Deterministic stick-slip dynamics in a one-dimensional random potential

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Analysis & numerics for rate-independent processes
Mathematisches Forschungsinstitut Oberwolfach
1 March 2007
Outline of talk

- Introduction and motivation.
- Brief review of (some) previous studies in the area.
- Statement of main result.
- Sketch of the proof of the main result.
- Directions for future research.
Introduction

Motivation

- Many physical processes exhibit “frictional/stick-slip behaviour”.
- Simple examples:
  - Ball rolling/person skiing down a slope with some bumps.
  - Progression of a dislocation line in a crystal.
  - Evolution of a magnetic domain under an applied field (Barkhausen effect).
Motivation

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Intuition suggests that stick-slip behaviour arises from microstructural variations.

Microstructure $\leadsto$ macroscopic observables, e.g. yield stresses, coefficients of friction & c.

These “macro” quantities can be used as parameters in (relatively) successful models, e.g. rate-independent differential inclusions.

Exactly how the microstructure determines macroscopic behaviour is still not generally understood.
Existing approaches

- Rate-independent solutions to differential inclusions:

\[-\nabla V(X_t) + f(t) \in \partial \psi(\dot{X}_t)\]

with \(\psi\) convex and homogeneous of degree one.

- Over-damped limit (neglect kinetic energy):

\[\dot{X}^\varepsilon_t = -\nabla V^\varepsilon(t, X^\varepsilon_t)\]

- Is it possible to extract the first model from the second as a suitable limit as \(\varepsilon \downarrow 0\)?
Previous one-dimensional studies

\[ \dot{X}^\varepsilon_t = -V'(X_t) - (\varepsilon G)' \left( \frac{X^\varepsilon_t}{\varepsilon} \right) + f(\varepsilon t). \]

- Abeyaratne-Chu-James (1996), Menon (2002): averaging methods for periodic perturbations of the potential; not rate-independent, but can extract a rate-independent corollary (limit satisfies a deterministic ordinary differential inclusion determined by bounds on \( G' \)).

- Grunewald (2005): perturbation is an (integrated) Ornstein-Uhlenbeck process; Fokker-Planck methods insufficient to establish stick-slip behaviour in the limit.
On the real line $\mathbb{R}$, consider:

- a potential $V(x) = \frac{\kappa}{2}x^2$, $\kappa > 0$;
- a $C^0$ gradient field $g := G': \Omega \times \mathbb{R} \to [\gamma^-, \gamma^+]$;
  - $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space;
  - wiggly potential $x \mapsto V(x) + \varepsilon G(\omega, \frac{x}{\varepsilon})$;
- a $C^0$ external loading $f: [0, +\infty) \to \mathbb{R}$;
Model in one dimension — random ODE

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Random gradient flow ODE with “landscape parameter” $\omega \in \Omega$:

$$\dot{X}_t^\varepsilon(\omega) = -V'(X_t^\varepsilon(\omega)) - G'(\omega, \frac{X_t^\varepsilon(\omega)}{\varepsilon}) + f(\varepsilon t).$$

Standard results give existence of solutions for all positive time. Later results remove any need for uniqueness.
Model in one dimension — random landscape
Model in one dimension — random landscape

\[ V(x), (V + G)(x) \]
Model

Model in one dimension — random landscape

\[ -V'(x) \]

\[ x \]

\[ -4 \quad -2 \quad 2 \quad 4 \]

\[ -15 \quad -10 \quad -5 \quad 5 \quad 10 \quad 15 \]
Model

Model in one dimension — random landscape

\[-V'(x), -(V + G)'(x)\]
Model in one dimension — limit process

Random ODE:

\[ \dot{X}_t^\varepsilon(\omega) = -\kappa X_t^\varepsilon(\omega) - g(\omega, \frac{X_t^\varepsilon(\omega)}{\varepsilon}) + f(\varepsilon t). \]

Limiting process as \( \varepsilon \downarrow 0 \):

\[ X_t^0 := \lim_{\varepsilon \downarrow 0} X_t^{\varepsilon}. \]

In principle, this limiting object is a stochastic process \( X^0 : \Omega \times [0, +\infty) \to \mathbb{R} \) dependent on the choice of process \( g \) and the "landscape parameter" \( \omega \in \Omega \), but...
Main theorem — first draft


Let \( g : \Omega \times \mathbb{R} \to [\gamma^-, \gamma^+] \) be a doubly-reflected Wiener process and let \( f \in C^0([0, +\infty); \mathbb{R}) \). Then, for \( \mathbb{P}\)-a.a. \( \omega \in \Omega \), \( X^0(\omega) \) satisfies the deterministic ordinary differential inclusion

\[
- V'(X^0_t) + f(t) \in \partial \psi^\gamma(\dot{X}^0_t),
\]

(ODI)

where the dissipation \( \psi^\gamma : \mathbb{R} \to [0, +\infty) \) is given by

\[
\psi^\gamma(\dot{x}) := \begin{cases} 
\gamma^- \dot{x}; & \dot{x} \leq 0; \\
\gamma^+ \dot{x}; & \dot{x} \geq 0.
\end{cases}
\]

Note that (ODI) is deterministic and has a unique deterministic solution, which can be easily visualised by the “drainpipe rule”.
“Drainpipe rule” for solutions of (ODI)

\[ X_t^0, \ f(t), \ \mathcal{A}^\gamma(f(t)) \]

A typical deterministic, rate-independent \( \mathbb{P} \)-a.s. limit \( X^0 \), shown in blue. Loading \( f(t) = \sin t + \cos 2t \) shown in red; “sticky attractor” \( \mathcal{A}^\gamma(f(t)) \) shown in green.
Hysteresis loops for the deterministic, rate-independent $\mathbb{P}$-a.s. limit $X^0$, shown in blue. Again, $f(t) = \sin t + \cos 2t$. 

\[ f(t) \]

\[ X^0_t \]
Sticky attractor for $X^0$ dynamics

- The **sticky attractor** $\mathcal{A}^\gamma : \mathbb{R} \rightarrow 2^\mathbb{R}$:

  $$\mathcal{A}^\gamma(F) := \left[ \frac{F - \gamma^+}{\kappa}, \frac{F - \gamma^-}{\kappa} \right].$$

- **Attractor** in the sense that all trajectories lie in $\mathcal{A}^\gamma(f(t))$ for all $t > 0$, regardless of initial condition.

- **Sticky** in the sense that if a trajectory can remain stationary and stay inside $\mathcal{A}^\gamma(f(t))$, it will.
Outline of proof

Strategy of proof of main theorem

- Identify the fixed-point set for the dynamics at scale $\varepsilon > 0$, some fixed landscape given by $\omega \in \Omega$, constant loading $f(t) \equiv 0$.
- Take a suitable limit of these sets as $\varepsilon \downarrow 0$, $\mathbb{P}$-almost surely losing $\omega$-dependence along the way.
- No loading $\rightsquigarrow$ constant loading $\rightsquigarrow$ variable loading.
- Show that the limit “tube” $t \mapsto A^\gamma(f(t))$ has the desired properties (sticky attractor) for the process $X^0$. 
Outline of proof

Limits of sets

Definition (Kuratowski (1966))

Let $(\mathbb{M}, d)$ be a metric space. Define the Kuratowski limit inferior of a family of subsets $\{A_\varepsilon \subseteq \mathbb{M}\}_{\varepsilon > 0}$ to be

$$\text{Li}_{\varepsilon \downarrow 0} A_\varepsilon := \left\{ x \in \mathbb{M} \left| \limsup_{\varepsilon \downarrow 0} d_H(x, A_\varepsilon) = 0 \right. \right\},$$

where $d_H(x, A_\varepsilon) := \inf_{y \in A_\varepsilon} d(x, y)$ is the usual Hausdorff semi-distance.

(The Kuratowski notions of limit superior (Ls) and limit (Lim) are not required in this analysis.)
Key lemma

\[ \dot{X}_t^\varepsilon(\omega) = -\kappa X_t^\varepsilon(\omega) - g\left(\omega, \frac{X_t^\varepsilon(\omega)}{\varepsilon}\right) + f(\varepsilon t). \]

Lemma

Let \( g : \Omega \times \mathbb{R} \to [\gamma^-, \gamma^+] \) be a doubly-reflected Wiener process and let

\[ A_{\varepsilon}^g(\omega)(0) := \left\{ x \in \mathbb{R} \mid -\kappa x - g\left(\omega, \frac{x}{\varepsilon}\right) = 0 \right\}, \]

the fixed-point set for the dynamics in the landscape \( V(\cdot) + \varepsilon G(\omega, \cdot/\varepsilon) \) at scale \( \varepsilon > 0 \) with no loading. Then

\[ \text{Li}_{\varepsilon \downarrow 0} A_{\varepsilon}^g(\omega)(0) = A^\gamma(0) \equiv \left[ \frac{-\gamma^+}{\kappa}, \frac{-\gamma^-}{\kappa} \right] \text{ for } \mathbb{P}\text{-a.a. } \omega \in \Omega. \]

“The attractors for \( \varepsilon > 0 \) fill up the correct interval as \( \varepsilon \downarrow 0 \).”
Sketch proof of key lemma

- Idea: intermediate value theorem + scaling.
Outline of proof

Sketch proof of key lemma

- Idea: intermediate value theorem + scaling.
- Define “first return separations” $D_n(\omega)$ from $\gamma^+$ to $\gamma^-$ and back to $\gamma^+$.
- Require (in both directions): sample-continuity of $g$, $D_n < +\infty \, \mathbb{P}\text{-a.s.}$, $\sum_n D_n = +\infty \, \mathbb{P}\text{-a.s.}$ and

$$\frac{D_n}{\sum_{i=0}^{n-1} D_i} \xrightarrow[n\to\infty]{} 0 \, \mathbb{P}\text{-a.s.}$$
Outline of proof

Sketch proof of key lemma

- Idea: intermediate value theorem + scaling.
- Define “first return separations” $D_n(\omega)$ from $\gamma^+$ to $\gamma^-$ and back to $\gamma^+$.
- Require (in both directions): sample-continuity of $g$, $D_n < +\infty$ $\mathbb{P}$-a.s., $\sum_n D_n = +\infty$ $\mathbb{P}$-a.s. and

$$\frac{D_n}{\sum_{i=0}^{n-1} D_i} \xrightarrow{n \to \infty} 0 \text{ $\mathbb{P}$-a.s.} \quad (\star)$$

- For $g : \Omega \times \mathbb{R} \to [\gamma^-, \gamma^+]$ a doubly-reflected Wiener process, all the conditions $(\star)$ are met ($g$ sample-continuous with $D_n$ IID, $\mathbb{E}[D_n] = 4|\gamma^+ - \gamma^-|^2$, $\text{Var}[D_n] = 32|\gamma^+ - \gamma^-|^4$).
Outline of proof

Sketch proof of key lemma

- Clearly, many more processes satisfy (✠), but a doubly-reflected Wiener process is a good prototype.
- In fact, something better is true: the conditions (✠) are necessary and sufficient to conclude that

\[
\begin{align*}
\text{Li}_{\epsilon \downarrow 0} A^{g(\omega)}_\epsilon (0) &= \left[ \frac{-\gamma^+}{\kappa}, \frac{-\gamma^-}{\kappa} \right] \text{ P-a.s.}
\end{align*}
\]

- Argue from contradiction.
  If any one of the conditions (✠) fails then there is a collection of “bad” landscapes of positive probability for which \(\text{Li}_{\epsilon \downarrow 0} A^{g(\omega)}_\epsilon (0)\) is not what we want.
Further lemmata

Lemma (Stickiness locally in time)

Let \( 0 \leq t_0 < t_1 < \infty \) and let \( I \) denote any interval from \( t_0 \) to \( t_1 \) with either end open or closed. \( \mathbb{P} \)-a.s., if \( f|_I \) is bounded, and

\[
X^0_{t_0} \in \mathcal{A}^\gamma(f(t)) \text{ for all } t \in I,
\]

then \( X^0_t = X^0_{t_0} \) for all \( t \in I \).

Lemma (Right limit property)

Let \( t_0 \geq 0 \) be such that \( f(t_0+) \) exists. Then

\[
X^0_{t_0+} = X^0_{t_0} \diamond \mathcal{A}^\gamma(f(t_0+)) \quad \mathbb{P}\text{-a.s.,}
\]

where \( y \diamond A \) denotes the closest point of the interval \( \bar{A} \) to \( y \).
Main theorem revisited


Let $f \in C^0([0, +\infty); \mathbb{R})$ and let $g : \Omega \times \mathbb{R} \to [\gamma^-, \gamma^+]$ be any stochastic process. Then $g$ satisfies (✠) if, and only if, $X^0 \mathbb{P}$-a.s. satisfies the deterministic ordinary differential inclusion

$$-V'(X^0_t) + f(t) \in \partial \psi^\gamma(\dot{X}_t^0), \quad \text{(ODI)}$$

where the dissipation $\psi^\gamma : \mathbb{R} \to [0, +\infty)$ is given by

$$\psi^\gamma(\dot{x}) := \begin{cases} 
\gamma^- \dot{x}; & \dot{x} \leq 0; \\
\gamma^+ \dot{x}; & \dot{x} \geq 0.
\end{cases}$$
Some conclusions

- If one subscribes to the idea that rate-independent evolutions like (ODI) should arise as small-scale limits of deterministic evolutions in wiggly energies, our theorem shows that the precise choice of wiggle is not so important.

- “Homogenization without periodicity of the fast (microscale) process.”
Further work

- Extension to more general spatial noise processes $g$?
- Extension to $\mathbb{R}^d$, $d \geq 1$? To infinite-dimensional spaces like $W^{k,p}(\mathcal{D}; \mathbb{R})$?

\[
\dot{X}_t = -\nabla V(X_t) - \nabla G(\omega, \frac{X_t}{\varepsilon}) + f(\varepsilon t).
\]

- Include the effects of a heat bath via a stochastic differential?

\[
\dot{X}^\varepsilon_t(\omega_1, \omega_2) = -\nabla V(X^\varepsilon_t) - \nabla G(\omega_1, \frac{X^\varepsilon_t}{\varepsilon}) + \sigma(\varepsilon)\dot{W}_t(\omega_2).
\]

Which “wins” as $\varepsilon \downarrow 0$? The diffusive or the stick-slip dynamics?