Thermalization of rate-independent processes by entropic regularization

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Many evolutionary problems of interest take the form of a gradient descent; a trajectory that is the steepest descent in a space $Q$ of an energy functional $E$ with respect to some dissipation functional $\Psi$. The state space $Q$ is, in general, a metric space that may lack any linear or differentiable structure [1]. Typically, the dissipation potential $\Psi$ is assumed to have superlinear growth at infinity; if instead $\Psi$ is homogeneous of degree 1, then the resulting evolutionary system is rate-independent (or quasi-static) in the sense that the solution operator commutes with strictly increasing reparametrizations of time. Rate-independent processes model hysteretic phenomena such as plasticity and phase transformations in elastic solids, electromagnetism, dry friction on surfaces, and pinning problems in superconductivity; such models are limiting models in the limit of vanishing inertia, relaxation times, and thermal effects [5].

We consider a model for the influence of a heat bath upon a rate-independent evolution that takes values in a (finite-dimensional, smooth) Riemannian manifold $(Q,g)$ over an interval of time $[0,T]$. The energetic potential is $E: [0,T] \times Q \to \mathbb{R}$; the dissipation potential is $\Psi: [0,T] \times \mathcal{T}Q \to [0,\infty)$ and is assumed to be continuous, convex, non-degenerate and homogeneous of degree 1 on each tangent space — i.e. $\Psi$ defines a time-dependent Finsler metric on $Q$.

Given $h > 0$ and discrete times $(t_i := ih)_{i=0}^{T/h}$, the incremental variational formulation of the rate-independent problem is to find states $(z_i)_{i=0}^{T/h} \subseteq Q$ such that each $z_{i+1}$ minimizes

\[
W(z_i, z_{i+1}) := E(t_{i+1}, z_{i+1}) - E(t_i, z_i) + h\Psi\left(\log_{z_i} z_{i+1}/h\right),
\]

where $\log_{z_i}$ denotes the inverse of the exponential map $\exp_{z_i}$ from $T_zQ$ into $Q$. The continuous-time rate-independent process $z: [0,T] \to Q$ is the limit as $h \to 0$ of the interpolants of the solutions to (1), and is the $\Psi$-gradient descent in $E$:

$D\Psi(t, z(t), \dot{z}(t)) \ni -DE(t, z(t))$.

We posit a Markov chain model for the effect of a heat bath working with constant power $\theta > 0$ (i.e. the heat bath supplies energy proportional to $\theta h$ over each subinterval $[t_i, t_{i+1}]$): we consider the $Q$-valued Markov chain $Z^h$ with transition probabilities having density

\[
\rho(z_{i+1}|z_i) \propto \exp\left(-W(z_i, z_{i+1})/\theta h\right)
\]

with respect to the Riemannian volume measure $d\text{Vol}_{(Q,g)}$. This density has the variational characterization that it minimizes

\[
\int_Q \left(W(z_i, \cdot) + \theta h \rho(\cdot|z_i) \log \rho(\cdot|z_i)\right) d\text{Vol}_{(Q,g)};
\]
and so the Markov chain model can be seen as a competition between the energetic considerations (1) of the original gradient descent and entropic considerations. The natural objective is to identify the continuous-time limit process of $Z^h$ as $h \to 0$.

If $\Psi$ is homogeneous of degree 2, then the scheme (2) corresponds to the addition of Itô noise to generate a stochastic gradient descent in $E$ as in [4]. Our main result is that when $\Psi$ is homogeneous of degree one, although $Z^h$ is a stochastic process with non-trivial distribution for each $h > 0$, the limit process as $h \to 0$ is a deterministic rate-dependent process. Furthermore, the limit process is, up to sign, a gradient descent in $E$ with respect to a new, nonlinear, dissipation potential $\tilde{\Psi}: [0, T] \times TQ \to [0, +\infty)$.

More precisely, let $\Psi$ denote the Cramer transform of $\Psi$, defined by

\[
\tilde{\Psi}^*(t, x, \ell) := \log \int_{T^*_x Q} \exp \left( - (\langle \ell, v \rangle + \Psi(t, x, v)) \right) dv \quad \text{for } \ell \in T^*_x Q,
\]

\[
\tilde{\Psi}(t, x, v) := \sup \left\{ (\ell, v) - \tilde{\Psi}^*(t, x, \ell) \big| \ell \in T^*_x Q \right\} \quad \text{for } v \in T_x Q.
\]

The Cramer transform $\tilde{\Psi}$ is a strict convexification and smoothing-out of $\Psi$. It exhibits quadratic behaviour near the origin (slow evolutionary rates) and linear growth at infinity (fast evolutionary rates); see Figure 1.

The limit process of $Z^h$ as $h \to 0$ is the solution $y: [0, T] \to Q$ of

\[
D\tilde{\Psi}(t, y(t), -\dot{y}(t)/\theta) = DE(t, y(t)),
\]

and, under suitable conditions, $Z^h$ converges to $y$ in probability as $h \to 0$, i.e.

\[
\lim_{h \to 0} P \left[ \sup_{t \in [0, T]} d(\mathcal{Q}, \mathcal{D}) (Z^h(t), y(t)) \geq \delta \right] = 0 \quad \text{for every } \delta > 0.
\]
The intuition behind this result is that, to a first approximation,
\[
E[\log_z Z_{i+1}^h | Z_i^h = z_i] \approx -\theta h \tilde{D} \Psi^*(t_i, z_i, DE(t_i, z_i)),
\]
and so the variance is expected to be negligible in the limit as \( h \to 0 \), leaving only the mean flow (5). The principal condition necessary to ensure the convergence (6) is that the curvature of \((Q, g)\) and the vector field \( f(t, x) := -\tilde{D} \Psi^*(t, x, DE(t, x)) \) be such that geodesics starting at \((t, x)\) and \((t, x')\) near to the trajectory of \( y \) with initial velocities given by \( f(t, x) \) and \( f(t, x') \) do not diverge too quickly. When \( Q = \mathbb{R}^n \) with its usual metric, this corresponds to the requirement that \( f \) be a monotone vector field \([7]\), at least near the trajectory of \( y \).

As shown in \([6]\), this result predicts rheological power laws such as the Andrade creep law for soft metals. Andrade \([2, 3]\) observed that soft metals exposed to constant subcritical applied stress at room temperature exhibited strain \( \sim t^{1/3} \) for short time, and \( \sim t \) in long time. Under the assumption of linear strain hardening and constant applied load — i.e. the dissipative potential \( \Psi(t, x, v) \) is \( x|v| \) on \( Q = (0, +\infty) \) and energetic potential \( E(t, x) = -\ell x \) with \(|\ell| < 1\) — Andrade’s law appears naturally, since in this setting solutions to (5) do indeed grow \( \sim t^{1/3} \).

This example justifies posing the problem on a manifold with state-dependent dissipation functional instead of the simpler setting of \( \mathbb{R}^n \) with a state-independent dissipation functional: the Andrade creep law would not be obtained in the simpler setting. Furthermore, some rate-independent problems on \( \mathbb{R}^n \) have a dissipation functional \( \Psi \) that is 0 or \( +\infty \) in some directions; restricting attention to a submanifold of \( \mathbb{R}^n \) on which \( \Psi \) is well-behaved can circumvent these difficulties.

It would be of interest to extend the above results to evolutions in spaces without a locally linear, smooth, or finite-dimensional structure, as in \([1]\).

References


