The Chain Rule and Changing Coordinates in Integration

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I think that now could be a good time to forestall any panic that you might be feeling at the horrible change of coordinates formulae you've just had thrown at you in 3D Geometry and Motion.

First, think of linear algebra. Consider $\mathbb{R}^n$ as a vector space. Remember that a choice of basis $e_1, \dots, e_n$ for $\mathbb{R}^n$ tells us how to write a vector $x \in \mathbb{R}^n$ in terms of coordinates $x^1, \dots, x^n$:

$$x = x^1 e_1 + \cdots + x^n e_n.$$

What happens if we change coordinates, i.e. pick a different basis $\tilde{e}_1, \dots, \tilde{e}_n$ for $\mathbb{R}^n$? We get new coordinates $\tilde{x}^1, \dots, \tilde{x}^n$:

$$x = \tilde{x}^1 \tilde{e}_1 + \cdots + \tilde{x}^n \tilde{e}_n.$$

As you now know, the map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ that takes e_i to $\tilde{e}_i$ is a linear map, and so it has a matrix.

Exercise. Write down the matrix of ϕ.

Now, you also know how to calculate the determinant of a matrix. Let C denote the unit cube in $\mathbb{R}^n$:

$$C := \{x \in \mathbb{R}^n | 0 \le x^j \le 1 \text{ for each } j\}.$$

Exercise. Convince yourself that $\det(\phi) =$ the volume of $\phi(C)$. In fact, if $S \subseteq \mathbb{R}^n$ has a volume, then $\det(\phi) = \operatorname{vol}(\phi(S))/\operatorname{vol}(S)$.

Example. This simple example should give you an idea of how to proceed. Consider $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\phi(1,0) = (1,0)$ and $\phi(0,1) = (a,b)$. The matrix of ϕ is

$$\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}$$

and so $\det(\phi) = b$. The "volume" (i.e. area) of C is 1. $\phi(C)$ is of course a parallelogram, which has area b.

Now let's change tack completely and think about change of variables in a nice 1-dimensional integral like

$$\int_a^b f(x)\, dx.$$

Back in school, you probably computed quite a few integrals by change of variables, setting $x = x(u)$ and

$$\int_a^b f(x) dx = \int_{\tilde{a}}^{\tilde{b}} f(x(u)) \frac{dx}{du}\, du,$$

where $x\left(\tilde{a}\right) = a$, $x\left(\tilde{b}\right) = b$.

Example.

$$\begin{aligned}
\int_0^1 \sqrt{1-x^2}\, dx &= \int_0^{\pi/2} \sqrt{1-\sin^2(u)} \frac{d(\sin(u))}{du}\, du \\
&= \int_0^{\pi/2} \cos(u)\cos(u)\, du \\
&= \int_0^{\pi/2} \frac{1}{2} + \frac{1}{2}\cos(2u)\, du \\
&= \frac{u}{2} + \frac{1}{4}\sin(2u)\Big|_{u=0}^{\pi/2} \\
&= \frac{\pi}{4}
\end{aligned}$$

where we have used the change of variables $x(u) = \sin(u)$ and the identity $\cos(2u) = \cos^2(u) - \sin^2(u)$, which implies that $\cos^2(u) = \frac{1}{2} + \frac{1}{2}\cos(2u)$.

Now, just ponder this, and think how it relates to the above comments we made about volume:

$$\cos(2u) = \det\left(\tfrac{d(x(u))}{du}\right).$$

We can generalize this: if $x^j = x^j(\tilde{x}^1, \ldots, \tilde{x}^n)$ for $j = 1, \ldots, n$, and

$$\begin{aligned}(x^1, \ldots, x^n) &= \phi(\tilde{x}^1, \ldots, \tilde{x}^n) \\ &= \big(x^1(\tilde{x}^1, \ldots, \tilde{x}^n), \ldots, x^n(\tilde{x}^1, \ldots, \tilde{x}^n)\big)\end{aligned}$$

then define the *Jacobian* of ϕ to be

$$D\phi(\tilde{x}^1, \ldots, \tilde{x}^n) = \begin{pmatrix} \frac{\partial x^1}{\partial \tilde{x}^1} & \cdots & \frac{\partial x^n}{\partial \tilde{x}^1} \\ \vdots & \ddots & \vdots \\ \frac{\partial x^1}{\partial \tilde{x}^n} & \cdots & \frac{\partial x^n}{\partial \tilde{x}^n} \end{pmatrix}.$$

So, when you next see a 2- or 3-dimensional change of coordinates in an integration, as in

$$\begin{aligned}&\int_\Omega f(x^1, \ldots, x^n)\, d(x^1, \ldots, x^n) \\ &= \int_{\phi^{-1}(\Omega)} f(\phi(\tilde{x}^1, \ldots, \tilde{x}^n)) \det(D\phi(\tilde{x}^1, \ldots, \tilde{x}^n))\, d(\tilde{x}^1, \ldots, \tilde{x}^n),\end{aligned}$$

where ϕ gives the old coordinates $x^1, \ldots, x^n$ in terms of the new coordinates $\tilde{x}^1, \ldots, \tilde{x}^n$, *don't panic!* Just remember your linear algebra and change of variables in one dimension.

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