A Spatial Discretization of the Wave Equation

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Introduction

- We are examining solutions to a *spatially discrete wave equation* – time remains continuous.
- We will see how solutions to this equation are similar to and differ from the classical continuum wave equation.

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Wave Equation in $\mathbb{R}^n$

Recall the continuous Laplacian and continuum wave equation:

$$\Delta f(x) := \sum_{j=1}^{n} \frac{\partial^2 f}{\partial (x^j)^2}(x)$$

$$\ddot{u}(x,t) = c^2 \Delta u(x,t)$$

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Wave Equation in $h\mathbb{Z}^n$

The discrete Laplacian (Newtonian approximation to second derivative in each coordinate) and the spatially discretized wave equation:

$$\Delta_h f(x) := \sum_{j=1}^{n} \frac{f(x+he_j) - 2f(x) + f(x-he_j)}{h^2}$$

$$\ddot{u}(x,t) = \omega^2 h^2 \Delta_h u(x,t)$$

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Oscillating Lattice

- Rectangular lattice $\Lambda = h\mathbb{Z}^n$.
- Particles of mass m.
- Springs of elasticity κ connecting nearest-neighbour particles.
- $u(x, t)$ = displacement of particle x from rest at time t.
- Discrete wave equation with $\omega^2 = \kappa / m$.
- Initial conditions: displacement of particle at origin by u_0 – corresponds to a δ-function at 0 in the continuum case.

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Numerical Simulation

- Fairly easy to simulate numerically for dimensions 1, 2, 3, e.g. using C++.
- Take care to make discrete time step small compared to lattice spacing (CFL condition).
- Both numerical and graphical output.

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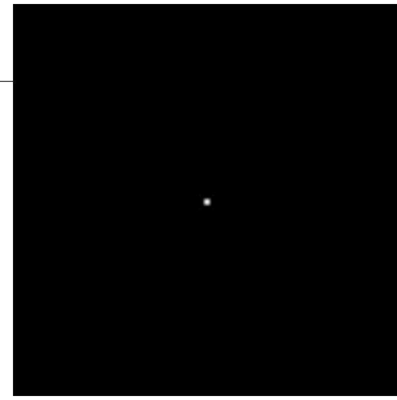
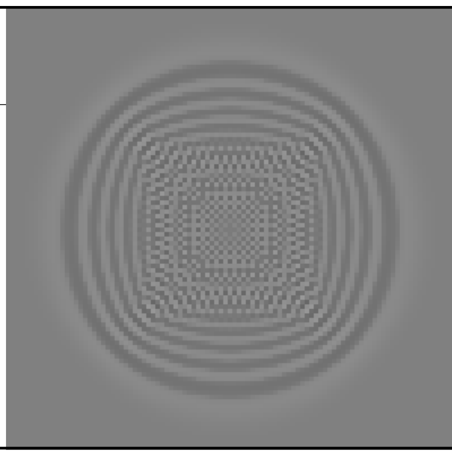
2D Video Clip

- Colour indicates direction of displacement – oscillating along a line so two colours: one ‘up’, one ‘down’. Black = no displacement.
- Higher brightness indicates greater displacement from equilibrium.
- Brightness scaled so that greatest displacement is always represented by brightest tone on palette.





Main Phenomena

- Monotone wavefront; group and phase velocities.
- Maximum displacement.
- Asymptotic synchronization.
- Equipartition and distribution (dispersal) of energy.

Wavespeeds

$$\ddot{q}(x,t) = c^2 \Delta q(x,t)$$

$$\ddot{q}(x,t) = \omega^2 h^2 \Delta_h q(x,t)$$

- The product ωh in the spatially discrete wave equation has taken the place of c in the continuum wave equation.
- We expect wavespeeds to depend on ωh.

The Monotone Wavefront

- Continuum case: can show that $u(x,t) = 0$ outside light cone with slope c, independent of dimension.
- Discrete case: we observe that $u(x,t) > 0$ outside the light cone with slope ωh, independent of dimension.

The Monotone Wavefront

- The notion of a true light cone (and the relativistic notion of "elsewhere") has been destroyed by the spatial discretization.
- This breakdown of finite wavespeed if caused by the instantaneous nearest-neighbour interactions.
- Not too bad – super-exponential decay beyond the light cone.

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Group and Phase Velocities

- "Velocity of energy transport" and "velocity of the constituent medium."
- In the continuum case, group and phase velocities are both precisely c for all wavenumbers (frequencies).
- In the discrete case, group and phase velocities vary with wavenumber, are bounded above by ωh and attain this bound.

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Consequences

- So, despite the disappearance of the light cone, the system still transports energy / information at a finite speed.
- Energy is carried by different wavenumbers at different speeds.

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Maximum Displacement

- Continuum case: can show that $|u(x_\star(t), t)|$ is maximal for $x_\star(t)$ on the light cone of slope c, regardless of dimension.
- Discrete case: find an $x_\star(t)$ with $|u(x_\star(t), t)|$ maximal. We observe that

$$d_{\mathbb{R}^n}(0, x_\star(t)) = \|x_\star(t)\| \approx \frac{\omega h t}{\sqrt{n}}$$

$$|x^1_\star(t)| = \cdots = |x^n_\star(t)| \approx \frac{\omega h t}{n}$$

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Maximum Displacement

- Why should maximum displacement live on the diagonals of the lattice?
- Why should it travel at speed $\omega h / \sqrt{n}$ (with respect to the ambient Euclidean metric)?
- This is not fully understood, but there is a nice intuitive argument.

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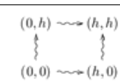
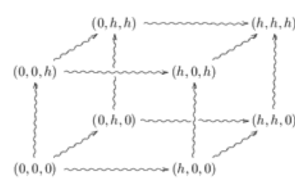
Maximum Displacement



$\|(h, h)\|_2 = \sqrt{2}h \quad \|(h, h)\|_1 = 2h$



$\|(h, h, h)\|_2 = \sqrt{3}h \quad \|(h, h, h)\|_1 = 3h$

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Maximum Displacement

- Intuitively:
- Energy travels along springs.
- Constructive interference on the diagonals.
- Lattice distance to maximum displacement is ωht, while ambient distance is $\omega ht/\sqrt{n}$. On the diagonal,

$$\|x_*(t)\|_2 \approx \frac{\omega ht}{\sqrt{n}} \Leftrightarrow \|x_*(t)\|_1 \approx \omega ht$$

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The Fourier Transform (1)

- The *Pontrjagin dual* of a commutative topological group G is the space of inequivalent irreducible unitary representations of G.

$$\widehat{G} = \frac{\{\chi : G \to S^1 | \chi(gh) = \chi(g)\chi(h)\}}{\sim}$$

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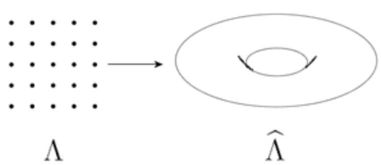
The Lattice's Dual Torus

$$\Lambda = h\mathbb{Z}^n \Leftrightarrow \widehat{\Lambda} = h^{-1}\mathbb{T}^n$$



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The Lattice Fourier Transform

$$\hat{f}(\xi) := \sum_{x \in \Lambda} f(x) e^{-i\xi \cdot x}$$

$$f(x) = \left(\frac{h}{2\pi}\right)^n \int_{\widehat{\Lambda}} \hat{f}(\xi) e^{i\xi \cdot x}\, d\xi$$

$$\sum_{x \in \Lambda} |f(x)|^2 = \left(\frac{h}{2\pi}\right)^n \int_{\widehat{\Lambda}} \left|\hat{f}(\xi)\right|^2 d\xi$$

$$\widehat{(T_y f)}(\xi) = e^{i\xi \cdot y} \hat{f}(\xi)$$

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The Fourier Transform Applied

- Apply our Fourier transform to the equation of motion (spatial partial transform):

$$\ddot{q}(x,t) = \omega^2 h^2 \Delta_h q(x,t)$$

$$\Rightarrow \hat{q}(\xi,t) = q_0 \cos(2\omega\psi(\xi)t)$$

$$\text{where } \psi(\xi) := \sqrt{\sum_{j=1}^{n} \sin^2 \frac{h\xi^j}{2}}$$

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Stationary Phase

- Want to evaluate integrals like

$$I(t) := \int_{\mathcal{D}} F(x) e^{if(x)t}\, dx$$

- Taylor expansion of exponent about points of stationary phase

$$\nabla f(x_s) = 0$$

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Stationary Phase

- For a single non-degenerate point of stationary phase with $\mathcal{D}$ a "nice" domain like $\mathbb{R}^n$ or $\mathbb{T}^n$: as $t \to \infty$,

$$I(t) \sim F(x_s) e^{if(x_s)t} \sqrt{\frac{(\pi i)^n}{t^n \det D^2 f(x_s)}}$$

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Stationary Phase Applied

- We can create a stationary phase expansion for the inverse Fourier transform integrals (using complex exponential form for trigonometric functions)

$$u(x,t) = \frac{u_0 h^n}{(2\pi)^n} \int_{\hat{\Lambda}} \cos\left(2\omega\psi(\xi)t\right) e^{i\xi\cdot x}\, d\xi$$

$$\dot{u}(x,t) = -\frac{u_0 h^n}{(2\pi)^n} \int_{\hat{\Lambda}} 2\omega\psi(\xi) \sin\left(2\omega\psi(\xi)t\right) e^{i\xi\cdot x}\, d\xi$$

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N_2 Synchronization

- Two immediate consequences of stationary phase expansion:
- Amplitude of oscillation for any given particle decays like $t^{-n/2}$.
- Particles separated by 2 lattice spacings asymptotically oscillate in phase and with the same amplitude (N_2 synchronization).

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Energies

- Kinetic and potential energies:

$$T(x,t) := \frac{m}{2} |\dot{u}(x,t)|^2$$

$$V(x,t) := \frac{\kappa}{4} \sum_{y \in N_1(x)} |u(y,t) - u(x,t)|^2$$

$$H(x,t) := T(x,t) + V(x,t)$$

$$\mathcal{L}(x,t) := T(x,t) - V(x,t)$$

$$T_\Omega(t) := \sum_{x\in\Omega} T(x,t) \text{ etc.}$$

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Local Energy Changes

- Using stationary phase methods, we can show that in any finite / bounded region Ω, as $t \to \infty$:

$$T_\Omega(t) \to 0,$$
$$V_\Omega(t) \to 0,$$
$$H_\Omega(t) \to 0,$$
$$\mathcal{L}_\Omega(t) \to 0.$$

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Global Energy Changes

- Using Plancherel and Riemann-Lebesgue, can show that over the whole lattice Λ, as $t \to \infty$, there is an *asymptotic equipartition of energy*:

$$T(t) \to \tfrac{1}{2} H(0),$$
$$V(t) \to \tfrac{1}{2} H(0),$$
$$H(t) \equiv H(0),$$
$$\mathcal{L}(t) \to 0.$$

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Distribution of Energy Theorem

- Ω any bounded region in the lattice Λ; as $t \to \infty$:

$$
\begin{aligned}
T_\Omega(t) &\to 0, & T(t) &\to \tfrac{1}{2}H(0),\\
V_\Omega(t) &\to 0, & V(t) &\to \tfrac{1}{2}H(0),\\
H_\Omega(t) &\to 0, & H(t) &\equiv H(0),\\
\mathcal{L}_\Omega(t) &\to 0, & \mathcal{L}(t) &\to 0.
\end{aligned}
$$

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What Have We Learned?

- In terms of the motion of individual particles Δ_h is a poor approximation to Δ.
- To correct this, we would need to introduce other terms with different weights.
- In terms of energy transport in long time, Δ_h is quite similar to Δ.

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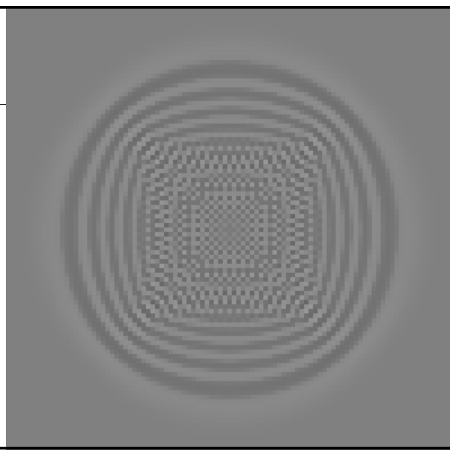

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