c_j u_j = f + 0 = f$$