A Hitchhiker's Guide to Γ-Convergence

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Outline

Why Do We Need Γ-Convergence?

- Γ-convergence, introduced by Ennio de Giorgi in 1975, is a notion of convergence that is appropriate for variational problems.

- Suppose that you are given a sequence of functionals $(F_n)_{n\in\mathbb{N}}$ on some space $\mathcal{X}$. Suppose that x_n minimizes F_n. Does $\lim_{n\to\infty} x_n$, if it exists at all, minimize anything?

- In what sense does F_n have to converge to some other functional F to ensure that minimizers of F_n converge to minimizers of F? Neither pointwise nor uniform convergence does the job, as will be shown.
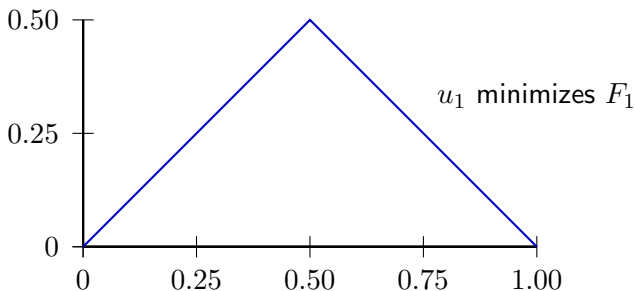
- Γ-convergence is the answer!

A Cautionary Tale: Microstructure

Define a functional $F\colon H_0^1([0,1];\mathbb{R}) \to [0,+\infty]$ by

$$F(u) := \int_0^1 \left(u'(x)^2 - 1\right)^2 \, \mathrm{d}x.$$

Define F_n by restricting F to the "polygonal functions" of mesh size $\frac{1}{2n}$:

$$F_n(u) := \begin{cases} F(u), & \text{if } u' \text{ is constant on } \left(\frac{i}{2n}, \frac{i+1}{2n}\right) \text{ for } i = 1, \ldots, 2n-1, \\ +\infty, & \text{otherwise.} \end{cases}$$

u_1 minimizes F_1

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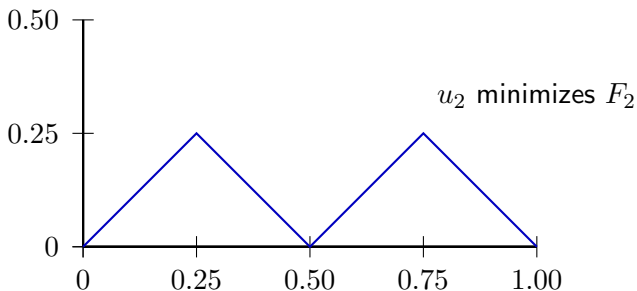

u_2 minimizes F_2

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Define F_n by restricting F to the "polygonal functions" of mesh size $\frac{1}{2n}$:

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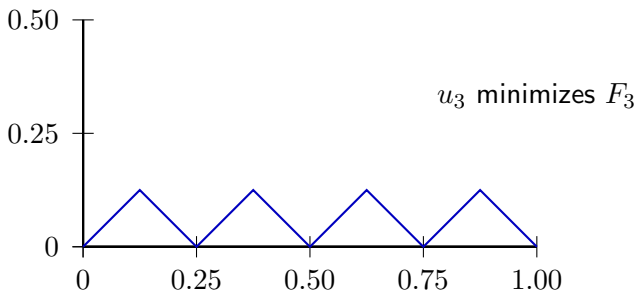
u_3 minimizes F_3

Well, This is Embarrassing. . .

- A (non-unique) minimizer for F_n is given by u_n, a succession of n "tent functions" of height $1/2n$ and width $1/n$:

$$F_n(u_n) = 0 \text{ for each } n \in \mathbb{N}.$$

- $u_n \to u$ uniformly, where $u(x) = 0$ for all $x \in [0, 1]$.
- But u is not a minimizer of F!

$$F(u) = 1 > 0 = F(u_1).$$

- What has gone wrong? In one sense, the problem is that F is not lower semicontinuous. A deeper reason is that F_n does not Γ-converge to F.

The Setting

- $\mathcal{X}$ will be a normed / metric / topological space.[1] Most of the examples later on will take $\mathcal{X} = \mathbb{R}$; in most applications, $\mathcal{X}$ is a space of functions, *e.g.* $H_0^1(\Omega; \mathbb{R}^3)$.
- $\overline{\mathbb{R}}$ denotes the extended real number line, $\mathbb{R} \cup \{-\infty, +\infty\}$.
- We will consider functions $F \colon \mathcal{X} \to \overline{\mathbb{R}}$. For historical reasons, and because $\mathcal{X}$ is often a space of functions, F is often called a functional.

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[1]To make things easy, $\mathcal{X}$ is assumed to be a sequential space, so everything can be phrased in terms of convergence of sequences. In general, we need to use nets (a.k.a. Moore–Smith sequences) and be more careful with terminology like "limit point" *vs.* "cluster point".

Lower and Upper Limits

Definition (Limit Points)

A limit point of a sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathcal{X}$ is a point $x \in \mathcal{X}$ such that some subsequence $(x_{n_k})_{k \in \mathbb{N}}$ has $\lim_{k \to \infty} x_{n_k} = x$.

Definition (Lower and Upper Limits)

The lower limit (or limit inferior) of a real sequence $(x_n)_{n \in \mathbb{N}}$ is the infimum of all its limit points:

$$\liminf_{n \to \infty} x_n := \inf \left\{ x \in \overline{\mathbb{R}} \,\middle|\, x \text{ is a limit point of } (x_n)_{n \in \mathbb{N}} \right\}.$$

The upper limit (or limit superior) is the supremum of all its limit points:

$$\limsup_{n \to \infty} x_n := \sup \left\{ x \in \overline{\mathbb{R}} \,\middle|\, x \text{ is a limit point of } (x_n)_{n \in \mathbb{N}} \right\}.$$

Lower and Upper Limits

Lemma (Basic Properties of Lower and Upper Limits)

- *For any real sequence $(x_n)_{n \in \mathbb{N}}$, the lower and upper limits are well-defined in $\overline{\mathbb{R}}$ and satisfy*

$$-\infty \leq \liminf_{n \to \infty} x_n \leq \limsup_{n \to \infty} x_n \leq +\infty.$$

- *The lower and upper limits are equal if, and only if, $(x_n)_{n \in \mathbb{N}}$ is a convergent sequence, in which case*

$$\liminf_{n \to \infty} x_n = \lim_{n \to \infty} x_n = \limsup_{n \to \infty} x_n.$$

Example

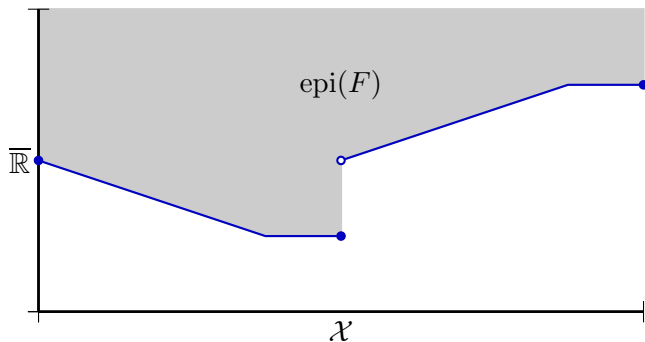$\liminf_{n \to \infty} \sin n = \liminf_{n \to \infty} (-1)^n = -1$ but $\limsup_{n \to \infty} \sin n = \limsup_{n \to \infty} (-1)^n = +1$

Epigraphs

Definition (Epigraph)

The epigraph of $F\colon \mathcal{X} \to \overline{\mathbb{R}}$ is the subset of $\mathcal{X} \times \overline{\mathbb{R}}$ that lies "on or above the graph" of F:

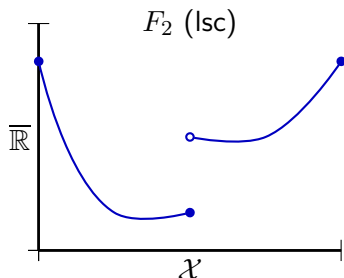$$\mathrm{epi}(F) := \left\{ (x, v) \in \mathcal{X} \times \overline{\mathbb{R}} \,\middle|\, v \geq F(x) \right\}.$$

Semicontinuity

Definition (Lower Semicontinuity)

$F \colon \mathcal{X} \to \overline{\mathbb{R}}$ is lower semicontinuous (or just lsc for short) if

$$x_n \to x \implies F(x) \leq \liminf_{n \to \infty} F(x_n);$$

equivalently, $\{x \in \mathcal{X} \mid F(x) \leq t\}$ is closed in $\mathcal{X}$ for each $t \in \overline{\mathbb{R}}$; also equivalently, $\mathrm{epi}(F)$ is closed in $\mathcal{X} \times \overline{\mathbb{R}}$.

F_1 (not lsc) F_2 (lsc)
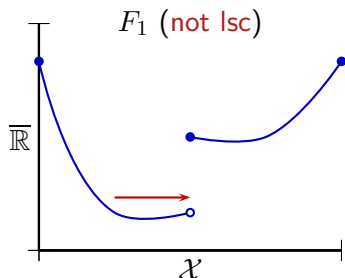
Semicontinuity

Lower semicontinuity is very important in variational problems, because (on compacta) it ensures the existence of minimizers:

Theorem

Let $\mathcal{X}$ be compact and let $F \colon \mathcal{X} \to \overline{\mathbb{R}}$ be lsc. Then there exists $x_* \in \mathcal{X}$ such that

$$F(x_*) = \inf_{x \in \mathcal{X}} F(x).$$

Proof.

Let $(x_n)_{n \in \mathbb{N}}$ be such that $\lim_{n \to \infty} F(x_n) = \inf_{x \in \mathcal{X}} F(x)$. Since $\mathcal{X}$ is compact, $(x_n)_{n \in \mathbb{N}}$ has a convergent subsequence $(x_{n_k})_{k \in \mathbb{N}}$ that converges to some $x_* \in \mathcal{X}$. Since F is lsc,

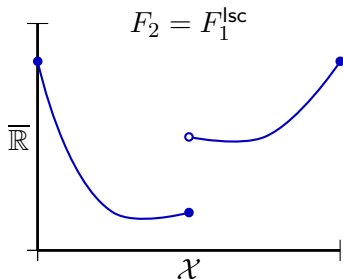$$F(x_*) \leq \liminf_{k \to \infty} F(x_{n_k}) = \inf_{x \in \mathcal{X}} F(x). \qquad \square$$

Semicontinuity

Definition (Lower Semicontinuous Envelope)

Given $F\colon \mathcal{X} \to \overline{\mathbb{R}}$, the lower semicontinuous envelope (or relaxation) of F is the "greatest lsc function bounded above by F":

$$F^{\text{lsc}}(x) := \sup \left\{ G(x) \,\middle|\, G\colon \mathcal{X} \to \overline{\mathbb{R}} \text{ is lsc and } G \leq F \text{ on } \mathcal{X} \right\}$$

$$= \inf \left\{ \liminf_{n\to\infty} F(x_n) \,\middle|\, (x_n)_{n\in\mathbb{N}} \subseteq \mathcal{X} \text{ and } x_n \to x \right\}.$$
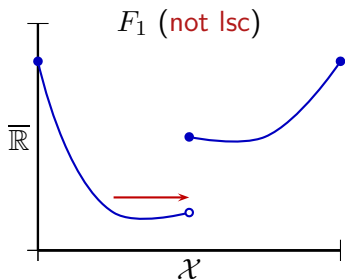
Pointwise and Uniform Convergence

Definition (Pointwise Convergence)

Let $F_n\colon \mathcal{X} \to \overline{\mathbb{R}}$ for each $n \in \mathbb{N}$, and $F\colon \mathcal{X} \to \overline{\mathbb{R}}$. $F_n \to F$ pointwise if

$$\text{for each } x \in \mathcal{X}, \ \lim_{n\to\infty} |F_n(x) - F(x)| = 0.$$

Definition (Uniform Convergence)

Let $F_n\colon \mathcal{X} \to \overline{\mathbb{R}}$ for each $n \in \mathbb{N}$, and $F\colon \mathcal{X} \to \overline{\mathbb{R}}$. $F_n \to F$ uniformly if

$$\lim_{n\to\infty} \sup_{x\in\mathcal{X}} |F_n(x) - F(x)| = 0.$$

Γ-Convergence

Definition (Γ-Convergence)

Let $F_n \colon \mathcal{X} \to \mathbb{R}$ for each $n \in \mathbb{N}$. We say that $(F_n)_{n \in \mathbb{N}}$ Γ-converges to $F \colon \mathcal{X} \to \mathbb{R}$, and write $\Gamma\text{-}\lim_{n \to \infty} F_n = F$ or $F_n \xrightarrow{\Gamma} F$, if

- (asymptotic common lower bound) for every $x \in \mathcal{X}$ and every $(x_n)_{n \in \mathbb{N}}$ such that $x_n \to x$ in $\mathcal{X}$,

$$F(x) \leq \liminf_{n \to \infty} F_n(x_n);$$

- (existence of recovery sequences) for every $x \in \mathcal{X}$, there exists some $(x_n)_{n \in \mathbb{N}}$ such that $x_n \to x$ in $\mathcal{X}$ and

$$F(x) \geq \limsup_{n \to \infty} F_n(x_n).$$

Remarks

- Note that Γ-convergence of functionals on $\mathcal{X}$ is phrased in terms of convergence of sequences of elements of $\mathcal{X}$. If we change the norm/metric/topology on $\mathcal{X}$, then we change the topology of Γ-convergence as well.

- One can even have "hybrid" notions of convergence, *e.g.* Mosco convergence, a.k.a. "weak Γ-$\lim\inf$ and strong Γ-$\lim\sup$" convergence.

- Modulo a few technicalities, Γ-convergence on a topological vector space $\mathcal{X}$ can be equivalently expressed in terms of the minimum energies of the F_n, having been "tilted" by an arbitrary continuous load: $F_n \xrightarrow{\Gamma} F$ if, and only if, for every continuous linear map $\ell \colon \mathcal{X} \to \mathbb{R}$,

$$\lim_{n \to \infty} \inf_{x \in \mathcal{X}} \left(F_n(x) + \langle \ell, x \rangle \right) = \inf_{x \in \mathcal{X}} \left(F(x) + \langle \ell, x \rangle \right).$$

Convergence of Sets: Kuratowski Convergence

Consider a metric space $(\mathcal{X}, d)$. The distance from a point $x \in \mathcal{X}$ to a set $A \subseteq \mathcal{X}$ is defined by

$$d(x, A) := \inf_{a \in A} d(x, a).$$

Definition (Kuratowksi limits)

Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of subsets of $\mathcal{X}$. Let

$$\operatorname*{K-lim\,inf}_{n \to \infty} A_n := \left\{ x \in \mathcal{X} \,\middle|\, \limsup_{n \to \infty} d(x, A_n) = 0 \right\},$$

$$\operatorname*{K-lim\,sup}_{n \to \infty} A_n := \left\{ x \in \mathcal{X} \,\middle|\, \liminf_{n \to \infty} d(x, A_n) = 0 \right\}.$$

If these two are equal, then their common value is called the Kuratowski limit $\operatorname*{K-lim}_{n \to \infty} A_n$.

Examples of Kuratowski Convergence

Examples

In $\mathcal{X} = \mathbb{R}$ with its usual Euclidean distance,

$$\underset{n \to \infty}{\mathrm{K\text{-}lim}} \frac{1}{n}\mathbb{Z} =$$

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$$\operatorname*{K-lim}_{n\to\infty} \frac{1}{n}\mathbb{Z} = \mathbb{R},$$

$$\operatorname*{K-lim}_{n\to\infty} n\mathbb{Z} = \{0\}.$$

Examples of Kuratowski Convergence

Examples

In $\mathcal{X} = \mathbb{R}^2$ with its usual Euclidean distance, let

$$A_n := \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; |x|^n + |y|^n < 1 \right\}.$$

Then

$$\underset{n \to \infty}{\text{K-lim}} \, A_n =$$

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$$\mathrm{K\text{-}lim}_{n \to \infty} A_n = \left\{ (x, y) \,\middle|\, |x| \le 1 \text{ and } |y| \le 1 \right\},$$

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and, sending $n \to 0$, i.e. $1/n \to \infty$,

$$\operatorname*{K-lim}_{n \to 0} A_n =$$

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and, sending $n \to 0$, i.e. $1/n \to \infty$,

$$\operatorname*{K-lim}_{n \to 0} A_n = (\{0\} \times [-1, 1]) \cup ([-1, 1] \times \{0\}).$$

Γ- and K-Convergence

The following theorem, which relates Γ-convergence and Kuratowski convergence, is very useful if you have some intuition about the *epigraphs* of the functionals under consideration:

Theorem

F_n *Γ-converges to* F *on* $\mathcal{X}$ *if, and only if, the epigraphs of the* F_n *K-converge to the epigraph of* F *in* $\mathcal{X} \times \overline{\mathbb{R}}$:

$$\Gamma\text{-}\lim_{n\to\infty} F_n = F \iff \text{K-}\lim_{n\to\infty} \text{epi}(F_n) = \text{epi}(F).$$

Properties of Γ-Limits

Lemma

Kuratowski limits are always closed sets.

Theorem

Γ-limits are always lower semicontinuous.

Proof.

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Proof.

Suppose that $F_n \xrightarrow{\Gamma} F$ on $\mathcal{X}$. Then $\mathrm{epi}(F_n) \xrightarrow{\mathrm{K}} \mathrm{epi}(F)$ in $\mathcal{X} \times \overline{\mathbb{R}}$. Since it is a Kuratowski limit, $\mathrm{epi}(F)$ has to be closed. Hence, F is lower semicontinuous. $\qquad \square$

Properties of Γ-Limits

Theorem (Γ and Pointwise Convergence)

If $F_n \xrightarrow{\Gamma} F$ and $F_n \to G$ pointwise, then $F \leq G$.

Theorem (Γ and Uniform Convergence)

If $F_n \xrightarrow{unif} F$, then $F_n \xrightarrow{\Gamma} F^{lsc}$.

Corollary

The Γ-limit of a constant sequence $(F)_{n \in \mathbb{N}}$ is ...

Properties of Γ-Limits

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If $F_n \xrightarrow{unif} F$, then $F_n \xrightarrow{\Gamma} F^{lsc}$.

Corollary

The Γ-limit of a constant sequence $(F)_{n \in \mathbb{N}}$ is F^{lsc}.

Properties of Γ-Limits

Theorem (Well-Definedness of Γ-Limits)

If $F_n \xrightarrow{\Gamma} F$, then every subsequence $(F_{n_k})_{k \in \mathbb{N}}$ of $(F_n)_{n \in \mathbb{N}}$ also Γ-converges to F.

Theorem (Stability of Γ-Limits under Continuous Perturbations)

If $F_n \xrightarrow{\Gamma} F$ and G is continuous, then

$$F_n + G \xrightarrow{\Gamma} F + G.$$

Properties of Γ-Limits

Theorem (Fundamental Theorem of Γ-Convergence)

If $F_n \xrightarrow{\Gamma} F$ and x_n minimizes F_n, then every limit point of the sequence $(x_n)_{n \in \mathbb{N}}$ is a minimizer of F.

- Compare this theorem with the microstructure example in the beginning: in that example, the functionals do not Γ-converge, since the minimizers $(u_n)_{n \in \mathbb{N}}$ of $(F_n)_{n \in \mathbb{N}}$ converge to a u that does not minimize F.

- Note, however, that this theorem may be vacuously true, *e.g.* if F_n has no minimizer, if $(x_n)_{n \in \mathbb{N}}$ has no convergent subsequences, or the space $\mathcal{X}$ is non-compact.

Examples / Exercises

Example

Define $F_n \colon \mathbb{R} \to \mathbb{R}$ by $F_n(x) := \cos(nx)$. What is the Γ-limit of this sequence?

$$\Gamma\text{-}\lim_{n \to \infty} \cos(nx) =$$

Examples / Exercises

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Example

Define $F_n \colon \mathbb{R} \to \mathbb{R}$ by $F_n(x) := \frac{1}{2}x^2 + \sin(nx)$. What is the Γ-limit of this sequence?

$$\Gamma\text{-}\lim_{n\to\infty} \left(\frac{1}{2}x^2 + \sin(nx) \right) =$$

Examples / Exercises

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$$\operatorname*{\Gamma\text{-}lim}_{n \to \infty} \left(\frac{1}{2}x^2 + \sin(nx) \right) = \frac{1}{2}x^2 - 1.$$

Examples / Exercises

Example (Regularization of a Step Function)

Define $F_n \colon \mathbb{R} \to \mathbb{R}$ by

$$F_n(x) := \begin{cases} 0, & x \leq -1/n, \\ \frac{nx+1}{2}, & -\frac{1}{n} \leq x \leq \frac{1}{n}, \\ 1, & x \geq \frac{1}{n}. \end{cases}$$

$$\lim_{n \to \infty} F_n(x) =$$

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Example (Regularization of a Step Function)

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$$\lim_{n \to \infty} F_n(x) = \begin{cases} 0, & x < 0, \\ \frac{1}{2}, & x = 0, \\ 1, & x > 0. \end{cases}$$

$$\Gamma\text{-}\lim_{n \to \infty} F_n(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0. \end{cases}$$

Examples / Exercises

Example

Define $F_n \colon \mathbb{R} \to \mathbb{R}$ by

$$F_n(x) := nx \exp(nx).$$

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Examples / Exercises

Example

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$$\lim_{n \to \infty} F_n(x) = \begin{cases} 0, & x \leq 0, \\ +\infty, & x > 0. \end{cases}$$

$$\Gamma\text{-}\lim_{n \to \infty} F_n(x) = \begin{cases} 0, & x < 0, \\ -e^{-1}, & x = 0, \\ +\infty, & x > 0. \end{cases}$$

Examples / Exercises

Example

Define $F_n \colon \mathbb{R} \to \mathbb{R}$ by

$$F_n(x) := nx \exp(-2n^2 x^2).$$

$$\lim_{n \to \infty} F_n(x) =$$

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Examples / Exercises

Example

Define $F_n \colon \mathbb{R} \to \mathbb{R}$ by

$$F_n(x) := nx \exp(-2n^2 x^2).$$

$$\lim_{n \to \infty} F_n(x) = 0 \text{ for all } x \in \mathbb{R}.$$

$$\operatorname*{\Gamma\text{-}lim}_{n \to \infty} F_n(x) = \begin{cases} -\frac{1}{2} e^{-1/2}, & x = 0, \\ 0, & x \neq 0. \end{cases}$$

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