A Brief Tour of Vector Spaces and their Duals

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This is intended to be a whistlestop tour of vector spaces and their duals, with a bit about direct sums and tensor products tacked on to the end. If you don't get that bit, don't worry: it's the least important part, included only on account of my bizarre thought processes. Next to nothing is proved, basically because you should be able to do it all anyway with a bit of thought – this is as much a memory aid as a guidebook.

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Vector Spaces

Okay, let's start with the basics. A *vector space* over a field $\mathbb{K}$ is an Abelian group $(V, +)$ together with scalar multiplication by elements of $\mathbb{K}$ so that for all $\alpha_1, \alpha_2 \in \mathbb{K}$ and $v_1, v_2 \in V$,

- $\alpha_1(v_1 + v_2) = \alpha_1 v_1 + \alpha_1 v_2$;
- $(\alpha_1 + \alpha_2)v_1 = \alpha_1 v_1 + \alpha_2 v_1$;
- $(\alpha_1 \alpha_2)v_1 = \alpha_1(\alpha_2 v_1)$;
- $1v_1 = v_1$.

In particular, $\mathbb{K}$ is a vector space over itself.

A subset $\{v_i | i \in I\} \subseteq V$ is *linearly independent* if

$$\sum_{i \in I} \alpha_i v_i = 0 \Rightarrow \alpha_i \equiv 0$$

for all sums $\sum_{i\in I}\alpha_i v_i$ with finitely many nonzero α_i. The *span* of $\{v_i|i\in I\}\subseteq V$ is

$$\operatorname{span}\{v_i|i\in I\} := \left\{\sum_{i\in I}\alpha_i v_i \middle| \alpha_i\in\mathbb{K}, \text{ finitely many } \alpha_i\neq 0\right\}. \tag{1}$$

A *basis* of V is a linealy independent set whose span is all of V.

Theorem 1. *Any two bases of a given vector space over a fixed field can be put into 1-to-1 correspondence.*

The number of elements required to form a bais of V is called the *dimension* of V, denoted $\dim_{\mathbb{K}} V$. (If the field $\mathbb{K}$ is obvious, we just write $\dim V$.)

Theorem 2. *Every vector space has a basis, and hence a dimension.*

This theorem you don't know how to prove – it requires a bit of mathematical black magic called Zorn's Lemma.

Of course, when $\dim V = n < \infty$ the linear independence and spanning conditions become simpler. $\{v_1,\dots,v_n\}$ is linearly independent if and only if

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = 0 \Rightarrow \alpha_i \equiv 0$$

and

$$\operatorname{span}\{v_1,\dots,v_n\} = \{\alpha_1 v_1 + \cdots + \alpha_n v_n | \alpha_i\in\mathbb{K}\}.$$

The field we use makes a difference to what the dimension of the space is:

Proposition 3. *Let V be a vector space over $\mathbb{C}$ with $\dim_{\mathbb{C}} V = n$. Then V can also be a vector space over $\mathbb{R}$ with $\dim_{\mathbb{R}} V = 2n$.*

You already know plenty of finite-dimensional vector spaces, such as $\mathbb{R}^n$, $\mathbb{C}^n$. There are also plenty of infinite-dimensional vector spaces:

- $\mathbb{K}[x]$ = all polynomials in x with coefficients from the field $\mathbb{K}$. This space has a basis given by $\{1, x, x^2, \dots, x^n, \dots\}$;
- the space of all infinite sequences in $\mathbb{R}$;
- the subspace of all convergent infinite sequences in $\mathbb{R}$.

Linear Maps

Given two vector spaces U, V over the same field $\mathbb{K}$, a function $T : U \to V$ is a *linear map* if for all $\alpha_1, \alpha_2 \in \mathbb{K}$ and $u_1, u_2 \in U$,

$$T(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 T(u_1) + \alpha_2 T(u_2).$$

Such maps are *homomorphisms* of vector spaces. The collection of all such maps $T : U \to V$ is denoted $\mathrm{Hom}_{\mathbb{K}}(U, V)$.

Lemma 4. *An element of* $\mathrm{Hom}_{\mathbb{K}}(U, V)$ *is determined by its values on a basis of* U*. I.e., if we know what* $T(e_i) \in V$ *is for a basis* $\{e_i | i \in I\}$ *of* U*, we know* $T(u)$ *for all* $u \in U$*.*

Proposition 5. $\mathrm{Hom}_{\mathbb{K}}(U, V)$ *is a vector space over* $\mathbb{K}$*. If* $\dim U = n < \infty$ *and* $\dim V = m < \infty$ *then* $\dim \mathrm{Hom}_{\mathbb{K}}(U, V) = nm$*.*

Hint for proof: Take bases of U and V and construct an associated basis of $\mathrm{Hom}_{\mathbb{K}}(U, V)$.

A linear map $T : U \to V$ that is also a bijection is an *isomorphism* of vector spaces. We say that U and V are *isomorphic* and write $U \cong V$.

Theorem 6. *If* U *and* V *are vector spaces over* $\mathbb{K}$*,* $U \cong V \Rightarrow \dim U = \dim V$*. Moreover, if* U *and* V *are finite-dimensional vector spaces over* $\mathbb{K}$*,* $U \cong V \Leftrightarrow \dim U = \dim V$*. .*

There's an important correspondence between linear maps and matrices[1]. Let U and V be finite-dimensional with bases $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_m\}$ respectively. Suppose that $T(e_i) = \sum_{j=1}^{m} \alpha_{ji} f_j$. Then if we write

$$\begin{pmatrix} u^1 \\ \vdots \\ u_n \end{pmatrix}$$

[1] $m \times n$ means m rows (horizontal), n columns (vertical). The (i, j)th entry is the one in the ith row down from the top, jth column across from the left.

for the vector $u = \sum_{i=1}^n u^i e_i \in U$, then $T(u) \in V$ is the column vector given by the matrix product

$$\begin{pmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \dots & \alpha_{mn} \end{pmatrix} \begin{pmatrix} u^1 \\ \vdots \\ u_n \end{pmatrix}.$$

To reiterate: How do we write down the matrix of $T : U \to V$ with respect to given bases $\{e_i | i \in I\}$ of U and $\{f_j | j \in J\}$ of V? Answer: Work out $T(e_i)$ in terms of the f_j, and write those coefficients as the ith column of the matrix of T.

Slogans. (1) The matrix of T is the array with the image of the ith basis vector as its ith column.
(2) Saying $T(e_i) = \sum_{j=1}^m \alpha_{ji} f_j$ is the same as saying that the ith column of the matrix of T reads $\alpha_{1i}, \alpha_{2i}, \dots$ and so on, down to α_{mi}.

Remember how to do the matrix product? The rule is: "the rows of the first dive onto the columns of the second", as in

$$\left(\begin{array}{ccc} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \hline \alpha_{i1} & \dots & \alpha_{in} \\ \hline \vdots & \ddots & \vdots \\ \alpha_{m1} & \dots & \alpha_{mn} \end{array}\right) \left(\begin{array}{cc|c|cc} \beta_{11} & \dots & \beta_{1j} & \dots & \beta_{1r} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \beta_{n1} & \dots & \beta_{nj} & \dots & \beta_{nr} \end{array}\right)$$

So the (i, j)th entry of $(\alpha_{ij})(\beta_{ij})$ is $\sum_{k=1}^n \alpha_{ik}\beta_{kj}$.

Theorem 7. *The operation of assigning a matrix A to a linear map $T : U \to V$ is an isomorphism from the vector space $\mathrm{Hom}_{\mathbb{K}}(U, V)$ to the vector space of $\dim U \times \dim V$ matrices over $\mathbb{K}$. Moreover, if $S : U \to V$ has matrix A and $T : V \to W$ has matrix B then $T \circ S : U \to W$ has matrix BA.*

The Dual Space

$\mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$ has a special name: the *dual space* to V, usually denoted V^*.

Now, you can almost certainly visualize $\mathbb{R}^1$, $\mathbb{R}^2$ and $\mathbb{R}^3$. It's even possible that you might have a pretty good intuitive picture of $\mathbb{R}^n$ in your head. What about $(\mathbb{R}^n)^*$? The answer to this question (at least in the author's opinion) is to think about the elements of V^* rather than the space V^* itself. In other words, if you see "$g \in V^*$", just say to yourself, "g is a linear function that eats vectors in V and gives me scalars from the field $\mathbb{K}$". Elements of V^* are often called *linear functionals* on V.

So, we have a vector space V and its dual V^*. We know that V has at least one basis: let $\{e_i | i \in I\}$ be one. V^* also admits a basis – but is there a basis of V^* that arises from the basis $\{e_i | i \in I\}$ of V? Here's a good attempt: for each $i \in I$, define $e^i \in V^*$ by $e^i(e_j) = \delta_{ij}$. δ_{ij} is the *Kronecker delta* defined by

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Just what is this function e^i? If we write $v \in V$ as $\sum_{i \in I} \alpha_i e_i$ with respect to the basis $\{e_i | i \in I\}$ then $e^i(v) = \alpha_i$. So e^i is the ith coordinate function.

Theorem 8. *$\{e^i | i \in I\}$ is linearly independent for any vector space V. $\operatorname{span}\{e^i | i \in I\} = V^*$ if $\dim V < \infty$, but that this is not necessarily true if $\dim V = \infty$.*

Hint for proof: try to write $g \in V^*$ given by

$$g\left(\sum_{i \in I} \alpha_i e_i\right) = \sum_{i \in I} \alpha_i$$

as $\sum_{i \in I} \beta_i e^i$. Remember equation (1).

Corollary 9. $\dim V = \dim V^*$.

Note: there are infinite-dimensional vector spaces for which $\{e^i | i \in I\}$ is indeed a basis of V^*, so the situation isn't completely hopeless.

Now suppose we have two vector spaces U and V and a linear map $T \in \operatorname{Hom}_{\mathbb{K}}(U, V)$. There is an associated map $T^* \in \operatorname{Hom}_{\mathbb{K}}(V^*, U^*)$ called the *adjoint map* to T. No, that isn't a typo: the adjoint T^* goes in the opposite direction to T.

$$\begin{array}{ccc} U & \xrightarrow{T} & V \\ \updownarrow & & \updownarrow \\ U^* & \xleftarrow{T^*} & V^* \end{array}$$

T^* is defined by

$$(T^*(g))(u) = g(T(u)).$$

So, to reiterate, T eats vectors in U and gives vectors in V. T^* eats functions on V and gives functions on U.

Let U and V be finite-dimensional with bases $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_m\}$ respectively. Since U and V are finite-dimensional, the dual bases $\{e^1, \ldots, e^n\}$ and $\{f^1, \ldots, f^m\}$ are indeed bases of U^* and V^*. What is the matrix of T^* in terms of the matrix of T?

Suppose $T(e_k) = \sum_{j=1}^{m} \alpha_{jk} f_j$. Now just calculate, plugging in an arbitrary basis function $f^i \in V^*$ in place of g, and an arbitrary e_k in place of u. Prove (i.e., do the calculations!) that

$$(T^*(f^i))(e_k) = \alpha_{ik}.$$

Now convince yourself that this means that the matrix of $T^*(f^i)$ with respect to the bases $\{e_1, \ldots, e_n\}$ of U and $\{1\}$ of $\mathbb{K}$ is

$$\begin{pmatrix} \alpha_{i1} & \ldots & \alpha_{in} \end{pmatrix}.$$

If that line mystifies you, just remember the rule: the matrix of a linear transformation is the array whose kth column is the image of the kth basis vector. Now remember it again and write the matrix of T^*, whose ith column is $T^*(f^i)$, i.e.

$$\begin{pmatrix} \alpha_{11} & \ldots & \alpha_{m1} \\ \vdots & \ddots & \vdots \\ \alpha_{1n} & \ldots & \alpha_{mn} \end{pmatrix}$$

In other words, if T has an $m \times n$ matrix A then the matrix of T^* is the $n \times m$ matrix given by changing the rows of A for the columns, and vice versa. This is called the *transpose* of A, denoted A^{T}. That is, if $A = (\alpha_{ij})$ and $A^{\mathrm{T}} = B = (\beta_{ij})$ then $\beta_{ij} = \alpha ji$.

New Spaces from Old

This is completely unrelated to the idea of dual spaces, except that taking the dual is one example of getting a new vector space from an old one. There's a whole family of such methods, and I think it would be good for you to have a selection of them written down, even if you don't use them for ages. You certainly don't have to understand this part for the exam.

The Dual. This you've met. Given V over $\mathbb{K}$, form $V^* = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$. $\dim V = \dim V^*$.

The Direct Sum. Start with two spaces U and V over $\mathbb{K}$. We form the *direct sum* $U \oplus V$ essentially by taking all vectors of the form $u + v$ with $u \in U$, $v \in V$. Here's a simple example: take two copies of $\mathbb{R}$, then $\mathbb{R}\oplus\mathbb{R}$ is just the familiar plane $\mathbb{R}^2$. The trick is to think of U and V as having nothing in common, even if they're the same space (as in the $\mathbb{R}\oplus\mathbb{R}$ example).

Remember the phrase "complementary subspace"? We say U' is complementary to U in V if $U + U' = V$ and $U \cap U' = \{0\}$. Well, the direct sum is just like that: take all possible sums and decree that the intersection is zero, even if it doesn't look like it.

So, try this: take U with basis $\{e_1, \ldots, e_n\}$ and V with basis $\{f_1, \ldots, f_m\}$. Convince yourself that $\{e_1, \ldots, e_n, f_1, \ldots, f_m\}$ is a basis for $U \oplus V$ and hence that $\dim(U \oplus V) = \dim U + \dim V$.

The Tensor Product. Start with two spaces U and V over $\mathbb{K}$ with bases $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_m\}$ respectively. We define the *tensor product* $U \otimes V$ to be the vector space with basis $\{e_i \otimes f_j | 1 \leq i \leq n, 1 \leq j \leq m\}$. Seem tricky? Just tell yourself, an element of $U \otimes V$ is a $\mathbb{K}$-linear combination of "things" $e_i \otimes f_j$, just like an element of U is a $\mathbb{K}$-linear combination of "things" e_i.

A couple of things to convince yourself of: first, go back to when you found a basis for $\mathrm{Hom}_{\mathbb{K}}(U, V)$ $(\dim U, \dim V < \infty)$ and see that $\mathrm{Hom}_{\mathbb{K}}(U, V) \cong U \otimes V$; secondly, see that $\dim(U \otimes V) = (\dim U)(\dim V)$.