The Gauge Integral of Denjoy, Luzin, Perron, Henstock and Kurzweil

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Ortiz Group Meeting
Caltech, California, U.S.A.
12 August 2011

Outline

Why All These Different Integrals?

"Does anyone believe that the difference between the Lebesgue and Riemann integrals can have physical significance, and that whether say, an airplane would or would not fly could depend on this difference? If such were claimed, I should not care to fly in that plane."

— Richard Wesley Hamming

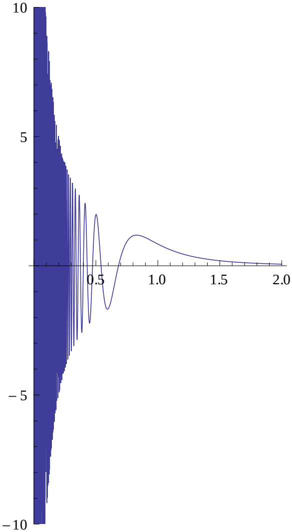
Denjoy's Headache

Around 1912, the French mathematician Arnaud Denjoy wanted to understand integrals like

$$\int_0^1 \frac{1}{x} \sin \frac{1}{x^3} \, \mathrm{d}x.$$

This integral is not absolutely convergent: the integrand has diverging contributions of both signs as $x \to 0$. Nevertheless, we do have the *improper* integral

$$\lim_{\alpha \to 0} \int_\alpha^1 \frac{1}{x} \sin \frac{1}{x^3} \, \mathrm{d}x = \frac{\pi}{6} - \frac{1}{3} \int_0^1 \frac{\sin t}{t} \, \mathrm{d}t.$$



Other Conditionally Convergent Integrals

- Denjoy (1912, 1915–1917) developed an integral that did not need absolute convergence (summability) of the integrand — many equivalent formulations were later supplied by Luzin (1912), Perron (1914), Henstock (1955) and Kurzweil (1957).
- This gauge integral has probably the strongest convergence theorems of any integral, yet the Henstock–Kurzweil formulation is a surprisingly simple modification of the Riemann integral.
- The gauge integral is an ideal tool for integrals with many cancellation effects, *e.g.* Feynman's path integrals

$$\int_X \exp\left(\frac{i}{\hbar}\int_0^T \mathcal{L}(t, x(t), \dot{x}(t))\,\mathrm{d}t\right)\mathrm{d}x$$

over some space X of paths defined for times $t \in [0, T]$.

Cauchy's (Much-Repeated) Mistake

Cauchy (1815) evaluated the convergent trigonometric integrals

$$\int_0^\infty \sin(y^2)\cos(xy)\,\mathrm{d}y = \frac{1}{2}\sqrt{\frac{\pi}{2}}\left(\cos\left(\frac{x^2}{4}\right) - \sin\left(\frac{x^2}{4}\right)\right),$$

$$\int_0^\infty \cos(y^2)\cos(xy)\,\mathrm{d}y = \frac{1}{2}\sqrt{\frac{\pi}{2}}\left(\cos\left(\frac{x^2}{4}\right) + \sin\left(\frac{x^2}{4}\right)\right).$$

Then, differentiating with respect to x, he obtained

$$\int_0^\infty y\sin(y^2)\sin(xy)\,\mathrm{d}y = \frac{x}{4}\sqrt{\frac{\pi}{2}}\left(\sin\left(\frac{x^2}{4}\right) + \cos\left(\frac{x^2}{4}\right)\right),$$

$$\int_0^\infty y\cos(y^2)\sin(xy)\,\mathrm{d}y = \frac{x}{4}\sqrt{\frac{\pi}{2}}\left(\sin\left(\frac{x^2}{4}\right) - \cos\left(\frac{x^2}{4}\right)\right).$$

Problem! The "differentiated" integrals do not converge!

Paradise Lost: The Leibniz–Newton Integral

"Since the time of Cauchy, integration theory has in the main been an attempt to regain the Eden of Newton. In that idyllic time [. . .] derivatives and integrals were [. . .] different aspects of the same thing."

— *Peter Bullen*

Definition

$f\colon [a,b] \to \mathbb{R}$ is integrable iff there exists $F\colon [a,b] \to \mathbb{R}$ such that $F' = f$ everywhere, in which case

$$\int_a^b f := F(b) - F(a).$$

When integration began to be defined in terms different from (anti-)differentiation, this simple relationship was lost. . .

Tagged Partitions

Definition

A tagged partition of $[a, b] \subseteq \overline{\mathbb{R}}$ is a finite collection

$$P = \{(t_1, I_1), (t_2, I_2), \dots, (t_K, I_K)\},$$

where the I_k are pairwise non-overlapping subintervals of $[a, b]$, with $\bigcup_{k=1}^{K} I_k = [a, b]$ and $t_k \in I_k$ are the tags. The mesh size of P is

$$\text{mesh}(P) := \max_{1 \leq k \leq K} \ell(I_k),$$

where

$$\ell(I) := \begin{cases} \text{length}(I), & \text{if } I \text{ is a bounded interval,} \\ 0, & \text{otherwise.} \end{cases}$$

Partition Sums

Definition

For $f\colon [a,b] \to \mathbb{R}$ and a tagged partition $P = \{(t_k, I_k)\}_{k=1}^{K}$ of $[a,b]$, let

$$\sum_P f := \sum_{k=1}^{K} f(t_k)\ell(I_k).$$

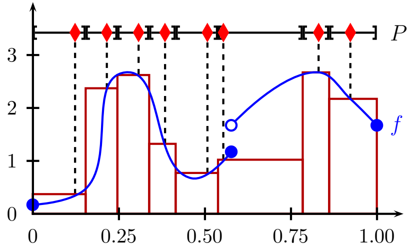


Figure: The partition sum $\sum_P f$ of $f\colon [0,1] \to \mathbb{R}$ over an 8-element tagged partition P of $[0,1]$, with tags shown by the red diamonds.

The Riemann Integral (1854)

Definition

Let $f\colon [a,b] \to \mathbb{R}$ be given. A number $v \in \mathbb{R}$ is called the Riemann integral of f over $[a,b]$ if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| \sum_P f - v \right| < \varepsilon$$

whenever P is a tagged partition of $[a,b]$ with $\mathrm{mesh}(P) \leq \delta$.

Theorem (Lebesgue's criterion for Riemann integrability)

$f\colon [a,b] \to \mathbb{R}$ is Riemann integrable iff it is bounded and continuous almost everywhere.

The Lebesgue Integral (1901–04)

- Integrate functions on any measure space $(X, \mathcal{A}, \mu)$.
- Start with simple functions: for $\alpha_k \in \mathbb{R}$ and pairwise disjoint $E_k \in \mathcal{A}$,

$$\int_X \sum_{k=1}^{K} \alpha_k \mathbb{1}_{E_k} \, \mathrm{d}\mu := \sum_{k=1}^{K} \alpha_k \mu(E_k).$$

- Then integrate non-negative functions by approximation from below:

$$\int_X f \, \mathrm{d}\mu := \sup \left\{ \int_X \varphi \, \mathrm{d}\mu \,\middle|\, \begin{array}{c} \varphi = \sum_{k=1}^{K} \alpha_k \mathbb{1}_{E_k} \colon X \to \mathbb{R} \text{ is simple and} \\ 0 \leq \varphi(x) \leq f(x) \text{ for } \mu\text{-a.e. } x \in X \end{array} \right\}.$$

- Then integrate real-valued functions:

$$\int_X f \, \mathrm{d}\mu := \int_X f_+ \, \mathrm{d}\mu - \int_X f_- \, \mathrm{d}\mu$$

provided that at least one of the integrals on the RHS is finite, which is not the case in Denjoy's example, $f(x) = x^{-1} \sin x^{-3}$.

Gauges

The construction of the gauge integral is very similar to that of the Riemann integral, but with one critical difference: instead of using the mesh size to measure the fineness of a tagged partition, we use a gauge, which need not be uniform in the integration domain.

Definition

A gauge on $[a, b] \subsetneq \mathbb{R}$ is a function

$$\gamma \colon [a, b] \to (0, +\infty).$$

A tagged partition $P = \{(t_k, I_k)\}_{k=1}^{K}$ of $[a, b]$ is said to be γ-fine if

$$I_k \subseteq \big(t_k - \gamma(t_k), t_k + \gamma(t_k)\big) \text{ for all } k \in \{1, \dots, K\}.$$

The idea is to choose γ according to the f that you want to integrate: where f is badly-behaved, γ is chosen to be small.

Gauges

In fact, a somewhat better definition that generalizes more easily to unbounded/multidimensional integration domains is the following:

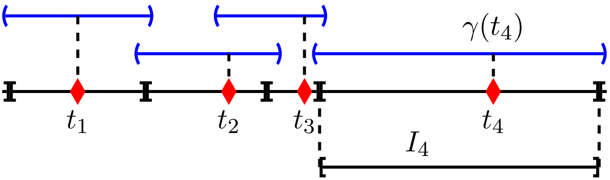
Definition

A gauge on $[a, b] \subseteq \overline{\mathbb{R}}$ is a function

$$\gamma \colon [a, b] \to \{\text{non-empty open intervals in } [a, b]\}.$$

A tagged partition $P = \{(t_k, I_k)\}_{k=1}^{K}$ of $[a, b]$ is said to be γ-fine if

$$I_k \subseteq \gamma(t_k) \text{ for all } k \in \{1, \ldots, K\}.$$



The (Henstock–Kurzweil) Gauge Integral

The (Henstock–Kurzweil[1]) definition of the gauge integral is now easy to state: it's just the Riemann integral with "mesh δ" replaced by "γ-fine".

Definition

Let $f\colon [a,b] \to \mathbb{R}$ be given. A number $v \in \mathbb{R}$ is called the gauge integral of f over $[a,b]$ if, for all $\varepsilon > 0$, there exists a gauge γ such that

$$\left| \sum_P f - v \right| < \varepsilon$$

whenever P is a γ-fine tagged partition of $[a,b]$. The gauge integral of f over $E \subseteq [a,b]$ is defined to be the integral of $f \cdot \mathbb{1}_E$ over $[a,b]$.

[1] Working independently: Henstock (1955) and Kurzweil (1957) developed the gauge formulation of the integral from the late 1950s onwards, both aware that they were simplifying the Denjoy–Luzin–Perron integral, but ignorant of each other's work.

Uniqueness and Cousin's Lemma

Lemma

The gauge integral of f, if it exists, is unique.

It also is important to check is that the condition "whenever P is a γ-fine tagged partition of $[a, b]$" is never vacuous:[2]

Lemma (Cousin)

Let γ be any gauge on $[a, b]$. Then there exists at least one γ-fine tagged partition of $[a, b]$.

The proof is a nice exercise, and basically boils down to a compactness argument on the open cover $\{\gamma(x) \,|\, x \in [a, b]\}$.

[2] Cousin was a student of Poincaré and got no credit for this when he proved it in 1895: instead, Lebesgue published it in his 1903 thesis, got the fame, and the result was known for a long time as the Borel–Lebesgue theorem. It's now sometimes called Thomson's lemma, since he generalized it to *full covers*.

How Gauges Beat Singularities

Example

Consider $f\colon [0,1] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1/\sqrt{x}, & \text{if } x > 0; \\ 0, & \text{if } x = 0. \end{cases}$$

This is gauge integrable with $\int_0^1 f = 2$. Given $\varepsilon > 0$, a choice of gauge γ that will ensure $|\sum_P f - 2| < \varepsilon$ for all γ-fine tagged partitions P of $[0,1]$ is

$$\gamma(x) := \begin{cases} \varepsilon x^2, & \text{if } x > 0; \\ \varepsilon^2, & \text{if } x = 0. \end{cases}$$

This has the neat effect of *forcing* 0 to be a tag.

Basic Properties of the Gauge Integral

Proposition

Let $f, g\colon [a,b] \to \mathbb{R}$ be gauge integrable, and let $\alpha, \beta \in \mathbb{R}$. Then

1. *$\alpha f + \beta g$ is gauge integrable with*
$$\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g,$$
i.e. the Denjoy space of gauge-integrable functions on $[a,b]$ is a vector space and the gauge integral is a linear functional;
2. *the gauge integral is monotone, i.e. $f \leq g$ a.e. $\implies \int_a^b f \leq \int_a^b g$;*
3. *the gauge integral is additive, i.e. for all $c \in (a,b)$,*
$$\int_a^b f = \int_a^c f + \int_c^b f.$$

The Gauge and Other Integrals

Proposition

1. *If f is Riemann integrable, then it is gauge integrable and the two integrals are equal.*[a]
2. *If f is Lebesgue integrable, then it is gauge integrable and the two integrals are equal.*[b]
3. *If f is non-negative, then it is Lebesgue integrable iff it is gauge integrable.*
4. *If f is gauge integrable, then it is Lebesgue measurable.*

[a] Good news for Hamming.
[b] Even more good news for Hamming.

$$f \text{ Lebesgue integrable} \iff |f| \text{ Lebesgue integrable}$$

$$|f| \text{ gauge integrable} \implies f \text{ gauge integrable} \not\Rightarrow |f| \text{ gauge integrable}$$

Improper Gauge Integrals

Recall that Denjoy was interested in making the improper integral

$$\lim_{\alpha\to 0}\int_\alpha^1 \frac{1}{x}\sin\frac{1}{x^3}\,\mathrm{d}x = \frac{\pi}{6} - \frac{1}{3}\int_0^1 \frac{\sin t}{t}\,\mathrm{d}t$$

into a legitimate integral. For the gauge integral, unlike the Riemann and Lebesgue integrals, there are *no* improper integrals:

Theorem (Hake)

$f\colon [a,b]\to\mathbb{R}$ *is gauge integrable iff it is gauge integrable over every* $[\alpha,\beta]\subseteq[a,b]$ *and the limit*

$$\lim_{\substack{\alpha\to a+\\ \beta\to b-}}\int_\alpha^\beta f$$

exists in $\mathbb{R}$, *in which case it equals* $\int_a^b f$.

Dominated Convergence Theorem

One of the most powerful results about the Lebesgue integral is the dominated convergence theorem, which gives conditions for pointwise convergence of integrands to imply convergence of the integrals. The gauge integral has a similar theorem, but we need two-sided bounds on the integrands:

Theorem (Dominated convergence theorem — Lebesgue integral)

Let $f_n \colon [a,b] \to \mathbb{R}$ be Lebesgue integrable for each $n \in \mathbb{N}$, and suppose that $(f_n)_{n\in\mathbb{N}}$ converges pointwise to $f \colon [a,b] \to \mathbb{R}$. If there exists a Lebesgue-integrable function $g \colon [a,b] \to \mathbb{R}$ such that $|f_n| \leq g$ for all $n \in \mathbb{N}$, then f is Lebesgue integrable with

$$\int_{[a,b]} f = \lim_{n\to\infty} \int_{[a,b]} f_n .$$

Dominated Convergence Theorem

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Theorem (Dominated convergence theorem — gauge integral)

Let $f_n \colon [a, b] \to \mathbb{R}$ *be gauge integrable for each* $n \in \mathbb{N}$*, and suppose that* $(f_n)_{n \in \mathbb{N}}$ *converges pointwise to* $f \colon [a, b] \to \mathbb{R}$*. If there exist gauge-integrable functions* $g, h \colon [a, b] \to \mathbb{R}$ *such that* $g \leq f_n \leq h$ *for all* $n \in \mathbb{N}$*, then* f *is gauge integrable with*

$$\int_a^b f = \lim_{n \to \infty} \int_a^b f_n.$$

Bounded Variation and Absolute Integrability

Definition

$F\colon [a,b] \to \mathbb{R}$ has bounded variation (is BV) if

$$V(F,[a,b]) := \sup \left\{ \sum_{k=1}^{K} |F(x_k) - F(y_k)| \;\middle|\; \begin{array}{c} \{[x_k,y_k]\}_{k=1}^{K} \text{ is a collection} \\ \text{of non-overlapping} \\ \text{subintervals of } [a,b] \end{array} \right\}$$

is finite.

Theorem (BV criterion for absolute gauge integrability)

Let $f\colon [a,b] \to \mathbb{R}$ be gauge integrable and let $F(x) := \int_a^x f$. Then $|f|$ is gauge integrable iff F is BV, in which case

$$\int_a^b |f| = V(F,[a,b]).$$

Absolute Continuity

Definition

$F\colon [a,b] \to \mathbb{R}$ is absolutely continuous (or AC) on $E \subseteq [a,b]$ if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{k=1}^{K} |F(x_k) - F(y_k)| < \varepsilon$$

whenever $\{[x_k, y_k]\}_{k=1}^{K}$ are pairwise non-overlapping subintervals of $[a,b]$ with endpoints in E and $\sum_{k=1}^{K} |x_k - y_k| < \delta$.

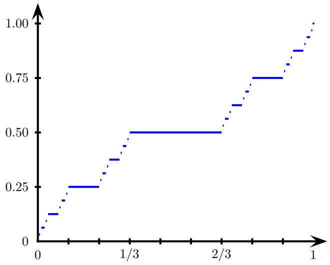

Figure: The Cantor–Lebesgue function, a.k.a. the Devil's Staircase.

Absolute continuity is a stronger requirement than ordinary continuity (which is just AC with $K = 1$) and is also stronger than having bounded variation (the Devil's Staircase is continuous and BV, but not AC).

Absolute Continuity and Lebesgue Integration

Theorem (Lebesgue)

Let $F\colon [a,b] \to \mathbb{R}$. The following are equivalent:

1. *F is AC;*
2. *F is differentiable almost everywhere, F' is Lebesgue integrable, and*
$$F(x) - F(a) = \int_{[a,x]} F' \text{ for all } x \in [a,b];$$
3. *there exists a Lebesgue-integrable $g\colon [a,b] \to \mathbb{R}$ such that*
$$F(x) - F(a) = \int_{[a,x]} g \text{ for all } x \in [a,b].$$

But... it is not true that every continuous and a.e.-differentiable F has Lebesgue-integrable derivative!

Absolute Continuity and Lebesgue Integration

Example

For $0 < \alpha < 1$ (Denjoy $\longleftrightarrow \alpha = \frac{2}{3}$), consider $F\colon [0, \frac{1}{\pi}] \to \mathbb{R}$ defined by

$$F(x) := \begin{cases} x^\alpha \sin x^{-1}, & \text{if } x > 0, \\ 0, & \text{if } x = 0. \end{cases}$$

This is continuous for all $\alpha > 0$, and differentiable except at $x = 0$, but not BV (and hence not AC).

$$F'(x) = x^{\alpha-1} \sin x^{-1} - x^{\alpha-2} \cos x^{-1} \text{ for a.e. } x \in [0, \tfrac{1}{\pi}],$$

which is not Lebesgue integrable over $[0, \frac{1}{\pi}]$, so

$$0 = F(\tfrac{1}{\pi}) - F(0) \neq \int_{[0, \frac{1}{\pi}]} F'.$$

Generalized Absolute Continuity

Denjoy (1912) defined his integral using “totalization”, a procedure of transfinite induction over possible singularities. The same year, Luzin made the connection with a generalized version of absolute continuity:

Definition

F is AC$_*$ on $E \subseteq [a, b]$ if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{k=1}^{K} \sup_{x,y \in [x_k, y_k]} |F(x) - F(y)| < \varepsilon$$

whenever $\{[x_k, y_k]\}_{k=1}^{K}$ are pairwise non-overlapping subintervals of $[a, b]$ with endpoints in E and $\sum_{k=1}^{K} |x_k - y_k| < \delta$. F is ACG$_*$ if it is continuous and E is a countable union of sets on which F is AC$_*$.

$$\left\{ F \,\middle|\, \begin{array}{c} F \text{ continuous and} \\ \text{differentiable n.e.} \end{array} \right\} \subsetneq \mathrm{ACG}_* \subsetneq \left\{ F \,\middle|\, \begin{array}{c} F \text{ continuous and} \\ \text{differentiable a.e.} \end{array} \right\}$$

Fundamental Theorem of Calculus

One of the gauge integral's best features is that it has the "ideal" fundamental theorem of calculus:

Theorem (Fundamental Theorem of Calculus)

1. *Let* $f\colon [a,b] \to \mathbb{R}$. *Then* $\int_a^b f$ *exists and* $F(x) = \int_a^x f$ *for all* $x \in [a,b]$ *iff* F *is ACG*$_*$ *on* $[a,b]$, $F(a) = 0$, *and* $F' = f$ *a.e. in* (a,b). *If* $\int_a^b f$ *exists and* f *is continuous at* $x \in (a,b)$, *then* $F'(x) = f(x)$.
2. *Let* $F\colon [a,b] \to \mathbb{R}$. *Then* F *is ACG*$_*$ *iff* F' *exists a.e. in* (a,b), F' *is gauge integrable on* $[a,b]$, *and* $\int_a^x F' = F(x) - F(a)$ *for all* $x \in [a,b]$.

Corollary

Let $F\colon [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable nearly everywhere on* (a,b). *Then* F' *is gauge integrable on* $[a,b]$ *and* $\int_a^x F' = F(x) - F(a)$ *for all* $x \in [a,b]$.

Integration by Parts

Theorem (Integration by parts)

Let $f\colon [a,b] \to \mathbb{R}$ be gauge integrable and let $G\colon [a,b] \to \mathbb{R}$ be AC; let $F\colon [a,b] \to \mathbb{R}$ be the indefinite gauge integral of f. Then fG is gauge integrable on $[a,b]$ with

$$\int_a^b fG = F(b)G(b) - \int_{[a,b]} FG'.$$

If G is not AC, but just BV, then the Lebesgue integral on the right becomes a Lebesgue–Stieltjes integral to allow for proper treatment of jumps and Cantor parts:

$$\int_a^b fG = F(b)G(b) - \int_{[a,b]} F \, \mathrm{d}G.$$

Differentiation Under the Integral Sign

The fundamental theorem of calculus for the gauge integral pretty quickly yields necessary and sufficient conditions for differentiation under the integral sign: the conditions depend on being able to integrate every derivative.

Theorem (Differentiation Under the Integral Sign)

Let $f \colon [\alpha, \beta] \times [a, b] \to \mathbb{R}$ *be such that* $f(\cdot, y)$ *is ACG*$_*$ *on* $[\alpha, \beta]$ *for almost all* $y \in [a, b]$. *Then* $F := \int_a^b f(\cdot, y)\, \mathrm{d}y$ *is ACG*$_*$ *on* $[\alpha, \beta]$ *and* $F'(x) = \int_a^b \partial_x f(x, y)\, \mathrm{d}y$ *for almost all* $x \in (\alpha, \beta)$ *iff*

$$\int_s^t \int_a^b \partial_x f(x, y)\, \mathrm{d}y\, \mathrm{d}x = \int_a^b \int_s^t \partial_x f(x, y)\, \mathrm{d}x\, \mathrm{d}y$$

for all $[s, t] \subseteq [\alpha, \beta]$.

Cauchy's Mistake Revisited

How does this theorem relate to Cauchy's mistake? Consider

$$f(x, y) := \sin(y^2) \cos(xy).$$

We have $\partial_x f(x, y) = -y \sin(y^2) \sin(xy)$, and the criterion from the theorem is that

$$\int_s^t \int_0^\infty \partial_x f(x, y) \, \mathrm{d}y \, \mathrm{d}x = \int_0^\infty \int_s^t \partial_x f(x, y) \, \mathrm{d}x \, \mathrm{d}y$$

for all $[s, t] \subseteq$ the range of differentiation. But

$$\int_s^t \underbrace{\int_0^\infty y \sin(y^2) \sin(xy) \, \mathrm{d}y}_{\text{not convergent!}} \, \mathrm{d}x \neq \int_0^\infty \int_s^t y \sin(y^2) \sin(xy) \, \mathrm{d}x \, \mathrm{d}y$$

$$= \int_0^\infty (\cos(sy) - \cos(ty)) \sin(y^2) \, \mathrm{d}y$$

$$= 0 \text{ if } t - s \in 2\pi\mathbb{Z}.$$

Linear Operators

In fact, we have

Theorem

Let S *be any set and let* Λ *be any linear functional defined on a subspace of* $S^{\mathbb{R}}$*. Let* $f\colon [\alpha,\beta]\times S \to \mathbb{R}$ *be such that* $f(\cdot,y)$ *is ACG*$_*$ *on* $[\alpha,\beta]$ *for almost all* $y\in[a,b]$*. Define* $F\colon[\alpha,\beta]\to\mathbb{R}$ *by* $F(x) := \Lambda[f(x,\cdot)]$*. Then* F *is ACG*$_*$ *on* $[\alpha,\beta]$ *and* $F'(x) = \Lambda\big[\partial_x f(x,\cdot)\big]$ *for almost all* $x\in(\alpha,\beta)$ *iff*

$$\int_s^t \Lambda\big[\partial_x f(x,\cdot)\big]\,\mathrm{d}x = \Lambda\left[\int_s^t \partial_x f(x,\cdot)\,\mathrm{d}x\right]$$

for all $[s,t]\subseteq[\alpha,\beta]$*.*

The previous theorem was the special case $S=[a,b]$ and

$$\Lambda[f(x,\cdot)] := \int_a^b f(x,y)\,\mathrm{d}y.$$

The McShane Integral

Definition

A γ-fine free tagged partition of $[a, b]$ is a finite collection

$$P = \{(t_1, I_1), (t_2, I_2), \dots, (t_K, I_K)\},$$

where the I_k are pairwise non-overlapping subintervals of $[a, b]$ such that $\bigcup_{k=1}^{K} I_k = [a, b]$ and $I_k \subseteq \gamma(t_k)$, but it is *not* required that $t_k \in I_k$.

The McShane integral of $f \colon [a, b] \to \mathbb{R}$ is the (unique) $v \in \mathbb{R}$ such that, for all $\varepsilon > 0$, there exists a gauge γ such that $|\sum_P f - v| < \varepsilon$ whenever P is a γ-fine free tagged partition of $[a, b]$.

Theorem

The McShane and Lebesgue integrals coincide.

Multidimensional Integrals

Banach-space valued functions

To integrate a function f taking values in a Banach space $(X, \| \cdot \|_X)$ to get an integral value $v \in X$, just replace absolute values by norms, *i.e.*

$$\left| \sum_P f - v \right| < \varepsilon \text{ by } \left\| \sum_P f - v \right\|_X < \varepsilon.$$

Division spaces

To integrate functions defined on more exotic domains than $[a, b] \subseteq \overline{\mathbb{R}}$, we need a collection of "basic sets" with known "size" that can be used to form non-overlapping partitions of the integration domain — an integration basis. Henstock did a lot of work in this area in his later career.

From Integrals Back to Measures

In Lebesgue's construction of the integral, one first defines Lebesgue measure λ on the Lebesgue σ-algebra $\mathcal{B}_0$ of $\mathbb{R}$ and the defines the integral step-by-step, starting with $\int_{\mathbb{R}} \mathbb{1}_E \, \mathrm{d}\lambda := \lambda(E)$. Why not go the other way?

Definition

Call $E \subseteq \mathbb{R}$ gauge measurable if $\mathbb{1}_E$ is gauge-integrable; let $\mathcal{G}$ denote the set of all gauge-measurable subsets of $\mathbb{R}$; and define

$$\mu(E) := \int_{-\infty}^{+\infty} \mathbb{1}_E \equiv \int_E 1 \text{ for each } E \in \mathcal{G}.$$

Theorem

$\mathcal{G} = \mathcal{B}_0$ and $\mu = \lambda$.

From Integrals Back to Measures

There are interesting generalizations by Thomson (1994) and Pfeffer (1999) that generate more interesting regular Borel measures on $\mathbb{R}$.

Definition

Given $F\colon [a,b] \to \mathbb{R}$, the variational measure μ_F is the measure defined by

$$\mu_F(E) := \inf_{\substack{\text{gauges} \\ \gamma \text{ on } [a,b]}} \sup_{\substack{\gamma\text{-fine } P \\ \text{with tags in } E}} \sum_{k=1}^{K} |F(I_k)|.$$

Theorem (Pfeffer)

If μ_F is an absolutely continuous measure (i.e. $\mu_F(E) = 0$ whenever $\lambda(E) = 0$), then F is the gauge integral of its derivative, which exists a.e. If, furthermore, μ_F is a finite measure, then F is the Lebesgue integral of its derivative.

Ongoing Gauge-Integral-Related Research

- **Applications to dynamical systems (Kurzweil's motivation)**
  P. Krejčí & M. Liero. "Rate independent Kurzweil processes." *Appl. Math.* **54**(2):117–145, 2009.
- **Gauge integrals in harmonic analysis**
  E. Talvila. "Henstock–Kurzweil Fourier transforms." *Illinois J. Math.* **46**(4):1207–1226, 2002
  F. J. Mendoza Torres *& al.* "About an existence theorem of the Henstock-Fourier transform." *Proyecciones* **27**(3):307–318, 2008.
- **The search for a "natural" topology on Denjoy space**
  A. Alexiewicz. "Linear functionals on Denjoy-integrable functions." *Colloq. Math.* **1**:289–293, 1948:
  $$\|f\|_{\mathsf{A}} := \sup_{[\alpha,\beta]\subseteq[a,b]} \left| \int_{\alpha}^{\beta} f \right| .$$
  J. Kurzweil. *Henstock–Kurzweil Integration: its Relation to Topological Vector Spaces.* World Scientific, River Edge, NJ, 2000.
- **Stochastic gauge integrals (strict generalization of Itō?)**
  E. J. McShane. *A Riemann-Type Integral that Includes Lebesgue–Stieltjes, Bochner and Stochastic Integrals.* AMS, Providence, RI, 1969.
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