Chapter 6
Laplace's Equation

6.1: Introduction

The Laplace equation (in 2D) is

$$u_{xx} + u_{yy} = 0$$

This equation is important in many applications. We shall see that it describes steady-state (or equilibrium) solutions to both the wave and diffusion equations.

6.2: Steady-State Configuration

We showed in Exercise Sheet 2 that in 2D the diffusion equation becomes

$$u_t - D(u_{xx} + u_{yy}) = 0$$

and claimed that the equation of motion of a tight membrane was

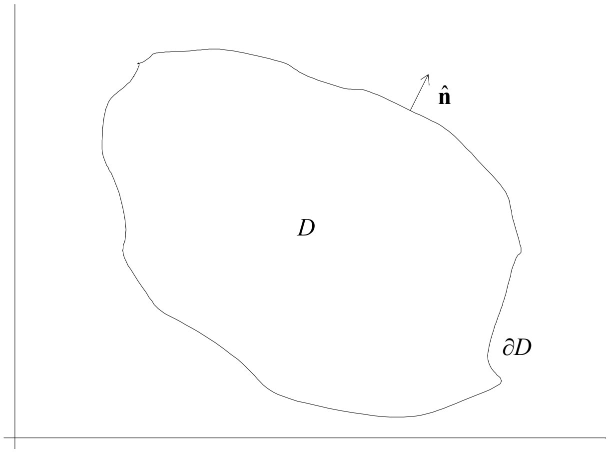
$$u_{tt} - D(u_{xx} + u_{yy}) = 0$$

In either case the steady-state solution (independent of t) satisfies Laplace's equation.

- For the diffusion of particles the solution will be the one satisfying boundary conditions for which there is no net flow.
- For the tight membrane the solution will be the one satisfying boundary conditions for which the net force on any element of the membrane is zero.

Boundary Conditions

A tight membrane or system of particles diffusing in 2D will usually have a boundary condition.



At the boundary we expect a constraint on u – the membrane might be fixed – giving rise to a boundary condition. There are two common types:

(i) *Dirichlet*

$$u = f \text{ for } (x, y) \in \partial D$$

Here f is an arbitrary function defined on ∂D.

(ii) *Neumann*

$$\nabla u \cdot \hat{\mathbf{n}} = f \text{ for } (x, y) \in \partial D$$

Again f is an arbitrary function defined on ∂D; $\hat{\mathbf{n}}$ is the outward unit normal.

6.3: Separation of Variables

Laplace's equation has been shown to separate in a number of co-ordinate systems.

Cartesian Co-ordinates

Try $u(x, y) = X(x)Y(y)$:

$$\Rightarrow \qquad \frac{X''}{X} = -\frac{Y''}{Y} = -k^2$$

$$\Rightarrow \qquad X(x) = \cos kx + A \sin kx$$

$$Y(y) = B \cosh kx + C \sinh kx$$

$$\Rightarrow \qquad u(x, y) = (\cos kx + A \sin kx)(B \cosh kx + C \sinh kx)$$

And so, by the principle of superposition, any linear combination of these solutions is also a solution.

Plane Polar Co-ordinates

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Try solutions of the form

$$u(r, \theta) = R(r)\Theta(\theta)$$

We shall require that $\Theta(\theta + 2\pi) = \Theta(\theta)$. So,

$$\Theta(\theta) = A\cos m\theta + B\sin m\theta$$

for $m = 0,1,2,\ldots$.

For $m = 0$:

$$R(r) = C \log \frac{r}{r_0}$$

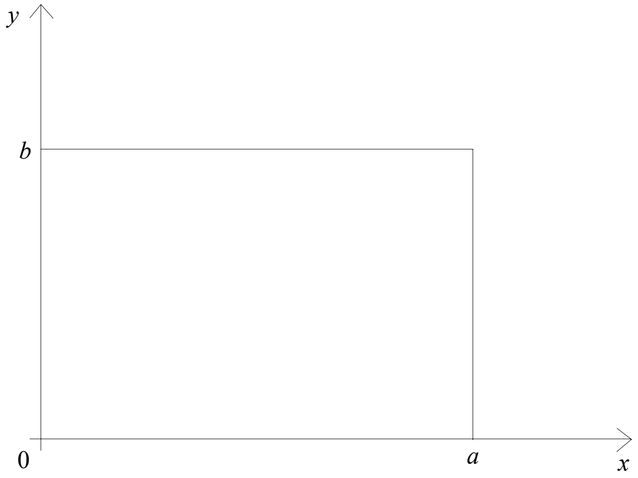
For $m \neq 0$:

$$R(r) = Dr^m + Er^{-m}$$

Problem

The problem is that separation of variables is not much help unless the boundary conditions are easily handled.

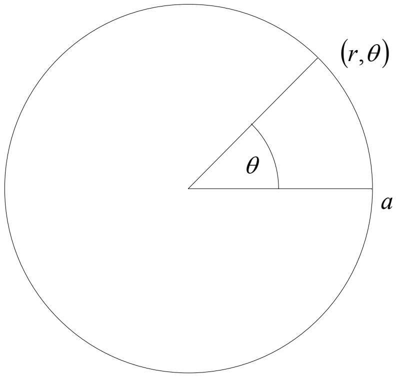
e.g. Boundary with rectangular symmetry.



$$\begin{aligned} u(x,0) &= f(x) & u(0,y) &= F(y) \\ u(x,b) &= g(x) & u(a,y) &= G(y) \end{aligned}$$

On each part of the boundary u is a function of one variable only.

e.g. Boundary with circular symmetry.



$$u(a,\theta)=f(\theta)$$

Again, the function is a 1-variable function on the boundary.

For an arbitrary boundary it is not in general possible to find a co-ordinate system in which Laplace's equation separates and for which the boundary conditions may be written in terms of one variable only.

6.4: Harmonic Functions and the Maximum Principle

A *harmonic function* is any function with continuous second-order derivatives satisfying Laplace's equation

$$u_{xx}+u_{yy}=0$$

in a certain domain $D\subset\mathbb{R}^2$.

The Maximum Principle

A harmonic function on a domain D attains its maximum and minimum on the boundary ∂D.

Intuitive argument:

Imagine that the harmonic function models a steady-state solution (i.e. $u_t=0$) for a system of diffusing particles. Suppose that the maximum occurs in the interior of D at (x_0,y_0). As the particles tend to diffuse from high to low concentrations they will move away from (x_0,y_0) thereby reducing $u(x_0,y_0)$. This contradicts the steady-state assumption. Hence $(x_0,y_0)\in D\setminus\partial D$ is not a maximum.

Idea of analytic argument:

At a maximum we need

$$u_{xx}\le 0,\ u_{yy}\le 0$$

by Taylor. Assuming $u_{xx}\neq 0$ we can write

$$u(x,y)-u(x_0,y_0)=\frac{1}{2}u_{xx}\big|_{(x_0,y_0)}X^2+\frac{1}{2}u_{yy}\big|_{(x_0,y_0)}Y^2+u_{xy}\big|_{(x_0,y_0)}XY+\ldots$$

where we have taken

$$X=x-x_0$$
$$Y=y-y_0$$

And so

$$u(x,y)-u(x_0,y_0)=\frac{1}{2}u_{xx}\left(X+\frac{u_{xy}Y}{u_{xx}}\right)^2+\frac{1}{2}Y^2\left(u_{yy}-\frac{u_{xy}^2}{u_{xx}}\right)$$

For a maximum we require that (i) $u_{xx}<0$ and (ii) $u_{yy}\le u_{xy}^2/u_{xx}\le 0$. However, $u_{xx}<0$ and $u_{yy}\le 0$, which contradicts our assumption that $u_{xx}+u_{yy}=0$ i.e. that u satisfies Laplace's equation.

To make this argument work for all cases we need the following lemma:

Lemma

Let $\rho(x,y)>0$ for $(x,y)\in D\subset\mathbb{R}^2$ and consider functions $v:\mathbb{R}^2\to\mathbb{R}$ with continuous second derivatives satisfying

$$v_{xx}+v_{yy}=\rho$$

Then v has no maximum in D.

Proof

By the previous argument v_{xx} and v_{yy} cannot both be zero.

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Proof of Maximum Principle

Suppose $u:\mathbb{R}^2\to\mathbb{R}$ satisfies $u_{xx}+u_{yy}=0$ for all $(x,y)\in D\subset\mathbb{R}^2$, and u does not attain its maximum value on ∂D. There must be at least one point $(x_0,y_0)\in D$ for which

$$u(x_0,y_0)>u_{\partial D}+\varepsilon$$

where

$$u_{\partial D}=\sup_{(x,y)\in\partial D}u(x,y)$$
$$\varepsilon>0$$

The function $v=u+\eta r^2$ ($r^2=(x-x_0)^2+(y-y_0)^2$) for a constant $\eta>0$ satisfies

$$v_{xx}+v_{yy}=u_{xx}+u_{yy}+4\eta=4\eta>0$$

Then

$$v(x_0,y_0)=u(x_0,y_0)>u_{\partial D}+\varepsilon=\left(v-\eta r^2\right)_{\partial D}+\varepsilon$$

If we choose η small enough that throughout D $\varepsilon - \eta r^2 > \varepsilon/2$ then we have

$$v(x_0, y_0) = v_{\partial D} + \varepsilon/2$$

Consequently v attains its maximum somewhere in D; but this contradicts the lemma. Hence, the supposition is false. So, u must attain its maximum somewhere on ∂D. Since $-u$ is also harmonic, u also attains its minimum on ∂D.

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Corollary

There is at most one harmonic function solving the Dirichlet problem for a closed domain D.

Proof

Suppose u_1, u_2 are solutions. Then $v = u_1 - u_2$ is also a solution.

$$v_{xx} + v_{yy} = 0$$
$$v(x, y) = 0 \quad \forall (x, y) \in \partial D$$

So, by the Maximum Principle, $\sup \operatorname{im}(v), \inf \operatorname{im}(v) \in \partial D$, so $\sup \operatorname{im}(v) = \inf \operatorname{im}(v) = 0$. So $v = 0$ everywhere. Hence, $u_1 = u_2$.

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