MA377 Rings and Modules

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1 Rings

1.1 Rings

Definitions 1.1.1. Let R be a non-empty set that has two laws of composition defined on it. (We call these laws *addition* and *multiplication* and use the familian notatation.) We say that R is a *ring* (with respect to the given addition and multiplication) if the following hold:

(i) $a+b$ and $ab \in R$ for all $a, b \in R$;

(ii) $a+b = b+a$ for all $a, b \in R$;

(iii) $a+(b+c) = (a+b)+c$ for all $a, b, c \in R$;

(iv) there exists an element $0 \in R$ such that $a+0 = a$ for all $a \in R$;

(v) given $a \in R$ there exists an element $-a \in R$ such that $a+(-a) = 0$;

(vi) $a(bc) = (ab)c$ for all $a, b, c \in R$;

(vii) $(a+b)c = ac+bc$ for all $a, b, c \in R$;

(viii) $a(b+c) = ab+ac$ for all $a, b, c \in R$.

Thus, a ring is an additive Abelian group on which an operation of multiplication is defined, this operation being associative and distributive (on both sides) with respect to the addition.

R is called a *commutative ring* if, in addition, it satisfies $ab = ba$ for all $a, b \in R$. The term *non-commutative ring* can be a little ambiguous. When applied to a particular example it clearly means that the ring is not commutative. However, when we discuss a class of "non-commutative rings" we mean "not necessarily commutative rings", and it is usually not intended to exclude the commutative rings in that class.

If there is an element $1 \in R$ such that $1a = a1 = a$ for all $a \in R$ we say R has an *identity*.

1.2 Examples of Rings

Example 1.2.1. The integers $\mathbb{Z}$, the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$ all with the usual operations.

Example 1.2.2. $R[x]$, the *polynomial ring* in an indeterminate x with coefficients in R, with $xr = rx$ for all $r \in R$.

Example 1.2.3. $M_n(R) := \{n \times n$ matrices over the ring $R\}$.

Example 1.2.4. $T_n(R) := \{n \times n$ upper-triangular matrices over the ring $R\}$.

Example 1.2.5. $U_n(R) := \{n \times n$ strictly upper-triangular matrices over the ring $R\}$.

Example 1.2.6. $F\langle x_1, \ldots, x_n\rangle$, the *free algebra* over a field F with generators $x_1, \ldots, x_n$. The generators do not commute, so $x_1x_2x_1x_3 \neq x_1^2x_2x_3$.

Example 1.2.7. $A_1(\mathbb{C})$, the *first Weyl algebra*, which is the ring of polynomials in x and y with coefficients in $\mathbb{C}$, where x, y do not commute but $xy - yx = 1$.

Example 1.2.8. Subrings of the above, such as $\mathcal{J} := \{a + ib | a, b \in \mathbb{Z}\}$.

1.3 Properties of Addition and Multiplication

We typically write $a - b$ for $a + (-b)$.

Proposition 1.3.1. *The following hold for any ring R:*

(i) the element $0 \in R$ is unique;

(ii) given $a \in R$, $-a$ is unique;

(iii) $-(-a) = a$ for all $a \in R$;

(iv) for any $a, b, c \in R$, $a + b = a + c \Leftrightarrow b = c$;

(v) given $a, b \in R$, the equation $x + a = b$ has a unique solution $x = b - a$;

(vi) $-(a + b) = -a - b$ for all $a, b \in R$;

(vii) $-(a-b) = -a+b$ *for all* $a, b \in R$*;*

(viii) $a0 = 0a = 0$ *for all* $a \in R$*;*

(ix) $a(-b) = (-a)b = -(ab)$ *for all* $a, b \in R$*;*

(x) $(-a)(-b) = ab$ *for all* $a, b \in R$*;*

(xi) $a(b-c) = ab - ac$ *for all* $a, b, c \in R$*.*

1.4 Subrings and Ideals

Definition 1.4.1. A subset S of a ring R is called a *subring* of R if S is itself a ring with respect to the laws of composition of R.

Proposition 1.4.2. *A non-empty subset* S *of a ring* R *is a subring of* R *if and only if* $a - b \in S$ *and* $ab \in S$ *whenever* $a, b \in S$*.*

Proof. If S is a subring then obviously the given condition is satisfied. Conversly, suppose that the condition holds. Take any $a \in S$: $a - a = 0 \in S$. For any $x \in S$, $0 - x = -x \in S$. So, if $a, b \in S$, $a - (-b) = a + b \in S$. So S is closed with respect to both addition and multiplication. Thus S is a subring since all the other axioms are automatically satisfied. □

Examples 1.4.3. (i) $2\mathbb{Z}$, the subset of even integers, is a subring of $\mathbb{Z}$.

(ii) $\mathbb{Z}$ is a subring of the polynomial ring $\mathbb{Z}[x]$.

Definition 1.4.4. A subset I of a ring R is called an *ideal* if

(i) I is a subring of R;

(ii) for all $a \in I$ and $r \in R$, $ar \in I$ and $ra \in I$.

If I is an ideal of R we denote this fact by $I \trianglelefteq R$.

Examples 1.4.5. (i) Let R be a non-zero ring. Then R has at least two ideals, namely R and $\{0\}$. We often write 0 for $\{0\}$.

(ii) $2\mathbb{Z}$ is an ideal of $\mathbb{Z}$.

Proposition 1.4.6. *Let* I *be a non-empty subset of a ring* R*. Then* $I \trianglelefteq R$ *if and only if for all* $a, b \in I$ *and* $r \in R$*,* $a - b \in I$*,* $ar \in I$*,* $ra \in I$*.*

Proof. Exercise. □

1.5 Cosets and Homomorphisms

Definition 1.5.1. Let I be an ideal of a ring R and $x \in R$. Then the set of elements $x + I := \{x + i | i \in I\}$ is the *coset* of x in R with respect to I.

When dealing with cosets, it is important to realise that, in general, a given coset can be represented in more than one way. The next lemma shows for the coset representatives are related.

Lemma 1.5.2. *Let R be a ring with an ideal I and $x, y \in R$. Then $x + I = y + I \Leftrightarrow x - y \in I$.*

Proof. Exercise. □

We denote the set of all cosets of R with respect to I by $\frac{R}{I}$. We can give $\frac{R}{I}$ the structure of a ring as follows: define

$$(x + I) + (y + I) := (x + y) + I$$

and

$$(x + I)(y + I) := xy + I$$

for $x, y \in R$. The key point here is that the sum and product on $\frac{R}{I}$ are well-defined; that is, they are independent of the coset representatives chosen. Check this and make sure that you understand why the fact that I is an ideal is crucial to the proof.

Definition 1.5.3. $\frac{R}{I}$ is called the *residue class ring* (or *quotient ring* or *factor ring*) of R with respect to I.

The zero element of $\frac{R}{I}$ is $0 + I = i + I$ for any $i \in I$.

If $I \subseteq S \subseteq R$ we denote by $\frac{S}{I}$ the subset $\{s + I | s \in S\} \subseteq \frac{R}{I}$.

Proposition 1.5.4. *Let $I \trianglelefteq R$. Then*

(i) every ideal of the ring $\frac{R}{I}$ is of the form $\frac{K}{I}$ where $K \trianglelefteq R$ and $K \supseteq I$. Conversely, $K \trianglelefteq R$, $K \supseteq I \Rightarrow \frac{K}{I} \trianglelefteq \frac{R}{I}$;

(ii) there is a one-to-one correspondence between the ideals of $\frac{R}{I}$ and the ideals of R containing I.

Proof. (i) If $K^* \trianglelefteq \frac{R}{I}$ then define $K := \{x \in R | x + I \in K^*\}$. Then $K \trianglelefteq R$, $K \supseteq I$ and $\frac{K}{I} = K^*$.

(ii) The correspondence is given by $K \leftrightarrow \frac{K}{I}$, where $I \subseteq K \trianglelefteq R$. □

Definitions 1.5.5. A map of rings $\theta : R \to S$ is a (*ring*) *homomorphism* if $\theta(x + y) = \theta(x) + \theta(y)$ and $\theta(xy) = \theta(x)\theta(y)$ for all $x, y \in R$. θ defined by $\theta(r) = 0$ for all $r \in R$ is a homomorphism; it is called the *zero homomorphism*. ϕ defined by $\phi(r) - r$ for all $r \in R$ is also a homomorphism; it is called the *identity homomorphism*. Let $I \trianglelefteq R$. Then $\sigma : R \to \frac{R}{I}$ defined by $\sigma(x) = x + I$ for $x \in R$ is a homomorphism of R onto $\frac{R}{I}$; it is called the *natural* (or *canonical*) *homomorphism* (of R onto $\frac{R}{I}$).

Proposition 1.5.6. *Let R, S be rings and $\theta : R \to S$ a homomorphism. Then*

(i) $\theta(0_R) = 0_S$;

(ii) $\theta(-r) = -\theta(r)$ *for all* $r \in R$;

(iii) the kernel $\ker\theta := \{x \in R | \theta(x) = 0_S\}$ *is an ideal of* R;

(iv) the image $\theta(R) := \{\theta(r) | r \in R\}$ *is a subring of* S.

Proof. Exercise. □

Definitions 1.5.7. Let $\theta : R \to S$ be a ring homomorphism. Then θ is called an *isomorphism* if θ is a bijection. We say that R and S are *isomorphic* rings and denote this by $R \cong S$.

1.6 The Isomorphism Theorems

Theorem 1.6.1. (The First Isomorphism Theorem.) *Let $\theta : R \to S$ be a homomorphism of rings. Then*

$$\theta(R) \cong \frac{R}{\ker\theta}.$$

Proof. Let $I := \ker\theta$ and define $\sigma : \frac{R}{I} \to \theta(R)$ by $\sigma(x + I) := \theta(x)$ for $x \in R$. The map σ is well-defined since for $x, y \in R$,

$$x + I = y + I \Rightarrow x - y \in I = \ker\theta \Rightarrow \theta(x - y) = 0 \Rightarrow \theta(x) = \theta(y).$$

σ is easily seen to be the required isomorphism. □

Theorem 1.6.2. (The Second Isomorphism Theorem.) *Let I be an ideal and L a subring of R. Then*

$$\frac{L}{L \cap I} \cong \frac{L+I}{I}.$$

Proof. Let σ be the natural homomorphism $R \to \frac{R}{I}$. Restrict σ to the ring L. We have $\sigma(L) = \frac{L+I}{I}$, a subring of $\frac{R}{I}$. The kernel of σ restricted to L is $L \cap I$. Now apply Theorem 1.6.1. □

Theorem 1.6.3. (The Third Isomorphism Theorem.) *Let $I, K \trianglelefteq R$ be such that $I \subseteq K$. Then*

$$\frac{R/I}{K/I} \cong \frac{R}{K}.$$

Proof. $\frac{K}{I} \trianglelefteq \frac{R}{I}$ and so $\frac{R/I}{K/I}$ is defined. Define a map $\gamma : \frac{R}{I} \to \frac{R}{K}$ by $\gamma(x+I) := x+K$ for all $x \in R$. The map γ is easily seen to be well-defined and a homomorphism onto $\frac{R}{K}$. Further,

$$\begin{aligned} \gamma(x+I) = K &\Leftrightarrow x+K = K \\ &\Leftrightarrow x \in K \\ &\Leftrightarrow x+I \in \frac{K}{I} \text{ since } K \supseteq I \end{aligned}$$

Therefore, $\ker \gamma = \frac{K}{I}$. Now apply Theorem 1.6.1. □

1.7 Direct Sums

Definitions 1.7.1. Let $\{I_\lambda\}_{\lambda \in \Lambda}$ be a collection of ideals of a ring R. We define their (*internal*) *sum* to be

$$\sum_{\lambda \in \Lambda} I_\lambda := \left\{ x \in R \,\middle|\, x = \sum_{i=1}^{k} x_i, x_i \in I_{\lambda_i}, k \in \mathbb{N} \right\},$$

the set of all finite sums of elements of the I_λ's. We say that the sum of the I_λ's is *direct* if each element of $\sum_{\lambda \in \Lambda} I_\lambda$ is uniquely expressible as $x_1 + \cdots + x_k$ with $x_i \in I_{\lambda_i}$. In this case we denote the sum as $\bigoplus_{\lambda \in \Lambda} I_\lambda$, or $I_1 \oplus \cdots \oplus I_n$ if Λ is finite.

Proposition 1.7.2. *The sum $\sum_{\lambda \in \Lambda} I_\lambda$ is direct if and only if $I_\mu \cap \sum_{\lambda \in \Lambda \setminus \{\mu\}} I_\lambda = 0$ for all $\mu \in \Lambda$.*

Proof. Exercise. □

Definition 1.7.3. Let $R_1, \ldots, R_n$ be rings. We define their *external direct sum* S to be the set of all n-tuples $\{(r_1, \ldots, r_n) | r_i \in R_i\}$. On S we define addition and multiplication componentwise, thus making S into a ring. We write $S = R_1 \oplus \cdots \oplus R_n$.

The set $(0, \ldots, 0, R_j, 0, \ldots, 0)$ is an ideal of S. Clearly S is the internal direct sum of the ideals $(0, \ldots, 0, R_j, 0, \ldots, 0)$ for $j = 1, \ldots, n$. But $(0, \ldots, 0, R_j, 0, \ldots, 0) \cong R_j$. Because of this S can be considered as a ring in which the R_j are ideals and S is their internal direct sum. Also, in Definitions 1.7.1 we can consider $I_1 \oplus \cdots \oplus I_n$ to be the external direct sum of the rings I_j. Hence, in practice, we do not need to distinguish between external and internal direct sums.

1.8 Division Rings

Definition 1.8.1. Let R be a ring with 1. An element $u \in U$ is a *unit* (or an *invertible element*) if there is a $v \in R$ such that $uv = vu = 1$. The element v is called the *inverse* of u and is denoted u^{-1}.

Definitions 1.8.2. A ring D with at least two elements is called a *division ring* (or a *skew field*) if D has an identity and every non-zero element of D has an inverse in D. A division ring in which the multiplication is commutative is called a *field*.

Example 1.8.3. (The Quaternions.) Let $\mathbb{H}$ be the set of all symbols $a_0 + a_1 i + a_2 j + a_3 k$ where $a_i \in \mathbb{R}$. Two such symbols $a_0 + a_1 i + a_2 j + a_3 k$ and $b_0 + b_1 i + b_2 j + b_3 k$ are considered to be equal if and only if $a_i = b_i$ for $i = 0, 1, 2, 3$.

We make $\mathbb{H}$ into a ring as follows: addition is componentwise and two elements of $\mathbb{H}$ are multiplied term-by-term using the relations $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. Then $\mathbb{H}$ is a non-commutative ring with zero $0 := 0 + 0i + 0j + 0k$ and identity $1 := 1 + 0i + 0j + 0k$.

Let $a_0 + a_1 i + a_2 j + a_3 k$ be a non-zero element of $\mathbb{H}$, so not all the a_i are zero. We have

$$(a_0 + a_1 i + a_2 j + a_3 k)(a_0 - a_1 i - a_2 j - a_3 k) = a_0^2 + a_1^2 + a_2^2 + a_3^2 \neq 0.$$

So, letting $n := a_0^2 + a_1^2 + a_2^2 + a_3^2$, the element $\frac{a_0}{n} - \frac{a_1}{n}i - \frac{a_2}{n} - \frac{a_3}{n}$ is the inverse of $a_0 + a_1 i + a_2 j + a_3 k$.

Thus, $\mathbb{H}$ is a division ring. It is called the division ring of *real quaternions*. *Rational quaternions* can be defined similarly where the coefficients are from $\mathbb{Q}$.

1.9 Matrix Rings

Definition 1.9.1. Let R be a ring with 1. Define $E_{ij} \in M_n(R)$ to be the matrix with 1 in the (i, j)th position and 0 elsewhere. The E_{ij} are called *matrix units.*

If $(a_{ij}) \in M_n(R)$ is arbitrary then clearly $(a_{ij}) = \sum_{i,j=1}^{n} a_{ij}E_{ij}$, $a_{ij} \in R$, and this expression is unique. We also have

$$E_{ij}E_{k\ell} = \begin{cases} E_{i\ell} & \text{if } j = k \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.9.2. *Let R be a ring with 1. Then*

(i) $I \trianglelefteq R \Rightarrow M_n(I) \trianglelefteq M_n(R)$;

(ii) conversely, every ideal of $M_n(R)$ is of the form $M_n(I)$ for some $I \trianglelefteq R$.

Proof. (i) Trivial.

(ii) Let $X \trianglelefteq M_n(R)$. We need an $I \trianglelefteq R$ such that $X = M_n(I)$. Let $A = (a_{ij}) = \sum_{i,j} a_{ij}E_{ij} \in X$. Consider fixed α, β, $1 \leq \alpha, \beta \leq n$. We have $E_{1\alpha}AE_{\beta 1} \in X$ since $X \trianglelefteq M_n(R)$. So $(a_{\alpha j}E_{1j})E_{\beta 1} \in X$. Hence

$$a_{\alpha\beta}E_{11} \in X, \tag{1.9.1}$$

that is, the matrix with $a_{\alpha\beta}$ in the $(1, 1)$ position and 0 elsewhere belongs to X.

Now let I be the set of all elements of R that occur in the $(1, 1)$ position of some matrix in X. We show that $I \trianglelefteq R$ and $X = M_n(I)$.

Let $a, b \in I$. Then a, b occur in the $(1, 1)$ positions of matrices $A, B \in X$. So $a - b = (A - B)_{11} \in I$. Let $a \in I$, $r \in R$. Let a be the $(1, 1)$ entry of

$A \in X$. Then $A = (a_{ij}) = \sum_{i,j} a_{ij}E_{ij}$ with $a_{11} = a$. Then $E_{11}A(rE_{11}) \in X$ since $X \trianglelefteq M_n(R)$. So $a_{11}rE_{11} \in X$, so $ar \in I$. Similarly, $ra \in I$. Thus $I \trianglelefteq R$.

Now let $C = (c_{ij}) = \sum_{i,j} c_{ij}E_{ij} \in X$, $c_{ij} \in R$. By (1.9.1), $c_{ij} \in I$. So $C \in M_n(I)$. So $X \subseteq M_n(I)$. Finally, let $D = (d_{ij}) = \sum_{i,j} d_{ij}E_{ij}$, $d_{ij} \in I$. By the definition of I, for each (i,j), $d_{ij}E_{11} \in X$. Therefore, $E_{i1}(d_{ij}E_{11})E_{1j} \in X$. So $d_{ij}E_{ij} \in X$ for each $1 \le i,j \le n$. Since X is an ideal, we have $D = \sum_{i,j} d_{ij}E_{ij} \in X$. So $M_n(I) \subseteq X$, so $M_n(I) = X$. □

Remark 1.9.3. The above does not hold for right ideals, e.g.

$$\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{pmatrix} \trianglelefteq_r M_2(\mathbb{Z}).$$

Definition 1.9.4. A ring R is said to be *simple* if 0 and R are the only ideals of R.

Theorem 1.9.5. *Let R be a ring with 1. If R is simple then so is the ring $M_n(R)$.*

Proof. 0 and R are the only ideals of R and so $M_n(0)$ and $M_n(R)$ are the only ideals of $M_n(R)$. So $M_n(R)$ is simple as well. □

Corollary 1.9.6. *Let D be a division ring. Then the ring $M_n(D)$ is a simple ring.*

Proof. The only ideals of D are 0 and D. □

1.10 The Field of Fractions

Definition 1.10.1. A (commutative) ring R is called an *integral domain* if $ab = 0 \Rightarrow a = 0$ or $b = 0$.

Example 1.10.2. $\mathbb{Z}$; $F[x]$, where F is a field.

Beware – non-commutative integral domains exist in advanced ring theory.

Definition 1.10.3. Let R be a (commutative) integral domain that is a subring of a field K. Then K is the *field of fractions* of R if every element of K is expressible as ab^{-1}, $b \neq 0$, $a, b \in R$. *We write* $K = \mathrm{Frac}(R)$.

We now show that every commutative integral domain has a field of fractions that is in some sense unique.

Example 1.10.4. $\mathbb{Q} = \text{Frac}(\mathbb{Z})$. Note that $\mathbb{Z} \subset \mathbb{R}, \mathbb{C}$ as well, but $\mathbb{R}, \mathbb{C} \neq \text{Frac}(\mathbb{Z})$.

Let R be a commutative integral domain, $R^* := R \backslash \{0\}$. Let $S = \{(a, b) | a \in R, b \in R^*\}$. Define a relation $\sim$ on S by

$$(a_1, b_1) \sim (a_2, b_2) \Leftrightarrow a_1 b_2 = a_2 b_1.$$

Lemma 1.10.5. $\sim$ *is an equivalence relation on* S.

Proof. Let $(a_i, b_i) \in S$, $i = 1, 2, 3$.

Reflexivity: $(a_1, b_1) \sim (a_1, b_1)$ since $a_1 b_1 = a_1 b_1$.

Symmetry:

$$\begin{aligned}(a_1, b_1) \sim (a_2, b_2) &\Leftrightarrow a_1 b_2 = a_2 b_1 \\ &\Leftrightarrow a_2 b_1 = a_1 b_2 \\ &\Leftrightarrow (a_2, b_2) \sim (a_1, b_1)\end{aligned}$$

Transitivity:

$$\begin{aligned}(a_1, b_1) \sim (a_2, b_2) \sim (a_3, b_3) &\Rightarrow a_1 b_2 = a_2 b_1, a_2 b_3 = a_3 b_2 \\ &\Rightarrow a_1 b_2 b_3 = a_2 b_1 b_3 \\ &\Rightarrow a_1 b_2 b_3 = b_1 a_3 b_2 \\ &\Rightarrow (a_1 b_3 - a_3 b_1) b_2 = 0 \\ &\Rightarrow a_1 b_3 = a_3 b_1\end{aligned}$$

since $b_2 \neq 0$ and R is an integral domain

$$\Rightarrow (a_1, b_1) \sim (a_3, b_3)$$

□

Theorem 1.10.6. *Every (commutative) integral domain with* 1 *has a field of fractions.*

Proof. Let R be an integral domain. Consider the equivalence relation $\sim$ as above. Denote the equivalence class of (a, b) by $\frac{a}{b}$. Let K be the set of all such equivalence classes. Define addition and multiplication in K by

$$\frac{a}{b} + \frac{c}{d} := \frac{ad + bc}{bd}$$

and

$$\frac{a}{b}\frac{c}{d} := \frac{ac}{bd}$$

for $\frac{a}{b}, \frac{c}{d} \in K$. We first make sure that these definitions are well-defined. Let $\frac{a}{b} = \frac{a'}{b'}$, $\frac{c}{d} = \frac{c'}{d'}$. Then $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$, so $ab' = ba'$, $cd' = dc'$. Hence

$$\begin{aligned}(ad + bc)b'd' &= adbd' + bcb'd' \\ &= a'bdd' + bb'c'd \\ &= (a'd' + b'c')bd\end{aligned}$$

So $(ad + bc, bd) \sim (a'd' + b'c', b'd')$. So $+$ is well-defined. Similarly for multiplication.

Note that $\frac{0}{b} = \frac{0}{d}$ for any $b, d \in R^*$, since $0d = b0 = 0$.

It can be checked that K is a commutative ring under these operations. $\frac{0}{b}$, for any $b \in R^*$, is the zero element of K; $\frac{-x}{y}$ is the additive inverse of $\frac{x}{y}$; the commutative, associative and distributive laws can be easily verified.

$\frac{1}{1} = \frac{b}{b}$ for any $b \in R^*$ is clearly the multiplicative identity in K. The multiplication is clearly commutative. Let $\frac{x}{y} \in K^*$, so $x \neq 0$, so $xy \neq 0$. So $\frac{y}{x}$ exists and

$$\frac{x}{y}\frac{y}{x} = \frac{xy}{xy} = \frac{1}{1} = 1_K$$

Thus every non-zero element of K has an inverse in K, so K is a field.

There is also a clear injective homomorphic embedding of R in K by $\theta : r \mapsto \frac{r}{1}$ for $r \in R$. Now we truly have K as a field of fractions for $\theta(R) \subseteq K$, and $R \cong \theta(R)$. □

Remark 1.10.7. There is no fundamental problem if R is without 1, since we can still have $1_K = \frac{b}{b}$ for any $b \in R^*$. Now embed $R \hookrightarrow K$ by $r \mapsto \frac{rb}{b}$.

Lemma 1.10.8. *Let $R, \bar{R}$ be commutative integral domains with fields of fractions $K, \bar{K}$ respectively. Then $R \cong \bar{R} \Rightarrow K \cong \bar{K}$.*

Proof. Let $\theta : R \to \bar{R}$ be an isomorphism. We have $K = \{ab^{-1} | (a, b) \in R \times R^*\}$. Define a map $\Theta : K \to \bar{K}$ by $\Theta(ab^{-1}) := \theta(a)\theta(b)^{-1}$.

Θ is well-defined: suppose that $ab^{-1} = cd^{-1}$, $(a, b), (c, d) \in R \times R^*$. Then $ad = bc$, so $\theta(ad) = \theta(bc)$, so $\theta(a)\theta(d) = \theta(b)\theta(c)$, so $\theta(a)\theta(b)^{-1} = \theta(c)\theta(d)^{-1}$, so $\Theta(ab^{-1}) = \Theta(cd^{-1})$.

Θ is a homomorphism: let $ab^{-1}, cd^{-1} \in K$. Then

$$\begin{aligned}
\Theta(ab^{-1} + cd^{-1}) &= \Theta((ad + bc)b^{-1}d^{-1}) \\
&= \theta(ad + bc)\theta(bd)^{-1} \\
&= (\theta(a)\theta(d) + \theta(b)\theta(c))\theta(b)^{-1}\theta(d)^{-1} \\
&= \theta(a)\theta(b)^{-1} + \theta(c)\theta(d)^{-1} \\
&= \Theta(ab^{-1}) + \Theta(cd^{-1}).
\end{aligned}$$

Similarly, $\Theta((ab^{-1})(cd^{-1})) = \Theta(ab^{-1})\Theta(cd^{-1})$.

Θ is surjective: every element of $\bar{K}$ is expressible as $\bar{a}\bar{b}^{-1}$, $(\bar{a}, \bar{b}) \in \bar{R} \times \bar{R}^*$. But θ is an isomorphism, so $\bar{a} = \theta(a)$ for some $a \in R$, and $\bar{b} = \theta(b)$ for some $b \in R^*$. So $\bar{a}\bar{b}^{-1} = \theta(a)\theta(b)^{-1} = \Theta(ab^{-1})$.

It is easy to check that Θ is injective. □

Corollary 1.10.9. *Let R be a commutative integral domain. Then its field of fractions is essentially unique, in that any two such fields of fractions are isomorphic.*

Proof. Take $R = \bar{R}$, $\theta = \mathrm{id}_R$ in the above. □

2 Modules

2.1 Modules

Definitions 2.1.1. Let R be a ring. A set M is called a *right R-module* if

(i) M is an Abelian additive group;

(ii) a law of composition $M \times R \to M : (m, r) \mapsto mr$ is defined that satisfies, for all $x, y \in M$ and $r, s \in R$,

$$\begin{aligned} (x+y)r &= xr + yr, \\ x(r+s) &= xr + xs, \\ x(rs) &= (xr)s. \end{aligned}$$

A *left R-module* is defined analagously. Here the composition law goes $R \times M \to M$ and is denoted rm.

Examples 2.1.2. (i) R and $\{0\}$ are both left and right R-modules.

(ii) Let V be a vector space over a field F. Then V is left (alternatively, a right) F-module. The module axioms are part of the vector space axioms.

(iii) Any Abelian group A can be considered as a left $\mathbb{Z}$-module: for $g \in A$, $k \in \mathbb{Z}$, define

$$kg := \begin{cases} g + \cdots + g & k \text{ times} \quad k > 0 \\ (-g) + \cdots + (-g) & k \text{ times} \quad k < 0 \\ 0_A & \qquad\qquad k = 0 \end{cases}$$

(iv) Let R be a ring. Then $M_n(R)$ becomes a left R-module under the action $r(x_{ij}) := (rx_{ij})$. Clearly, we can also make a similar right R-module action.

For technical reasons, it is easier to work with right modules in the theory of semi-simple Artinian rings.

Let R be a ring. The symbol M_R will denote M, a right R-module; similarly, ${}_RM$ will denote M, a left R-module.

Proposition 2.1.3. *Let M be a right R-module. Then*

(i) $0_M r = 0_M$ *for all* $r \in R$*;*

(ii) $m 0_R = 0_M$ *for all* $m \in M$*;*

(iii) $(-m)r = m(-r) = -(mr)$ *for all* $m \in M$, $r \in R$.

Proof. (i), (ii). Exercises.

(iii) By (ii),

$$mr + m(-r) = m(r + (-r)) = m0_R = 0_M.$$

So $m(-r) = -(mr)$ by the uniqueness of $-(mr)$ in the Abelian group M. Similarly $(-m)r = -(mr)$. □

Definition 2.1.4. Let $K \subseteq M_R$. Then K is a *right R-submodule* (or just *submodule*) if K is also a right R-module under the law of composition for M.

Proposition 2.1.5. *Let $\varnothing \neq K \subseteq M_R$. Then K is a submodule of M if and only if for all $x, y \in K$, $r \in R$, $x - y \in K$ and $xr \in K$.*

Proof. Exercise. □

Definitions 2.1.6. Submodules of the module R_R are called *right ideals*. Submodules of ${}_RR$ are called *left ideals*.

2.2 Factor Modules and Homomorphisms

Let K be a submodule of M_R. Consider the factor group $\frac{M}{K}$. Elements of $\frac{M}{K}$ are cosets of the form $m + K$ for $m \in M$. We can make $\frac{M}{K}$ into a right R-module by defining, for $m \in M$, $r \in R$,

$$[m + K]r := [mr + K].$$

Check that this action is well-defined and that the module axioms are satisfied.

Definition 2.2.1. $\frac{M}{K}$ with this action is called the *factor* (or *quotient*) *module* of M by K.

Example 2.2.2. Let $n \in \mathbb{Z}$, $n \geq 2$. Then $\frac{\mathbb{Z}}{n\mathbb{Z}}$ is a natural $\mathbb{Z}$-module.

Definitions 2.2.3. Let M, M' be right R-modules. A map $\theta : M \to M'$ is an *R-homomorphism* if

(i) $\theta(x+y) = \theta(x) + \theta(y)$ for all $x, y \in M$;

(ii) $\theta(xr) = \theta(x)r$ for all $x \in M$, $r \in R$.

(Similarly for left R-modules.) If K is a submodule of M_R then the map $\sigma : M \to \frac{M}{K}$ defined by $\sigma(m) = [m + K]$ is an R-homomorphism of M onto $\frac{M}{K}$. It is called the *canonical* (or *natural*) *R-homomorphism*.

Proposition 2.2.4. *Let $\theta : M_R \to M'_R$ be an R-homomorphism. Then*

(i) $\theta(0_M) = 0_{M'}$*;*

(ii) the kernel $\ker\theta := \{x \in M | \theta(x) = 0_{M'}\}$ *is a submodule of M;*

(iii) the image $\theta(M) := \{\theta(m) | m \in M\}$ *is a submodule of M';*

(iv) θ is injective if and only if $\ker\theta = \{0_M\}$.

Definition 2.2.5. Let $\theta : M_R \to M'_R$ be an R-homomorphism. If θ is bijective then it is an *R-isomorphism* and we write $M \cong M'$.

2.3 The Isomorphism Theorems

These are similar to those for rings and have similar proofs.

Theorem 2.3.1. *Let $\theta : M_R \to M'_R$ be an R-homomorphism. Then*

$$\theta(M) \cong \frac{M}{\ker\theta}.$$

Theorem 2.3.2. *If K, L are submodules of M_R then*

$$\frac{L+K}{K} \cong \frac{L}{L \cap K}.$$

Theorem 2.3.3. *If K, L are submodules of M_R and $K \subseteq L$ then $\frac{L}{K}$ is a submodule of $\frac{M}{K}$ and*

$$\frac{M/K}{L/K} \cong \frac{M}{L}.$$

When K is a submodule of M_R and $L \supseteq K$ a submodule of M, then $\frac{L}{K}$ is a submodule of $\frac{M}{K}$. Conversely, every submodule of $\frac{M}{K}$ is $\frac{L}{K}$ for L a submodule of M containing K. Thus

$$\left\{\text{submodules of } \frac{M}{K}\right\} \leftrightarrow \{\text{submodules of } M \text{ containing } K\}\,.$$

2.4 Direct Sums of Modules

Definition 2.4.1. Let $M_1, \ldots, M_n$ be right R-modules. The set of all n-tuples $\{(m_1, \ldots, m_n) | m_i \in M_i\}$ becomes a right R-module if we define

$$\begin{aligned}(m_1, \ldots, m_n) + (m_1', \ldots, m_n') &:= (m_1 + m_1', \ldots, m_n + m_n'),\\ (m_1, \ldots, m_n)r &:= (m_1 r, \ldots, m_n r),\end{aligned}$$

for $m_i, m_i' \in M_i$, $r \in R$. This is the *external direct sum* of the M_i, which we denote $\bigoplus_{i=1}^n M_i$ or $M_1 \oplus \cdots \oplus M_n$.

Definition 2.4.2. Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be a collection of subsets of M_R. We define their (*internal*) *sum* by

$$\sum_{\lambda \in \Lambda} M_\lambda := \{\, m_{\lambda_1} + \cdots + m_{\lambda_k} \,|\, m_{\lambda_i} \in M_{\lambda_i} \text{ for finite subsets } \{\lambda_1, \ldots, \lambda_k\} \subseteq \Lambda\}\,.$$

Thus $\sum_{\lambda \in \Lambda} M_\lambda$ is the set of all finite sums of elements from the M_λ. It is easy to see that $\sum_{\lambda \in \Lambda} M_\lambda$ is a submodule of M.

Definition 2.4.3. $\sum_{\lambda \in \Lambda} M_\lambda$ is *direct* if each $m \in \sum_{\lambda \in \Lambda} M_\lambda$ has a unique representation as $m = m_{\lambda_1} + \cdots + m_{\lambda_k}$ for some $m_{\lambda_i} \in M_{\lambda_i}$.

We can show that $\sum_{\lambda \in \Lambda} M_\lambda$ is direct if and only if

$$M_\mu \cap \left(\sum_{\lambda \in \Lambda \setminus \{\mu\}} M_\lambda \right) = \{0\}$$

for all $\mu \in \Lambda$. If the sum is direct we use the same $\oplus$ notation as above.

As before with rings, there really is no difference between (finite) internal and external direct sums of modules.

Definition 2.4.4. Let R be a ring with 1. M_R is *unital* if $m1 = m$ for all $m \in M$. Similarly for ${}_RM$.

Exercise 2.4.5. Let R be a ring with 1, M a right R-module. Show that M has submodules M_1 and M_2 such that $M = M_1 \oplus M_2$ with M_1 unital and $m_2r = 0$ for all $m_2 \in M_2$, $r \in R$.

Since modules like M_2 give us no information about R, whenever R has 1 we assume that all R-modules are unital.

2.5 Products of Subsets

Let K, S be non-empty subsets of M_R and R respectively. Define their product KS to be

$$KS := \left\{ \sum_{i=1}^{n} k_i s_i \,\middle|\, k_i \in K, s_i \in S, n \in \mathbb{N} \right\}.$$

I.e., KS consists of all finite sums of elements of type ks for $k \in K$, $s \in S$. If $\varnothing \neq K \subseteq M$, $S \trianglelefteq_r R$, then KS is a submodule of M – we need finiteness to make this work.

This definition applies, in particular, with $M = R$. Thus, if $\varnothing \neq S \subseteq R$,

$$S^2 := \left\{ \sum_{i=1}^{n} s_i t_i \,\middle|\, s_i, t_i \in S, n \in \mathbb{N} \right\}.$$

Extending this inductively, S^n consists of all finite sums of elements of type $s_1 s_2 \dots s_n$, $s_i \in S$. Note that $S \trianglelefteq_r R \Rightarrow S^n \trianglelefteq_r R$.

3 Zorn's Lemma

3.1 Definitions and Zorn's Lemma

Definition 3.1.1. A non-empty set $\mathcal{S}$ is said to be *partially ordered* if there is a binary relation $\leq$ on $\mathcal{S}$, defined for certain pairs of elements, such that for all $a, b, c \in \mathcal{S}$,

(i) $a \leq a$;

(ii) $a \leq b$ and $b \leq c \Rightarrow a \leq c$;

(iii) $a \leq b$ and $a \leq b \Rightarrow a = b$.

Definition 3.1.2. Let $\mathcal{S}$ be a partially ordered set, a non-empty subset $\mathcal{T}$ is said to be *totally ordered* if for all $a, b \in cT$, $a \leq b$ or $b \leq a$.

Definitions 3.1.3. Let $\mathcal{S}$ be a partially ordered set. An element $x \in \mathcal{S}$ is called *maximal* if $x \leq y$ and $y \in \mathcal{S} \Rightarrow x = y$. Similarly for *minimal*.

Definition 3.1.4. Let $\mathcal{T}$ be a totally ordered subset of a partially ordered set $\mathcal{S}$. We say $\mathcal{T}$ has an *upper bound (in $\mathcal{S}$)* if $\exists c \in \mathcal{S}$ such that $x \leq c$ for all $x \in \mathcal{T}$.

Axiom 3.1.5. (Zorn's Lemma.) If a partially ordered set $\mathcal{S}$ has the property that every totally ordered subset of $\mathcal{S}$ has an upper bound then $\mathcal{S}$ contains a maximal element.

Remark 3.1.6. There may in fact be several maximal elements. Zorn's Lemma guarantees the existence of at least one such element.

3.2 The Well-Ordering Principle

Definition 3.2.1. A non-empty set $\mathcal{S}$ is said to be *well-ordered* if it is totally ordered and every non-empty subset of $\mathcal{S}$ has a minimal element.

Axiom 3.2.2. (The Well-Ordering Principle.) Any non-empty set can be well-ordered.

3.3 The Axiom of Choice

Axiom 3.3.1. (The Axiom of Choice.) Given a class of non-empty sets there exists a "choice function", i.e. a function that assigns to each of the sets one of its elements.

It can shown that

$$\text{Axiom of Choice} \Leftrightarrow \text{Zorn's Lemma} \Leftrightarrow \text{Well-Ordering Principle.}$$

3.4 Applications

Definitions 3.4.1. let M be a right ideal of a ring R. M is said to be a *maximal right ideal* if $M \neq R$ and $M \subset M' \trianglelefteq_r R \Rightarrow M' = R$, Similarly for *maximal left ideal* and *maximal two-sided ideal.*

Theorem 3.4.2. *Let R be a ring with 1. Let $I \neq R$ be a (right) ideal of R. Then R contains a maximal (right) ideal M such that $I \subseteq M$.*

Proof. We prove this for $I \trianglelefteq_r R$. Consider $\mathcal{S} := \{X \trianglelefteq_r R | X \supseteq I, X \neq R\}$. $\mathcal{S} \neq \varnothing$ since $I \in \mathcal{S}$. Partially order $\mathcal{S}$ by inclusion. Let $\mathcal{T} := \{X_\alpha\}_{\alpha \in \Lambda}$ be a totally ordered subset of $\mathcal{S}$.

Consider $\bar{X} := \bigcup_{\alpha \in \Lambda} X_\alpha$. If $x_1, x_2 \in \bar{X}$ then $x_1 \in X_{\alpha_1}$, $x_2 \in X_{\alpha_2}$ for some $\alpha_1, \alpha_2 \in \Lambda$. Since $\mathcal{T}$ is totally ordered, we can assume that $X_{\alpha_1} \subseteq X_{\alpha_2}$. So $x_1, x_2 \in X_{\alpha_2}$, so $x_1 - x_2 \in X_{\alpha_2} \subseteq \bar{X}$. Clearly, also $x \in \bar{X}$, and $r \in R \Rightarrow xr \in \bar{X}$. Thus $\bar{X} \trianglelefteq_r R$. Also $\bar{X} \neq R$ since

$$\begin{aligned} \bar{X} = R &\Rightarrow 1 \in \bar{X} \\ &\Rightarrow 1 \in X_\alpha \text{ for some } \alpha \\ &\Rightarrow X_\alpha = R, \end{aligned}$$

which is a contradiction. Trivially, $\bar{X} \supseteq I$ so $\bar{X} \in \mathcal{S}$. Also clearly, $X_\alpha \subseteq \bar{X}$ for all $\alpha \in \Lambda$.

Thus, $\bar{X}$ is an upper bound in $\mathcal{S}$ for $\mathcal{T}$. So Zorn's Lemma applies and hence $\mathcal{S}$ contains a maximal element M. Clearly M is a maximal right ideal of R and contains I.

The proof is similar for left ideals and two-sided ideals. □

Remark 3.4.3. This result is false if R is without 1.

Corollary 3.4.4. *A ring with* 1 *contains a maximal (right) ideal.*

Proof. Take $I = \{0\}$ in the above. □

Note that $p\mathbb{Z}$ is a maximal ideal of $\mathbb{Z}$ for each prime p.

Theorem 3.4.5. *Every vector space has a basis.*

Proof. Exercise. Hint: Apply Zorn's Lemma to obtain a maximal set of linearly independent vectors. Note that a set of vectors is defined to be linearly independent if every finite subset is linearly independent. □

Exercises 3.4.6. Let R be a commutative ring with 1. Show that

(i) if R is a finite integral domain then R is a field;

(ii) if $M \trianglelefteq R$ and $M \neq R$ then M is maximal if and only if $\frac{R}{M}$ is a field.

4 Completely Reducible Modules

4.1 Irreducible Modules

Definition 4.1.1. A right R-module M is *irreducible* if

(i) $MR \neq 0$;

(ii) M has no submodules other than 0 and M.

If R has 1 and M is unital then (i) can be replaced by $M \neq 0$.

Examples 4.1.2. (i) Let p be a prime; then $\frac{\mathbb{Z}}{p\mathbb{Z}}$ is an irreducible $\mathbb{Z}$-module.

(ii) Every ring R with 1 has an irreducible right R-module. By Theorem 3.4.2, R has a maximal right ideal M; $\frac{R}{M}$ is an irreducible right R-module.

(iii) Let V be a vector space over a field F. Then any 1-dimensional subspace of V is an irreducible F-module.

The vector space V has the following interesting property: V is a sum of 1-dimensional irreducible submodules/subspaces, i.e. has a basis; this sum is direct. Not all modules over arbitrary rings have this property. Consider $\frac{\mathbb{Z}}{4\mathbb{Z}}$ as a $\mathbb{Z}$-module. $\frac{2\mathbb{Z}}{4\mathbb{Z}}$ is the only (irreducible) submodule of $\frac{\mathbb{Z}}{4\mathbb{Z}}$. So $\frac{\mathbb{Z}}{4\mathbb{Z}}$ is not expressed as a sum of irreducible submodules.

4.2 Completely Reducible Modules

Definition 4.2.1. M_R is said to be *completely reducible* if M is expressible as a sum of irreducible submodules.

Examples 4.2.2. (i) Let F be a field. Then every F-module is completely reducible, i.e. every vector space has a basis.

(ii) $\frac{\mathbb{Z}}{6\mathbb{Z}}$ is completely reducible as a $\mathbb{Z}$-module: $\frac{\mathbb{Z}}{6\mathbb{Z}} = \frac{2\mathbb{Z}}{6\mathbb{Z}} + \frac{3\mathbb{Z}}{6\mathbb{Z}}$.

Definition 4.2.3. Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of submodules of M_R. The family is *independent* if the sum $\sum_{\lambda\in\Lambda} M_\lambda$ is direct. Thus $\{M_\lambda\}_{\lambda\in\Lambda}$ is independent if and only if $M_\mu \cap \left(\sum_{\lambda\in\Lambda\setminus\{\mu\}} M_\lambda\right) = 0$ for all $\mu \in \Lambda$.

Lemma 4.2.4. *Suppose $\{M_\lambda\}_{\lambda\in\Lambda}$ is a family of irreducible submodules of M_R and let $M := \sum_{\lambda\in\Lambda} M_\lambda$. Let K be a submodule of M. Then there is an independent subfamily $\{M_\lambda\}_{\lambda\in\Lambda'}$ such that $M = K \oplus \left(\bigoplus_{\mu\in\Lambda'} M_\mu\right)$.*

Proof. We apply Zorn's Lemma to the independent families of the form $\{K\} \cup \{M_\mu\}_{\mu\in X}$, $X \subseteq \Lambda$.

Partially order the set $\mathcal{S}$ of all such families by inclusion. Let $\mathcal{T}$ be a totally ordered subset of $\mathcal{S}$, C the union of all the families in $\mathcal{T}$. Each member of $\mathcal{T}$ has the form $\{K\} \cup \{M_\mu\}_{\mu\in X\subseteq\Lambda}$, so we have the same form for C.

We need to show that $C \in \mathcal{S}$, i.e. C is an independent family. Let I be any submodule in C and suppose Σ is the sum of all other submodules in C. Let $x \in I \cap \Sigma$. Then $x = x_1 + \cdots + x_n$, $x_j \in I_j \neq I$, I_j a submodule in C for $j = 1, \ldots, n$.

Now $I, I_1, \ldots, I_n$ are all in C so each comes from some family in $\mathcal{T}$. But $\mathcal{T}$ is totally ordered, so $I, I_1, \ldots, I_n$ lie in some one family in $\mathcal{T}$. But this family is independent, so $x = x_1 = \cdots = x_n = 0$. So $I \cap \Sigma = 0$.

So C is independent. Also C has the form $\{K\} \cup \{M_\mu\}_{\mu\in X\subseteq\Lambda}$, so $C \in \mathcal{S}$. Clearly C is an upper bound for $\mathcal{T}$. So, by Zorn's Lemma, $\mathcal{S}$ contains a maximal element, say $\{K\} \cup \{M_\mu\}_{\mu\in\Lambda'\subseteq\Lambda}$. $(*)$.

We claim $M = K \oplus \left(\bigoplus_{\mu\in\Lambda'} M_\mu\right)$. Suppose $\exists\alpha \in \Lambda$ such that $M_\alpha \cap \left(K \oplus \left(\bigoplus_{\mu\in\Lambda'} M_\mu\right)\right) = 0$. Then $\{K\} \cup \{M_\alpha\} \cup \{M_\mu\}_{\mu\in\Lambda'}$ would be an independent family, contradicting $(*)$. Thus $M_\alpha \cap \left(K \oplus \left(\bigoplus_{\mu\in\Lambda'} M_\mu\right)\right) = M_\alpha$ for all $\alpha \in \Lambda$, since M_α is irreducible and $M_\alpha \cap \left(K \oplus \left(\bigoplus_{\mu\in\Lambda'} M_\mu\right)\right)$ is a submodule of M_α. So $M_\alpha \subseteq K \oplus \left(\bigoplus_{\mu\in\Lambda'} M_\mu\right)$ for each $\alpha \in \Lambda$. As $M = \sum_{\lambda\in\Lambda} M_\lambda$,

$$M = K \oplus \left(\bigoplus_{\mu\in\Lambda'} M_\mu\right).$$

□

Lemma 4.2.5. (Dedekind Modular Law.) *Let A, B, C be submodules of M_R such that $B \subseteq A$. Then $A \cap (B + C) = B + A \cap C$.*

Proof. Elementary. □

Theorem 4.2.6. *Let M be a non-zero right R-module. The following are equivalent:*

(i) M is completely reducible;

(ii) M is a direct sum of irreducible submodules;

(iii) $mR = 0$, $m \in M \Rightarrow m = 0$ and every submodule of M is a direct summand of M.

Proof. (i)⇒(ii). Take $K = 0$ in Lemma 4.2.4 above.

(ii)⇒(iii). Suppose that $mR = 0$ for some $m \in M$. Let $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$, M_λ irreducible. Then $m = m_1 + \cdots + m_k$ for some $m_i \in M_{\lambda_i}$.

$$\begin{aligned} mr = 0, r \in R &\Rightarrow m_1 r + \cdots + m_k r = 0 \\ &\Rightarrow m_i r = 0 \text{ for all } i, \end{aligned}$$

since the sum of the M_λ is direct.

Define $K_j := \{x \in M_{\lambda_j} | xR = 0\}$ for $j = 1, \ldots, k$. Then K_j is a submodule of M_{λ_j}. So $K_j = 0$ or M_{λ_j} since M_{λ_j} is irreducible. But $K_j \neq M_{\lambda_j}$ since $M_{\lambda_j} R \neq 0$ by definition of irreducible submodule. So $K_j = 0$ for $j = 1, \ldots, k$. Thus $m_j = 0$ for $j = 1, \ldots, k$, so $m = 0$. The second part follows from Lemma 4.2.4.

(iii)⇒(i). Our first aim is to show that M has an irreducible submodule.

Note that by the Dedekind Modular Law the hypothesis on M is inherited by every submodule of M.

Let $0 \neq y \in M$. Let $\mathcal{S}$ be the set of all submodules K of M such that $y \notin K$. $\mathcal{S} \neq \varnothing$ since $\{0\} \in \mathcal{S}$. Partially order $\mathcal{S}$ by inclusion; let $\mathcal{T}$ be a totally ordered subset of $\mathcal{S}$. Let C be the union of all the submodules in $\mathcal{T}$. Then $y \notin C$ and C is a submodule of M. So $C \in \mathcal{S}$ and C is an upper bound for $\mathcal{T}$. By Zorn's Lemma, $\mathcal{S}$ has a maximal element B. $y \notin B$ so $B \neq M$. Hence, by hypothesis, there is a $B' \neq 0$ such that $B \oplus B' = M$. We claim that B' is irreducible.

$B'R \neq 0$ by hypothesis. Suppose B' contains a proper submodule $B_1 \neq 0$. Then there is a submodule $B_2 \neq 0$ such that $B' = B_1 \oplus B_2$. Now $y \in B_1 \oplus B_2$ by the maximality of $B \in \mathcal{S}$. So $y \in (B \oplus B_1) \cap (B \oplus B_2) = B$, a contradiction. So B' is irreducible, and thus M has an irreducible. Let K be the sum of all these.

If $K \neq M$ there is a non-zero submodule L of M such that $M = K \oplus L$. But the above applied to L gives an irreducible in L, a contradiction since K contains all the irreducible submodules of M and $K \cap L = 0$. Thus $K = M$ and M is completely reducible. □

Remark 4.2.7. The first part of condition (iii) holds automatically when R has 1 and M is unital.

4.3 Examples of Completely Reducible Modules

Example 4.3.1. Let D be a division ring and $R := M_n(D)$. Then both ${}_RR$ and R_R are completely reducible.

Proof. Let

$$I_j := \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ D & \dots & D \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix} \leftarrow j\text{th row}$$

$$= \left\{ \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ a_{j1} & \dots & a_{jn} \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix} \middle| \, a_{jk} \in D \right\}$$

I_j is the set of all matrices in $M_n(D)$ where all rows except the jth are zero. Then $I_j \trianglelefteq_r R$ and $I_j = E_{jj}R$, where E_{jj} is the (j, j) matrix unit.

We claim that each I_j is an irreducible R-module.

Suppose that $0 \subsetneq X \subseteq I_j$, $X \trianglelefteq_r R$. Then X contains a non-zero matrix $A = (a_{\alpha\beta})$. A must have a non-zero entry, and since $A \in I_j$, $a_{jk} \neq 0$ for some k. Let B be he matrix with a_{jk}^{-1} in the (k, j)th place and 0 elsewhere. Then $AB = E_{jj}$. So $E_{jj} \in X$, since $A \in X \trianglelefteq_r R$. So $E_{jj}R \subseteq X$ since $X \trianglelefteq_r R$. Thus $I_j = X$. Since R has 1, $I_j R \neq 0$, each I_j is an irreducible right R-module.

It is clear that $R = I_1 \oplus \cdots \oplus I_n$, so R_R is completely reducible. Similarly for ${}_R R$. $\square$

Example 4.3.2. Let $R := R_1 \oplus \cdots \oplus R_m$ be a direct sum of rings, where $R_i := M_{n_i}(D_i)$, D_i division rings, $n_i \in \mathbb{N}$. Again ${}_R R$ and R_R are completely reducible.

Proof. Since each $R_i \triangleleft R$, each R_i can be viewed as an R_i-module or an R-module. Further, the R_i-submodules and R-submodules coincide. Note that $R_i R_j = 0$ for $i \neq j$.

By the previous example, each R_i is a sum of irreducible R_i-submodules. So each R_i is a sum of irreducible R-submodules. So R is a sum of irreducible R-submodules. Hence ${}_R R$ and R_R are completely reducible. $\square$

The significance of this example is that we now aim to show that if R is a ring with 1 and R_R is completely reducible then it is necessarily a ring of the type given in the second example above. As a consequence we shall have R_R completely reducible $\Leftrightarrow$ ${}_R R$ completely reducible.

5 Chain Conditions

5.1 Cyclic and Finitely Generated Modules

Definitions 5.1.1. Let $\varnothing \neq T \subseteq M_R$. By the *submodule of M generated by T* we mean the intersection of all submodules of M that contain T. We denote this by (T). Thus (T) is the "smallest" submodule of M that contains T.

When T consists of a single element $a \in M$ we have

$$(a) = \{ar + \lambda a | r \in R, \lambda \in \mathbb{Z}\}$$

since the RHS

(i) is a submodule of M;

(ii) contains a;

(iii) lies inside any submodule of M containing a.

If R has 1 and M is unital then $(a) = aR$.

If $M = (a)$ for some $a \in M$ then M is said to be a *cyclic module generated by a*. A module M is said to be *finitely generated* if $M = (a_1) + \cdots + (a_k)$ for some finite collection $\{a_1, \ldots, a_k\} \subseteq M$. The a_i are *generators* of M

If R has 1 and M_R is unital then M_R finitely generated $\Rightarrow M = a_1R + \cdots + a_kR$ for some $a_i \in M$.

A cyclic submodule of R_R (respectively ${}_RR$) is called a *principal right* (respectively *left*) *ideal* of R. Thus $a\mathbb{Z} \trianglelefteq \mathbb{Z}$ is principal.

5.2 Chain Conditions

Definition 5.2.1. A set A is called an *algebra* over a field F if

(i) A is a vector space over F;

(ii) A is a ring with the same addition as in (i);

(iii) the ring and vector space products satisfy

$$\lambda(ab) = \lambda(ab) = a(\lambda b)$$

for all $a, b \in A$ and $\lambda \in F$.

Example 5.2.2. $M_n(F)$ is an n^2-dimensional algebra over the field F.

Substructures, homomorphisms etc. for algebras can be defined in the usual ways. Thus

Definition 5.2.3. $I \subseteq A$ is an *(algebra) right ideal* if

(i) $I \trianglelefteq_r A$ as rings;

(ii) I is a subspace of the vector space A.

If A has identity 1 then the vector space structure is automatically preserved.

Example 5.2.4. Let $I \trianglelefteq_r A$, $K \trianglelefteq_\ell A$. Then for $\lambda \in F$, $x \in I$, $\lambda x = \lambda(x1) = x(\lambda 1) \in I$, since I is a right ideal of the ring A and $\lambda 1 \in A$. Similarly, for $\lambda \in F$, $y \in K$, $\lambda y = \lambda(1y) = (\lambda 1)y \in K$. In general, if A is an algebra over a field F and $\lambda \in F$ we cannot immediately say that $\lambda \in A$.

However, if A has 1 we can overcome this problem: define

$$\bar{F} := \{\lambda 1 | \lambda \in F\}.$$

Clearly $\bar{F}$ is a subalgebra of A and a field isomorphic to F. If we identify $F \leftrightarrow \bar{F}$ we can assume $F \hookrightarrow A$.

Example 5.2.5. For $M_n(F)$,

$$F \ni \lambda \leftrightarrow \begin{pmatrix} \lambda & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda \end{pmatrix} \in \bar{F}.$$

Now let A be an n-dimensional algebra with 1. Let $I_1 \subseteq I_2 \subseteq \dots$ be an ascending chain of right ideals in A. Since each I_j is a subspace of A we have

$$\dim I_j = \dim I_{j+1} \Leftrightarrow I_j = I_{j+1}.$$

Hence, the chain can have at most $n+1$ terms. Similarly for descending chains. Many properties of algebras can be deduced from these facts alone. Moreover, there are rings (for example, $\mathbb{Z}$) that are not algebras but that still satisfy something like the above property.

Definitions 5.2.6. (i) A module M_R has the *ascending chain condition* on submodules if every ascending chain of submodules $M_1 \subseteq M_2 \subseteq \dots$ has equal terms after a finite number of steps. Similarly for the *descending chain condition*.

(ii) M_R has the *maximum condition* if every non-empty set S of submodules of M contains a maximal element with respect to inclusion. Similarly for the *minimum condition*.

Remark 5.2.7. The ascending chain condition or descending chain condition alone does not imply that all chains stop after a fixed n terms. For example, $\mathbb{Z}$ has the ascending chain condition (it's a principal ideal domain) but ascending chains of arbitrary length can be constructed:

$$2^k\mathbb{Z} \subset 2^{k-1}\mathbb{Z} \subset \cdots \subset 2\mathbb{Z} \subset \mathbb{Z}.$$

However, if we have both the ascending chain condition and descending chain condition then such a "global" n does exist. This follows from the theory of composition series.

Theorem 5.2.8. *Let M_R be a right R-module. The following are equivalent*

(i) M has the maximum condition on submodules;

(ii) M has the ascending chain condition on submodules;

(iii) every submodule of M is finitely generated.

Proof. (i)⇒(iii). Suppose that K is a submodule of M that is not finitely generated. Choose $x_1 \in K$ and let $K_1 := (x_1)$. Then $K \neq K_1$. So $\exists x_2 \in K$ with $x_2 \notin K_1$. Let $K_2 := (x_1) + (x_2)$. Then $K_2 \neq K$. So $\exists x_3 \in K$ such that $x_3 \notin K_2$. $K_3 := (x_1) + (x_2) + (x_3)$. Define K_i inductively like this for positive integers i. Let $\mathcal{S} := \{K_i | i \in \mathbb{N}\}$; $\mathcal{S}$ has no maximal element. So, by the contrapositive, if M has the maximum condition on submodules then every submodule is finitely generated.

(iii)⇒(ii). Let $K_1 \subseteq K_2 \subseteq \dots$ be an ascending chain of submodules of M. Let $K := \bigcup_{i=1}^{\infty} K_i$; then K is a submodule of M and K is finitely generated, generated by $x_1, \dots, x_n \in K$, say. Then $\exists t \in \mathbb{N}$ such that $x_1, \dots, x_n \in K_t$. So $K = (x_1) + \cdots + (x_n) \subseteq K_t$. Hence $K_t = K_{t+j}$ for all $j \geq 0$.

(ii)$\Rightarrow$(i). Let $\mathcal{S}$ be a non-empty collection of submodules of M, Choose $K_1 \in \mathcal{S}$. If K_1 is not maximal in $\mathcal{S}$, $\exists K_2 \in \mathcal{S}$ such that $K_1 \subset K_2$. If K_2 is not maximal $\exists K_3 \in \mathcal{S}$ such that $K_2 \subset K_3$. So, by the Axiom of Choice, we obtain an ascending chain $K_1 \subset K_2 \subset K_3 \subset \ldots$ of submodules of M. □

Theorem 5.2.9. *Let M_R be a right R-module. The following are equivalent*

(i) M has the minimum condition on submodules;

(ii) M has the descending chain condition on submodules;

Proof. Similar to the above. □

Examples 5.2.10. (i) A finite ring has both the ascending chain condition and descending chain condition on right and left ideals.

(ii) A finite-dimensional algebra with 1 has both the ascending chain condition and descending chain condition on right and left ideals.

(iii) A commutative principal ideal domain has the ascending chain condition on ideals. $\mathbb{Z}$, $F[x]$ and $\mathcal{J}$ are commutative principal ideal domains. So, in particular, these have the ascending chain condition on ideals; they do not have the descending chain condition on ideals.

Theorem 5.2.11. *Let K be a submodule of a module M_R. Then M has the ascending chain condition (respectively the descending chain condition) on submodules if and only if $\frac{M}{K}$ has the ascending chain condition (respectively the descending chain condition) on submodules.*

Proof. ($\Rightarrow$). Easy.

($\Leftarrow$). Let $M_1 \subseteq M_2 \subseteq \ldots$ be an ascending chain of submodules of M. Consider the chains

$$M_1 \cap K \subseteq M_2 \cap K \subseteq \ldots,$$
$$M_1 + K \subseteq M_2 + K \subseteq \ldots.$$

The first chain consists of submodules of K, so $\exists j \in \mathbb{N}$ such that $M_j \cap K = M_{j+i} \cap K$ for all $i \geq 0$. The second chain consists of submodules of M containing K. These are in one-to-one correspondence with submodules of

$\frac{M}{K}$. So $\exists k \in \mathbb{N}$ such that $M_k + K = M_{k+i} + K$ for all $i \geq 0$. Let $n = \max\{j, k\}$. Now

$$\begin{aligned} M_{n+i} &= M_{n+i} \cap (M_{n+i} \cap K) \\ &= M_{n+i} \cap (M_n + K) \\ &= M_n + (M_{n+i} \cap K) \text{ by the Dedekind Modular Law} \\ &= M_n + (M_n \cap K) \\ &= M_n \end{aligned}$$

So M has the ascending chain condition on submodules. Similarly for the descending chain condition. □

Corollary 5.2.12. *Let $M_1, \ldots, M_n$ be submodules of M_R. If each M_i has the ascending chain condition (respectively, the descending chain condition) on submodules then so does $K := M_1 + \cdots + M_n$.*

Proof. Let $K_1 := M_1 + M_2$. Then by the Second Isomorphism Theorem,

$$\frac{K_1}{M_1} = \frac{M_1 + M_2}{M_1} \cong \frac{M_2}{M_1 \cap M_2}.$$

Now $\frac{M_2}{M_1 \cap M_2}$ has the ascending chain condition on submodules, since it is a factor of M_2. So $\frac{K_1}{M_1}$ has the ascending chain condition on submodules. Thus M_1 and $\frac{K_1}{M_1}$ have the ascending chain condition on submodules. So, by Theorem 5.2.11, K_1 has the ascending chain condition on submodules. Extend to K by induction.

Analagously for the descending chain condition. □

Corollary 5.2.13. *Let R have 1 and the ascending chain condition (respectively, the descending chain condition) on ideals. Let M_R be a (unital) finitely generated R-module. Then M has the ascending chain condition (respectively, the descending chain condition) on submodules.*

Proof. Since M_R is unital and finitely generated, there exist $m_1, \ldots, m_k \in M$ such that $M = m_1R + \cdots + m_kR$. By Corollary 5.2.12, it is enough to show that each m_iR has the ascending chain condition on submodules. Let $\theta_i : R_R \to m_iR : r \mapsto m_ir$. Then θ_i is an R-homomorphism onto m_iR. So each m_iR is a factor module of R_R. Since R_R has the ascending chain condition on submodules it follows that m_iR has the ascending chain condition on submodules.

Similarly for the descending chain condition. □

Remark 5.2.14. For the ascending chain condition the above result is still true even if R does not have 1; for the descending chain condition the result is false.

Corollary 5.2.15. *If R has the ascending chain condition (respectively, the descending chain condition) on right ideals then so does the ring $M_n(R)$.*

Proof. Consider $M_n(R)$ as a right R-module in the natural way. Let $T_{ij} := \{(r_{k\ell}) \in M_n(R) | r_{k\ell} \neq 0 \Rightarrow k = i, \ell = j\}$. Then each T_{ij} is an R-submodule of $M_n(R)$ that is isomorphic to R_R. So each T_{ij} has the ascending chain condition on R-submodules. But $M_n(R) = \bigoplus_{i,j=1}^{n} T_{ij}$. So by Corollary 5.2.12, $M_n(R)$ has the ascending chain condition on R-submodules. Clearly, however, a right ideal of $M_n(R)$ is also an R-submodule of $M_n(R)$. So $M_n(R)$ has the ascending chain condition on right ideals.

Similarly for the descending chain condition. □

Example 5.2.16. Let D be a division ring. Then $M_n(D)$ has both the ascending chain condition and descending chain condition on right ideals, since 0 and D are the only (right) ideals of D.

Exercise 5.2.17. Let S be a subring of $M_n(\mathbb{Z})$ such that S contains the identity of $M_n(\mathbb{Z})$. Show that S is a ring with the ascending chain condition on right ideals.

Definitions 5.2.18. A module with the ascending chain condition on submodules is called a *Noetherian*[1] *module.* A module with the descending chain condition on submodules is called an *Artinian*[2] *module.* A ring with the ascending chain condition on right ideals is called a *right Noetherian ring.* A ring with the descending chain condition on right ideals is called a *right Artinian ring.* Similarly for *left Noetherian ring* and *left Artinian ring.*

Theorem 5.2.19. (The Hilbert Basis Theorem.) *If R is a right Noetherian ring then so is the polynomial ring $R[x]$.*

[1]Amalie (Emmy) Noether (1883–1935)
[2]Emil Artin (1898–1962)

6 Semi-Simple Artinian Rings

6.1 Nil and Nilpotent Subsets

Definitions 6.1.1. Let R be a ring.

(i) $x \in R$ is *nilpotent* if $x^n = 0$ for some $n \geq 1$.

(ii) A subset $S \subseteq R$ is a *nil subset* if every element of S is nilpotent. Thus S nil, $x \in S \Rightarrow \exists n(x) \in \mathbb{N}$ such that $x^{n(x)} = 0$.

(iii) S is a *nilpotent subset* if $S^n = 0$ for some $n \geq 1$. Recall that

$$S^n = \left\{ \sum_{\text{finite}} s_1 s_2 \dots s_n \,\middle|\, s_i \in S \right\}.$$

Examples 6.1.2. (i) In $M_2(\mathbb{Z})$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is a nilpotent element.

(ii) In $\frac{\mathbb{Z}}{4\mathbb{Z}}$, $\frac{2\mathbb{Z}}{4\mathbb{Z}}$ is a nilpotent ideal.

Lemma 6.1.3. *(i) If I, K are nilpotent right ideals than so are $I+K$ and RI.*

(ii) Every nilpotent right ideal is contained in a nilpotent ideal.

Proof. (i) There are positive integers r, s such that $I^r = K^s = 0$. Consider $(I+K)^{r+s-1} = (I+K)(I+K)\dots(I+K)$. This, when expanded, has 2^{r+s-1} terms, each of which has the form $T = A_1 A_2 \dots A_{r+s-1}$, where each $A_i = I$ or K. So in a typical term either I occurs $\geq r$ times or K occurs $\geq s$ times.

Suppose I occurs $\geq r$ times. Then $T \subseteq K^i I^k$ where $i \geq 0$, $k \geq r$, with the convention $K^0 I = I$. We have used the fact that $IK \subseteq I$. So $T = 0$. So every term in the expansion of $(I+K)^{r+s-1}$ vanishes. So $I+K$ is nilpotent.

$$(RI)^r = (RI)(RI)\dots(RI) = R(IR)\dots(IR)I \subseteq RI^r = 0.$$

(ii) Let $I \trianglelefteq_r R$. If I is nilpotent then so is $I + RI$ by (i). Clearly $I + RI \trianglelefteq R$ and $I \subseteq I + RI$. □

Definition 6.1.4. The sum of all nilpotent ideals of R is calles the *nilpotent radical* of R, usually denoted $\mathcal{N}(R)$.

It follows from Lemma 6.1.3 that

$$\mathcal{N}(R) = \sum \text{nilpotent right ideals} = \sum \text{nilpotent left ideals.}$$

Clearly $\mathcal{N}(R)$ is a nil ideal. It is not, in general, nilpotent.

Exercise 6.1.5. Let R be a commutative ring. Show that $\mathcal{N}(R) =$ the set of all nilpotent elements of R. (Hint: use the Binomial Theorem.) Give an example to show that this is false in general for non-commutative rings.

Example 6.1.6. (A Zassenhaus Algebra.) Let F be a field, I the interval $(0, 1)$, R the vector space over F with basis $\{x_i | i \in I\}$. Define multiplication on R by extending the following product on basis elements:

$$x_i x_j := \begin{cases} x_{i+j} & \text{if } i + j < 1, \\ 0 & \text{if } i + j \geq 1. \end{cases}$$

Thus every element of R can be written uniquely in the form $\sum_{i \in I} a_i x_i$, $a_i \in F$, with $a_i = 0$ for all but a finite number of indices i. Check that R is nil but not nilpotent and that $\mathcal{N}(R) = R$.

6.2 Idempotent Elements

Definition 6.2.1. An element $e \in R$ is *idempotent* if $e = e^2$.

Examples 6.2.2. (i) In any ring, 0 is an idempotent element. If 1 exists, it is idempotent.

(ii) In $M_2(\mathbb{Z})$, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are idempotents.

Lemma 6.2.3. *Let e be an idempotent in a ring R. Then $R = eR \oplus K$, where $K = \{x - ex | x \in R\} \trianglelefteq_r R$.*

Proof. Clearly $K \trianglelefteq_r R$. Now $x \in R \Rightarrow x = ex + (x - ex) \in eR + K$. If $z \in eR \cap K$ then $z = ea = eb - b$ for some $a, b \in R$. Then

$$e^2 a = e^2 b - eb = eb - eb = 0$$

and

$$e^2 a = ea = 0.$$

So $z = 0$. So $R = eR \oplus K$ ∎

Corollary 6.2.4. (Peirce Decomposition.) *Let R be a ring with 1 and $e \in R$ an idempotent. Then $R = eR \oplus (1-e)R$*

Proof. $K = (1-e)R$ in the above if R has 1. □

Remark 6.2.5. e is idempotent $\Leftrightarrow$ $1-e$ is idempotent.

Exercise 6.2.6. Take an idempotent in $M_2(\mathbb{Z})$ and write down a Peirce decomposition.

Proposition 6.2.7. *Let R be a ring with 1. Suppose $R = \bigoplus_{j=1}^{n} I_j$, a direct sum of right ideals. Then we can write $1 = e_1 + \cdots + e_n$ with $e_j \in I_j$ having the following properties:*

(i) *each e_j is an idempotent;*

(ii) *$e_i e_j = 0$ for all $i \neq j$;*

(iii) *$I_j = e_j R$ for $j = 1, \ldots, n$;*

(iv) *$R = Re_1 \oplus \cdots \oplus Re_n$, a direct sum of left ideals.*

Proof. (i) and (ii). For each j we have

$$e_j = 1e_j = e_1e_j + \cdots + e_{j-1}e_j + e_j^2 + e_{j+1}e_j + \cdots + e_ne_j.$$

So

$$e_j - e_j^2 = e_1e_j + \cdots + e_{j-1}e_j + e_{j+1}e_j + \cdots + e_ne_j \in I_j \cap \sum_{s\neq j} I_s = 0$$

by directness. So $e_j = e_j^2$ and $\sum_{s\neq j} e_s e_j = 0$. Since the sum of the I_j is direct, we have $e_i e_j = 0$ for all $i \neq j$.

(iii), (iv). Exercises. □

Example 6.2.8. Take $R = M_n(\mathbb{Z})$, $e_j = E_{jj}$ matrix unit. Then $1 = e_1 + \cdots + e_n$ and $R = e_1R \oplus \cdots \oplus e_nR = Re_1 \oplus \cdots \oplus Re_n$.

Definition 6.2.9. Let R be a ring. The *centre* of R is

$$C(R) := \{x \in R | \forall r \in R, xr = rx\}.$$

$C(R)$ is a subring of R but not, in general, an ideal.

Exercise 6.2.10. Let F be a field. Find $C(M_n(F))$. Show that $C(M_n(F)) \cong F$ as rings.

Proposition 6.2.11. *Let R be a ring with 1, with $R = A_1 \oplus \cdots \oplus A_k$ a direct sum of ideals. Let $1 = e_1 + \cdots + e_k$, $e_j \in A_j$. Then*

(i) $e_j \in C(R)$ *for* $j = 1, \ldots, k$;

(ii) $e_j^2 = e_j$ *for all* j; $e_i e_j = 0$ *for* $i \neq j$;

(iii) $A_j = e_j R = R e_j$;

(iv) e_j *is the identity of the ring* A_j.

Proof. (ii) follows from Proposition 6.2.7.

(iii) $A_j = e_j R = R e_j$ as in Proposition 6.2.7 since $A_j \trianglelefteq_r R$ and $A_j \trianglelefteq_\ell R$.

(iv) Let $x \in A_j$. Then $x = e_j t_1 = t_2 e_j$ for some t_1, t_2. Then $e_j x = e_j^2 t_1 = e_j t_1 = x$ and $x e_j = t_2 e_j^2 = t_2 e_j = x$. Thus $x e_j = e_j x = x$ for all $x \in A_j$. Since $e_j \in A_j$, it follows that e_j is the identity of the ring A_j.

(i) Let $x \in R$. Then $e_j x = e_j^2 x = e_j(e_j x) = (e_j x) e_j$ since $e_j x \in A_j$ and e_j is the identity of A_j. Also $x e_j = x e_j^2 (x e_j) e_j = e_j (x e_j)$ similarly. By associativity, $e_j x = x e_j$ for all $x \in R$, so $e_j \in C(R)$. ☐

Definition 6.2.12. Such an $e_j \in C(R)$ is called a *central idempotent*.

6.3 Annihilators and Minimal Right Ideals

Definitions 6.3.1. Let $\varnothing \neq S \subseteq M_R$. We define the *right annihilator* of S to be $r(S) := \{r \in R | Sr = 0\}$. Clearly, $r(S) \trianglelefteq_r R$. When S is a submodule of M, $r(S) \trianglelefteq R$. $\ell(S)$, the *left annihilator* of S, is defined analagously when M is a left R-module.

In most applications, $S \subseteq R$ itself, and so we can consider both $r(S)$ and $\ell(S)$.

Definition 6.3.2. A non-zero right ideal M of a ring R is a *minimal right ideal* if whenever $M' \subsetneq M$, $M' \trianglelefteq_r R$, it follows that $M' = 0$.

If R has 1 then the minimal right ideals of R are precisely the irreducible submodules of R_R.

Lemma 6.3.3. *Let M be a minimal right ideal of a ring R. Then either $M^2 = 0$ or $M = eR$ from some $e = e^2 \in M$.*

Proof. Suppose $M^2 \neq 0$. Then $\exists a \in M$ such that $aM \neq 0$. $aM \trianglelefteq_r R$ and $aM \subseteq M$ since $a \in M$. So $aM = M$. Thus $\exists e \in M$ such that $a = ae$. $a \neq 0 \Rightarrow e \neq 0$. Also $a = ae = ae^2$. So $a(e - e^2) = 0$.

Now consider $M \cap r(a)$. $M \cap r(a) \trianglelefteq_r R$ and $M \cap r(a) \subseteq M$. So $M \cap r(a) = 0$ or M. Suppose for a contradiction that $M \cap r(a) = M$. So $M \subseteq r(a)$, so $aM = 0$. Thus $M \cap r(a) = 0$.

But $e - e^2 \in M \cap r(a) = 0$, so $e = e^2$. Now $0 \neq e^2 \in eR$. So $eR \neq 0$. But $eR \trianglelefteq_r R$ and $eR \subseteq M$ since $e \in M$. Thus $eR = M$ as required. □

Example 6.3.4. Take $R := \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$. Consider $M_1 := \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{pmatrix}$, $M_2 := \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{Q} \end{pmatrix}$. Both M_1, M_2 are minimal right ideals. Now $M_1^2 = 0$, $M_2 = eR$ where $e := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Definition 6.3.5. A ring with no non-zero nilpotent ideal and the descending chain condition on right ideals is called a *semi-simple Artinian ring.*

Note that by Lemma 6.1.3(ii) and symmetry such a ring has no non-zero nilpotent right or left ideals.

Remarks 6.3.6.

(i) The left-right symmetry of semi-simple Artinian rings will be established later.

(ii) We will justify the term "Artinian" by showing the existence of an identity.

(iii) We shall not define "semi-simple" on its own, but in this context it can be thought of as meaning a direct sum of simple rings.

Proposition 6.3.7. *Let R be a semi-simple Artinian ring and $I \trianglelefteq_r R$. Then $I = eR$ for some $e = e^2 \in I$.*

Proof. By the minimum condition every non-zero right ideal of R contains a minimal right ideal. Hence, by Lemma 6.3.3, since R contains non non-zero nilpotent right ideal, every non-zero right ideal of R contains a non-zero idempotent.

Now if $I = 0$ then the result is trivial with $e = 0$, so assume that $I \neq 0$. Let $\mathcal{E}$ be the set of all non-zero idempotents in I. By the above, $\mathcal{E} \neq \varnothing$. We claim that there is an idempotent $e \in \mathcal{E}$ such that $I \cap r(e) = 0$.

Suppose not. Let $I \cap r(z)$ be minimal in the set $\mathcal{S} := \{I \cap r(x) | x \in \mathcal{E}\}$. By assumption, $I \cap r(z) \neq 0$. So $I \cap r(z)$ contains a non-zero idempotent z'. So $(z')^2 = z'$, $zz' = 0$. Consider $z_1 = z + z' - z'z$. $z_1 \in I$ since $z, z' \in I$. We have

$$z_1 z = (z + z' - z'z)z = z + z'z - z'z = z.$$

So, in particular, $z_1 \neq 0$.

$$z_1 z' = (z + z' - z'z)z' = (z')^2 = z',$$

so

$$z_1^2 = z_1(z = z' - z'z) = z + z' - z'z = z_1.$$

Thus $z_1 \in \mathcal{E}$. We shall now show that

$$r(z_1) \cap I \subsetneq r(z) \cap I \tag{6.3.1}$$

$t \in r(z_1) \cap I \Rightarrow z_1 t = 0 \Rightarrow z z_1 t = 0 \Rightarrow zt = 0$, since $zz_1 = z \Rightarrow t \in r(z) \cap I$.

Also, $z' \in r(z_1) \cap I$ but $z' \notin r(z) \cap I$ since $z_1 z' = z' \neq 0$. This establishes (6.3.1). But (6.3.1) contradicts the minimality of $r(z) \cap I$. This proves our claim. So there is an $e \in \mathcal{E}$ such that $I \cap r(e) = 0$.

Now define $K := \{x - ex | x \in I\}$. Then $K \trianglelefteq_r R$, $K \subseteq I$, and $eK = 0$. So $K \subseteq I \cap r(e) = 0$. Thus $x = ex$ for all $x \in I$. Hence $I \subseteq eR$. But clearly $eR \subseteq I$ since $e \in I$ and $I \trianglelefteq_r R$. So $I = eR$ as required. □

Corollary 6.3.8. *Let R be a semi-simple Artinian ring and $A \trianglelefteq R$. Then there is an $e = e^2 \in A$ such that $A = eR = Re$.*

Proof. By Proposition 6.3.7, $A = eR$ for some $e = e^2 \in A$, since $A \trianglelefteq_r R$. Let $K := \{x - xe | x \in A\}$. Then $K \trianglelefteq_\ell R$, since $A \trianglelefteq R$. Also, $Ke = 0$, so

$KeR = 0$ and $K^2 = 0$ since $K \subseteq A = eR$. So $K = 0$ as R has no non-zero nilpotent left ideal. Thus $x = xe$ for all $x \in A$. So $A \subseteq Re$. But $Re \subseteq A$ since $e \in A$ and $A \trianglelefteq_\ell R$. Thus $A = eR = Re$. □

Corollary 6.3.9. *A semi-simple Artinian ring has identity.*

Proof. Take $A = R$ in Corollary 6.3.8. □

Theorem 6.3.10. *The following are equivalent for any ring R:*

(i) R is semi-simple Artinian ring;

(ii) R has 1 and R_R is completely reducible.

Proof. (i)⇒(ii). By Corollary 6.3.9 R has 1. Let $I \trianglelefteq_r R$. By Proposition 6.3.7 $I = eR$ for some $e = e^2 \in I$. By Peirce Decomposition, Corollary 6.2.4, I is a direct summand of R. So every submodule of R_R is a direct summand of R_R. So by Theorem 4.2.6, R_R is completely reducible.

(ii)⇒(i). We have $R = \bigoplus_{\lambda \in \Lambda} I_\lambda$, I_λ an irreducible submodule of R_R. Then $1 = x_1 + \cdots + x_n$ for some $x_i \in I_{\lambda_i}$. Now for any $x \in R$,

$$x = 1x = x_1 x + \cdots + x_n x \in I_{\lambda_1} \oplus \cdots \oplus I_{\lambda_n},$$

so $R = I_{\lambda_1} \oplus \cdots \oplus I_{\lambda_n}$ and $|\Lambda| < \infty$. So by Corollary 5.2.12, R_R has the descending chain condition on R-submodules, i.e. R has the descending chain condition on right ideals. Now let T be a nilpotent right ideal of R. Then $R = T \oplus K$ for some $K \trianglelefteq_r R$ by Theorem 4.2.6. We have $1 = t + k$ for some $t \in T$, $k \in K$. t is nilpotent so $t^n = 0$ for some $n \geq 1$. Thus $(1 - k)^n = 0$. So $1 - nk + \cdots \pm k^n = 0$, $1 = nk - \cdots \mp k^n \in K$. So $K = R$, so $T = 0$ as required. □

Remark 6.3.11. Note that we have shown $R = I_1 \oplus \cdots \oplus I_n$, a finite direct sum of minimal right ideals, when R is semi-simple Artinian.

Corollary 6.3.12. *A direct sum of matrix rings over division rings is a semi-simple Artinian ring.*

Proof. It was shown in Theorem 4.2.6 that for such a ring R_R is completely reducible. □

6.4 Ideals in Semi-Simple Artinian Rings

Proposition 6.4.1. *Let R be a semi-simple Artinian ring:*

(i) Every ideal of R is generated by an idempotent lying in the centre of R.

(ii) There is a 1-1 correspondence between ideals of R and idempotents in $C(R)$.

Proof. (i) See proofs of of Corollary 6.3.8 and Proposition 6.2.11.

(ii) For each $e \in C(R)$ define $f(e) := eR \trianglelefteq R$. Check that f is the required 1-1 correspondence. □

Definition 6.4.2. $I \trianglelefteq R$ is a *minimal ideal* if $I \neq 0$, $I' \subsetneq I$, $I' \trianglelefteq R \Rightarrow I' = 0$.

Theorem 6.4.3. *Let R be a semi-simple Artinian ring. Then R has a finite number of minimal ideals, their sum is direct, and is R.*

Proof. Note that at least one minimal ideal exists since R has the descending chain condition on right ideals . Let S_1 be a minimal ideal of R. Then by Proposition 6.4.1, $S_1 = e_1R = Re_1$, $e_1 = e_1^2 \in C(R)$. Note that $(1 - e_1)^2 = 1 - e_1$, $1 - e_1 \in C(R)$, and we have a direct sum of ideals $R = S_1 \oplus T_1$, $T_1 := (1 - e_1)R = R(1 - e_1)$. (Note that T_1 is two-sided.)

If $T_1 \neq 0$, T_1 contains a minimal $S_2 \trianglelefteq R$. As above, $R = S_2 \oplus K$, $K \trianglelefteq R$. Now

$$T_1 = T_1 \cap R = T_1 \cap (S_2 \oplus K) = S_2 \oplus (T_1 \cap K) = S_2 \oplus T_2$$

by the Dedekind Modular Law. We have $R = S_1 \oplus T_1 = S_1 \oplus S_2 \oplus T_2$. If $T_2 \neq 0$ proceed inductively.

We have $T_1 \supsetneq T_2 \supsetneq T_3 \supsetneq \dots$, so by descending chain condition this process must terminate. It can only stop when some $T_m = 0$. At this stage we have $R = S_1 \oplus \cdots \oplus S_m$, a finite direct sum of minimal ideals.

Now let S be a minimal ideal of R. We have $SR \neq 0$ since R has 1. So $SS_j \neq 0$ for some j, $1 \leq j \leq m$. Now $SS_j \trianglelefteq R$ and $SS_j \subseteq S$, $SS_j \subseteq S_j$. Since both S and S_j are minimal, we have $S = SS_j = S_j$. □

6.5 Simple Artinian Rings

Let R be a simple ring. Consider R^2. $R^2 \trianglelefteq R$ so $R^2 = 0$ or R. Suppose $R^2 = 0$. Then $xy = 0$ for all $x, y \in R$. So any additive subgroup of R is an ideal of R. Thus R has no additive subgroups other than 0 and R. Thus the additive group of R must be cyclic of prime order p; the ring structure of R is completely determined: $R = \{0, 1, \ldots, p-1\}$ with addition modulo p, multiplication identically 0.

Thus, when studying simple rings, we assume that $R^2 = R$. Therefore, in this case, $\mathcal{N}(R) = 0$.

Note that a simple Artinian ring is semi-simple Artinian.

Lemma 6.5.1. *Let R be a semi-simple Artinian ring and $0 \neq I \trianglelefteq R$. Then I itself is a semi-simple Artinian ring. In particular, when I is a minimal ideal, I is a simple Artinian ring.*

Proof. We claim that $K \trianglelefteq_r I \Rightarrow K \trianglelefteq_r R$.

We have $I = eR = Re$ where $e = e^2 \in I$ by Corollary 6.3.8. Now $k \in K$, $r \in E \Rightarrow kr = (ke)r$ since e is the identity of I and $k \in I$. So $kr = k(er) \in K$ since $er \in eR = I$. This proves the claim.

It follows that I considered as a ring has the descending chain condition on right ideals and no non-zero nilpotent ideal.

If I is minimal then by the above it must be a simple Artinian ring. (Note that $I^2 = I$ since $I^2 \neq 0$ – i.e., I is a simple ring of the type that we are considering. □

Theorem 6.5.2. *Let R be a semi-simple Artinian ring. Then R is a direct sum of simple Artinian rings and this representation is unique.*

Proof. By Theorem 6.4.3 and Lemma 6.5.1 above we have $R = S_1 \oplus \cdots \oplus S_m$, where the S_i are minimal ideals of R, hence simple Artinian rings. The uniqueness follows from the fact that the S_i are precisely the minimal ideals of R. □

6.6 Modules over Semi-Simple Artinian Rings

Proposition 6.6.1. *Let R be a semi-simple Artinian ring. Let M_R be unital and irreducible. Then $M \cong I$ as R-modules, where $I \trianglelefteq_r R$.*

Proof. $M \cong \frac{R}{K}$, K a maximal right ideal of R, by Exercise Sheet 3. By Corollary 6.2.4 and Proposition 6.3.7 (or by Theorem 6.3.10 and Theorem 4.2.6), $R = K \oplus I$ for some $I \trianglelefteq_r R$. Therefore, $\frac{R}{K} \cong I$ (given $r \in R$, $r = k + x$, $k \in K$, $x \in I$ unique; consider $r \mapsto x$). So $M \cong \frac{R}{K} \cong I$ as R-modules. □

Theorem 6.6.2. *Every non-zero unital module over a semi-simple Artinian ring is completely reducible.*

Proof. Let R be semi-simple Artinian. We have $R = I_1 \oplus \cdots \oplus I_n$, a direct sum of minimal right ideals of R (see Remark 6.3.11). Let $0 \neq M_R$ be unital and let $m \in M$. Then $m = m1 \in mI_1 + \cdots + mI_n$. We claim that each mI_j is either irreducible or 0.

Consider the map $\theta : I_j \to mI_j$ given by $\theta(x) = mx$ for $x \in I_j$. Clearly θ is an R-homomorphism onto mI_j. $\ker\theta$ is a submodule of $(I_j)_R$. So $\ker\theta = 0$ or I_j. This implies that either θ is an isomorphism or the zero map. This proves the claim.

Thus, $m \in \sum$ irreducible submodules. So $M = \sum$ irreducible submodules. So M is completely reducible. □

6.7 The Artin-Wedderburn Theorem

Definitions 6.7.1. Let M be a right R-module. An R-homomorphism $\theta : M \to M$ is called an *R-endomorphism*. The set of all R-endomorphisms of M is denoted by $\mathcal{E}_R(M)$ or simply $\mathcal{E}(M)$. On $\mathcal{E}_R(M)$ we define a sum and product as follows. Let $\theta, \phi \in \mathcal{E}_R(M)$. Define $\theta + \phi$ and $\theta\phi$ by

$$(\theta + \phi)(m) := \theta(m) + \phi(m)$$
$$(\theta\phi)(m) := \theta(\phi(m))$$

for $m \in M$. Check that $(\theta + \phi), (\theta\phi) \in \mathcal{E}_R(M)$. It is routine to check that $\mathcal{E}_R(M)$ is a ring under these operations.

Exercise 6.7.2. Show that if $M_1 \cong M_2$ as R-modules then $\mathcal{E}_R(M_1) \cong \mathcal{E}_R(M_2)$ as rings.

More generally,

Definitions 6.7.3. For right R-modules X and Y denote the set of all R-homomorphisms $X \to Y$ by $\mathrm{Hom}_R(X,Y)$. For $\alpha, \beta \in \mathrm{Hom}_R(X,Y)$ define $\alpha + \beta$ as above. $\mathrm{Hom}_R(X,Y)$ is easily seen to be an Abelian group.

Definitions 6.7.4. Let $V = V_1 \oplus \cdots \oplus V_n$, $W = W_1 \oplus \cdots \oplus W_m$ be right R-modules. Let $\varepsilon_j : V_j \to V$ be the *injection map* $\varepsilon_j(v_j) := (0, \ldots, 0, v_j, 0, \ldots, 0)$ for $v_j \in V_j$ (the v_j is in the jth place).

Let $\pi_i : W \to W_i$ be the *projection map*, $\pi_i(w_1, \ldots, w_m) = w_i$, $w_k \in W_k$.

For a module M, write $M^{(n)}$ for $M \oplus \cdots \oplus M$ (n times).

Lemma 6.7.5. *Let $V = V_1 \oplus \cdots \oplus V_n$, $W = W_1 \oplus \cdots \oplus W_m$ be right R-modules.*

(i) If $\phi_{ij} \in \mathrm{Hom}_R(V_j, W_i)$ are given for each $1 \le i \le m$, $1 \le j \le n$, then we can define $\phi \in \mathrm{Hom}_R(V,W)$ by

$$\phi(v_1, \ldots, v_m) := \begin{pmatrix} \phi_{11} & \ldots & \phi_{1n} \\ \vdots & \ddots & \vdots \\ \phi_{m1} & \ldots & \phi_{mn} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$
$$= (\phi_{11}v_1 + \cdots + \phi_{1n}v_n, \ldots, \phi_{m1}v_1 + \cdots + \phi_{mn}v_n).$$

(ii)

$$\mathrm{Hom}_R(V,W) \cong \begin{pmatrix} \mathrm{Hom}_R(V_1,W_1) & \ldots & \mathrm{Hom}_R(V_n,W_1) \\ \vdots & \ddots & \vdots \\ \mathrm{Hom}_R(V_1,W_m) & \ldots & \mathrm{Hom}_R(V_n,W_m) \end{pmatrix}$$

as Abelian groups.

(iii) In particular, for a right R-module M we have

$$\mathcal{E}_R(M^{(n)}) \cong \begin{pmatrix} \mathcal{E}_R(M) & \ldots & \mathcal{E}_R(M) \\ \vdots & \ddots & \vdots \\ \mathcal{E}_R(M) & \ldots & \mathcal{E}_R(M) \end{pmatrix}$$

as rings.

Proof. (i) Easy to see that ϕ as defined above does indeed belong to $\mathrm{Hom}_R(V, W)$.

(ii) Let $\psi \in \mathrm{Hom}_R(V, W)$. Define $\psi_{ij} := \pi_i \psi \varepsilon_j$. Then $\psi_{ij} \in \mathrm{Hom}_R(V_j, W_i)$. Define a map

$$\Theta : \mathrm{Hom}_R(V, W) \to \begin{pmatrix} \mathrm{Hom}_R(V_1, W_1) & \dots & \mathrm{Hom}_R(V_n, W_1) \\ \vdots & \ddots & \vdots \\ \mathrm{Hom}_R(V_1, W_m) & \dots & \mathrm{Hom}_R(V_n, W_m) \end{pmatrix}$$

by $\Theta(\psi) := (\psi_{ij})$. It is easy to see that Θ is an additive group homomorphism. Θ is injective:

$$\begin{aligned} \psi_{ij} = 0 \forall i, j &\Rightarrow \pi_i \psi \varepsilon_j = 0 \text{ for all } i, j \\ &\Rightarrow \psi \varepsilon_j = 0 \text{ for all } j \\ &\Rightarrow \psi = 0 \end{aligned}$$

Θ is surjective: let $(\theta_{ij}$ be given with $\theta_{ij} \in \mathrm{Hom}_R(V_j, W_i)$. Define $\theta : V \to W$ by

$$\phi(v_1, \dots, v_m) := \begin{pmatrix} \phi_{11} & \dots & \phi_{1n} \\ \vdots & \ddots & \vdots \\ \phi_{m1} & \dots & \phi_{mn} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

Then $\theta \in \mathrm{Hom}_R(V, W)$ by part (i). For $v_j \in V_j$ we have

$$\begin{aligned} \pi_i \theta \varepsilon_j(v_j) &= \pi \theta(0, \dots, 0, v_j, 0, \dots, 0) \\ &= \pi_i(\theta_{1j}(v_j), \dots, \theta_{mj}(v_j)) \\ &= \theta_{ij}(v_j) \end{aligned}$$

So $\pi_i \theta \varepsilon_j = \theta_{ij}$. Hence Θ is an isomorphism.

(iii) We must check that when $V = W = M^{(n)}$ then Θ defined above is a ring homomorphism. Let $\theta, \phi \in \mathcal{E}_R(M^{(n)})$. We have

$$\begin{aligned}
(\theta\phi)_{ij} &= \pi_i(\theta\phi)\varepsilon_j \\
&= \pi_i\theta\,\mathrm{id}_{M^{(n)}}\,\phi\varepsilon_j \\
&= \pi_i\theta\left(\sum_{k=1}^{n}\varepsilon_k\pi_k\right)\phi\varepsilon_j \\
&= \sum_{k=1}^{n}(\pi_i\theta\varepsilon_k)(\pi_k\phi\varepsilon_j) \\
&= \sum_{k=1}^{n}(\theta_{ik})(\phi_{kj}) \\
\Rightarrow ((\theta\phi)_{ij}) &= \sum_{k=1}^{n}(\theta_{ik})(\phi_{kj}) \\
\Rightarrow \Theta(\theta\phi) &= \Theta(\theta)\Theta(\phi)
\end{aligned}$$

So Θ is a ring homomorphism. □

Note that if $\theta : M_R \to K_R$ is an isomorphism then the inverse map exists and is an isomorphism from K onto M.

Corollary 6.7.6. (Schur's Lemma.) *Let R be a ring and M_R an irreducible module. Then $\mathcal{E}_R(M)$ is a division ring.*

Proof. Let $0 \neq \theta \in \mathcal{E}_R(M)$. We must show that θ is an isomorphism. θ is injective since $\ker\theta$ is a submodule of M, so $\ker\theta = 0$ or M. But $\ker\theta = M \Rightarrow \theta = 0$, a contradiction, so $\ker\theta = 0$ and θ is injective.

θ is surjective since $\theta(M)$ is a submodule of M. As above, $\theta(M) \neq 0$, so $\theta(M) = M$ and θ is surjective.

Thus θ is an isomorphism. Thus every non-zero element of $\mathcal{E}_R(M)$ has an inverse, and so $\mathcal{E}_R(M)$ is a division ring. □

Lemma 6.7.7. *Let R be a simple ring with $R^2 = R$. Then any two minimal right ideals of R are isomorphic as R-modules.*

Proof. Let I_1, I_2 be two minimal right ideals of R. Then $r(I_1) \trianglelefteq R$ and $r(I_1) \neq R$. So $r(I_1) = 0$, so $I_1 I_2 \neq 0$. So $\exists x \in I_1$ such that $xI_2 \neq 0$. $xI_2 \trianglelefteq_r R$

and $xI_2 \subseteq I_1$ since $I_1 \trianglelefteq_r R$. So $xI_2 = I_1$. Now define a map $\theta : I_2 \to I_1$ by $\theta(r) = xr$ for $r \in I_2$. Check that $\ker\theta = 0$ so that θ is an isomorphism from I_2 onto $xI_2 = I_1$. $\square$

Lemma 6.7.8. *Let R be a ring with 1. Then $R \cong \mathcal{E}_R(R_R)$ as rings.*

Proof. Let $x \in R$. Define $\rho_x \in \mathcal{E}_R(R_R)$ by $\rho_x(r) = xr$ for $r \in R$. Fopr $r, s \in R$,

$$\rho_x(r+s) = x(r+s) = xr + xs = \rho_x(r) + \rho_x(s)$$

and

$$\rho_x(rt) = x(rt) = (xr)t = \rho_x(r)t.$$

So we do indeed have that $\rho_x \in \mathcal{E}_R(R_R)$.

Now define $\Theta : R \to \mathcal{E}_R(R_R)$ by $\Theta(x) = \rho_x$ for $x \in R$. We claim that Θ is an isomorphism of rings. Let $x_1, x_2 \in R$. Then for any $r \in R$,

$$\rho_{x_1+x_2}(r) + (x_1 + x_2)r = x_1r + x_2r = \rho_{x_1}(r) + \rho_{x_2}(r) = (\rho_{x_1} + \rho_{x_2})(r),$$

so $\rho_{x_1+x_2} = \rho_{x_1} + \rho_{x_2}$, and hence $\Theta(x_1 + x_2) = \Theta(x_1) + \Theta(x_2)$. Also

$$\rho_{x_1x_2}(r) + (x_1x_2)r = x_1(x_2r) = \rho_{x_1}(\rho_{x_2}(r)) = (\rho_{x_1}\rho_{x_2})(r),$$

so $\Theta(x_1x_2) = \Theta(x_1)\Theta(x_2)$. Θ is injective since $\rho_x = 0 \Rightarrow \rho_x(1) = x1 = 0 \Rightarrow x = 0$. Θ is surjective: let $\phi \in \mathcal{E}_R(R_R)$. Let $y := \phi(1) \in R$. Then $\phi(r) = \phi(1r) = yr = \rho_y(r)$, for all $r \in R$. So $\phi = \rho_y$. Thus Θ is an isomorphism of rings. $\square$

Remark 6.7.9. If we work with left modules then we would have to define $\rho_x(r) = rx$. But then we get an anti-isomorphism between R and $\mathcal{E}_R(R_R)$: $\Theta(xy) = \Theta(y)\Theta(x)$. To get an isomorphsim we need to write our maps on the right.

Theorem 6.7.10. (The Artin-Wedderburn Theorem.) *R is semi-simple Artinian if and only if $R = S_1 \oplus \cdots \oplus S_m$, where $S_i \cong M_{n_i}(D_i)$ for some integers n_i and division rings D_i.*

Proof. $R = S_1 \oplus \cdots \oplus S_m$, S_i simple Artinian by Theorem 6.5.2. $S_i = I_1 \oplus \cdots \oplus I_{n_i}$, a direct sum of minimal right ideal, for some integer n_i, by Remark

6.3.11. But $I_j \cong I_k$ for all j, k, by Lemma 6.7.7. Thus $S_i = I_1 \oplus \cdots \oplus I_1$ (n_i summands). Thus,

$$\begin{aligned} S_i &\cong \mathcal{E}_{S_i}\left((S_i)_{S_i}\right) \text{ by Lemma 6.7.8} \\ &\cong \mathcal{E}_{S_i}\left(I_1^{(n_i)}\right) \\ &\cong M_{n_i}\left(\mathcal{E}_{S_i}(I_1)\right) \text{ by Lemma 6.7.5} \\ &= M_{n_i}(D_i), \end{aligned}$$

where $D_i := \mathcal{E}_{S_i}(I_1)$ is a division ring by Schur's Lemma. □

Theorem 6.7.11. *A semi-simple Artinian ring is left-right symmetric.*

Proof. Right-hand conditions ⇔ R is a direct sum of matrix rings over division rings ⇔ left-hand conditions.

For another proof, see Exercise Sheet 5, Question 7. □

7 Wedderburn's Theorem on Finite Division Rings

In this chapter we prove that every finite division ring is a field.

Our strategy is to let D be a finite division ring. We show that $|D| = q^n$ for some $q \geq 2$, $n > 1$. If we set $D^* := D\backslash\{0\}$ then D^* is not an Abelian group. Counting elements in each conjugacy class of D^* we get an equation

$$q^n - 1 = q - 1 + \sum_{n(a)|n, n(a)\neq n} \frac{q-1}{q^{n(a)}-1}.$$

We then show that such an equation is impossible on number theoretic grounds.

7.1 Roots of Unity

Definitions 7.1.1. (i) θ is called a *primitive nth root of unity* if $\theta^n = 1$ and $\theta^m \neq 1$ for all $m < n$, where m and n are integers.

(ii) $\Phi_n(x) := \prod(x - \theta)$, where the product is taken over all primitive nth roots of unity is called the *nth cyclotomic polynomial.*

We note that the primitive nth roots of unity exist because of $y = e^{2\pi k i/n} \in \mathbb{C}$. Thus

$$\begin{aligned}
\Phi_1(x) &= x - 1 \\
\Phi_2(x) &= x + 1 \\
\Phi_3(x) &= x^2 + x + 1 \\
\Phi_4(x) &= x^2 + 1
\end{aligned}$$

Lemma 7.1.2. *Every cyclotomic polynomial is monic with integer coefficients.*

Proof. First note that

$$x^n - 1 = \prod_{d|n} \Phi_d(x). \tag{7.1.1}$$

Now we prove our claim by induction on n. If $n = 1$, $\Phi_1(x) = x - 1$. So, assume all $\Phi_k(x)$ monic with integer coefficients for $k < n$. Now we can write

$$x^n - 1 = \Phi_n(x) \prod_{d|n, d \neq n} \Phi_d(x).$$

By the induction hypothesis we can write $x^n - 1 = \Phi_n(x) f(x)$, where $f(x)$ is monic with coefficients in $\mathbb{Z}$. Therefore, we may assume that

$$f(x) = x^\ell + a_{\ell-1}x^{\ell-1} + \cdots + a_1 x + a_0,$$

$a_i \in \mathbb{Z}$, and

$$\Phi_n(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0,$$

$b_i \in \mathbb{C}$. We see that

$$x^n - 1 = b_m x^{m+\ell} + (b_{m-1} + b_m a_{\ell-1})x^{m+\ell-1} + \cdots + a_0 b_0.$$

Comparing these two polynomials, we have $b_m = 1$, so Φ_n is monic; $m+\ell = n$; $b_{m-1} + b_m a_{\ell-1} = 0$, so $b_{m-1} = -1 \cdot a_{\ell-1} \in \mathbb{Z}$. By continuing this method, we see that $b_i \in \mathbb{Z}$ for $0 \leq i \leq m$. □

Lemma 7.1.3. *If $d|n$ and $d \neq n$ then*

$$\Phi_n(x) \,\Big|\, \frac{x^n - 1}{x^d - 1}$$

in the sense that the quotient is polynomial with integer coefficients.

Proof. By (7.1.1) we can write

$$\frac{x^n - 1}{x^d - 1} = \frac{\prod_{d|n} \Phi_d(x)}{\prod_{k|d} \Phi_k(x)}.$$

Because every divisor of d is a divisor of n, we have

$$\frac{x^n - 1}{x^d - 1} = \Phi_n(x) \prod_{d'|n, d' \nmid d} \Phi_{d'}(x)$$

and so $\frac{x^n-1}{x^d-1} = \Phi_n(x) f'(x)$, where

$$f'(x) = \prod_{d'|n, d' \nmid d} \Phi_{d'}(x).$$

So $f'(x)$ is a monic polynomial in $\mathbb{Z}[x]$ by the previous lemma. So

$$\Phi_n(x) \left| \frac{x^n - 1}{x^d - 1} \right.$$

□

Lemma 7.1.4. *Let q, n, m be positive integers, $q > 1$. Then*

$$q^m - 1 | q^n - 1 \Leftrightarrow m | n.$$

Proof. ($\Leftarrow$). Evident.

($\Rightarrow$). We may assume that $n > m$. Then $n = km + r$, $0 \leq r < m$, $k > 0$. Now

$$\begin{aligned} \frac{q^n - 1}{q^m - 1} &= \frac{q^{km} q^r - 1}{q^m - 1} \\ &= \frac{q^{km} q^r q^r + q^r - 1}{q^m - 1} \\ &= \frac{q^r((q^m)^k - 1}{q^m - 1} + \frac{q^r - 1}{q^m - 1} \end{aligned}$$

By our assumption, the LHS is an integer, and $\frac{q^r((q^m)^k-1}{q^m-1} \in \mathbb{Z}$, so $\frac{q^r-1}{q^m-1} \in \mathbb{Z}$, so $r = 0$. Thus $m|n$. □

7.2 Group Theory

Definitions 7.2.1. Let G be a group. We say that $x, y \in G$ are *conjugate* if $\exists a \in G$ such that $x = a^{-1} y a$, and so we can define an equivalence relation of conjugacy, and the corresponding *conjugacy classes*: $x^G := \{a^{-1} x a | a \in G\}$. We define

$$C^*(x) = \{a \in G | ax = xa\}$$

to be the *centralizer* of x in G. The *centre* of G is

$$Z(G) := \{g \in G | \forall h \in G, gh = hg\}.$$

Proposition 7.2.2. *For G a group, $x \in G$, $C^*(x)$ is a subgroup of G.*

Theorem 7.2.3. *Let G be a finite group. Then $|x^G| = |G : C^*(x)|$.*

Proof. Let $a, b \in G$. Then

$$\begin{aligned} a^{-1}xa = b^{-1}xb &\Leftrightarrow xab^{-1} = ab^{-1}x \\ &\Leftrightarrow (ab^{-1}) \in C^*(x) \\ &\Leftrightarrow C^*(x)a = C^*(x)b \end{aligned}$$

So there are as many elements in x^G as there are cosets of $C^*(x)$. □

7.3 Finite Division Rings

Lemma 7.3.1. *Let K be a non-zero subring of a finite division ring D. Then K is also a division ring.*

Proof. Exercise. Need $1_D \in K$ and $x^{-1} \in K$ for $x \in K$, $x \neq 0$. □

Corollary 7.3.2. *The centre of a finite division ring is a field.*

Lemma 7.3.3. *Let D be a finite division ring with centre C. Then $|D| = q^n$ where $q = |C| > 1$ and n is some positive integer.*

Proof. C is a field. We can view D as a vector space over C. Let $n = \dim_C D$, with $d_1, \ldots, d_n$ a basis for D over C. So every element of D is uniquely expressible as $c_1d_1 + \cdots + c_nd_n$ with $c_i \in C$. So we have $|D| = q^n$. □

Theorem 7.3.4. (Wedderburn 1905.) *A finite division ring is necessarily a field.*

Proof. Let D be a finite division ring with centre C, $|C| = q$. Then $|D| = q^n$, $q \geq 2$, $n \geq 1$, by Lemma 7.3.3. We want to show that $D = C$, or, equivalently, that $n = 1$.

Assume that $n > 1$. Let $a \in D^*$, $C(a) := \{x \in D | xa = ax\}$. Then $C(a)$ is a subring of D. By Lemma 7.3.1, it is a division ring with $C(A) \supseteq C$. So $|C(a)| = q^{n(a)}$ for some $n(a) \geq 1$. $C^*(a) := C(a) \backslash \{0\}$ is a multiplicative subgroup of D^*. We have $|C^*(a)| = q^{n(a)} - 1$, $|D^*| = q^n - 1$. By Lagrange's Theorem, $q^{n(a)} - 1 | q^n - 1$. Lemma 7.1.4 implies that $n(a)|n$. Theorem 7.2.3 applied to D^* implies that the number of elements conjugate to a = the index of $C^*(a)$ in $D^* = \frac{q^n-1}{q^{n(a)}-1}$. Now $a \in C^*(a) \Leftrightarrow n(a) = n$. By counting elements of D^*:

$$q^n - 1 = q - 1 + \sum_{n(a)|n, n(a) \neq n} \frac{q^n - 1}{q^{n(a)} - 1}, \tag{7.3.1}$$

where the sum is carried out for one a in each conjugacy class for elements not in the centre. Now $\Phi_n(q) = q^n - 1$ by Lemma 7.1.2; $n(a) \neq n \Rightarrow \Phi_n(q) \,\Big|\, \frac{q^n-1}{q^{n(a)}-1}$ by Lemma 7.1.3. By (7.3.1),

$$\Phi_n(q) = q - 1. \tag{7.3.2}$$

We have $\Phi_n(q) = \prod(q - \theta)$, θ a primitive nth root of 1. So $|\Phi_n(q)| = \prod |q - \theta| > q - 1$ since $n > 1$. This contradicts (7.3.2). So the assumption $n > 1$ is false, and $D = C$ is a field. □

To answer the question of what finite fields look like, we need Galois Theory.

8 Some Elementary Homological Algebra

In this section all rings have 1 and all modules are unital.

8.1 Free Modules

Definitions 8.1.1. A right R-module F is *free* if

(i) F is generated by a subset $S \subseteq F$;

(ii) $\sum_i s_i r_i = 0 \Leftrightarrow r_i = 0$ for all such finite sums with $s_i \in S$, $r_i \in R$.

We say that *free basis* for F. (Convention: $\{0\}$ is the free module generated by $\varnothing$.)

An element of F has a unique epression as $s_1 r_1 + \cdots + s_k r_k$. A typical free module is isomorphic to $(R \oplus \cdots \oplus R \oplus \dots)_R$. F_R free and finitely generated $\Rightarrow F_R \cong (R \oplus \cdots \oplus R)_R$ (a finite direct sum).

The $\mathbb{Z}$-module $\frac{\mathbb{Z}}{n\mathbb{Z}}$, $n > 1$, cannot be free: for suppose that $\bar{a} = a + n\mathbb{Z}$ is an element of a free basis. Then $\bar{a}n = \bar{0}$, with $n \neq 0$, a contradiction.

8.2 The Canonical Free Module

Definition 8.2.1. Let A be a set indexed by Λ. Let F_A be the set of all formal sums

$$\left\{ \sum_{\lambda \in \Lambda} a_\lambda r_\lambda \,\middle|\, \begin{array}{c} a_\lambda \in A, r_\lambda \in R, \text{finitely} \\ \text{many } r_\lambda \neq 0 \end{array} \right\}$$

with $\sum_{\lambda \in \Lambda} a_\lambda r_\lambda = \sum_{\lambda \in \Lambda} a_\lambda s_\lambda \Leftrightarrow r_\lambda = s_\lambda$ for all $\lambda \in \Lambda$. We make F_A into the *canonical free right R-module* by defining

$$\sum a_\lambda r_\lambda + \sum a_\lambda s_\lambda := \sum a_\lambda (r_\lambda + s_\lambda)$$

and

$$\left(\sum a_\lambda r_\lambda \right) r := \sum a_\lambda (r_\lambda r)$$

A is a free basis for F_A; we identify $a \in A$ with $a1 \in F_A$.

Proposition 8.2.2. *Every right R-module is the homomorphic image of a free right R-module.*

Proof. Let M be a free right R-module. Index the elements of M and form the free module F_M, considering M merely as a set. Elements of F_M are formal sums of the form $\sum(m_i)r_i$, $m_i \in M$, $r_i \in R$. Define $\theta : F_M \to M : \sum(m_i)r_i \mapsto \sum m_i r_i \in M$. This map is well-defined and is an R-homomorphism by the definition of F_M □

8.3 Exact Sequences

Definitions 8.3.1. Let M_i be a sequence of right R-modules and f_i a sequence of R-homomorphisms $M_i \to M_{i-1}$. The sequence (which may be finite or infinite)

$$\dots \xrightarrow{f_{i+2}} M_{i+1} \xrightarrow{f_{i+1}} M_i \xrightarrow{f_i} M_{i-1} \xrightarrow{f_{i-1}} \dots$$

is said to be *exact* if $\operatorname{im} f_{i+1} = \ker f_i$ for all i. A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0.$$

In a short exact sequence, since $0 \longrightarrow M' \xrightarrow{f} M$ is exact, $\ker f = 0$ and so f is a monomorphism (an injective homomorphism). Since $M \xrightarrow{g} M'' \longrightarrow 0$ is exact, $\operatorname{im} g = M''$ and g is an epimorphism (a surjective homomorphism).

$M' \cong \operatorname{im} f = f(M') \subseteq M$, so M is isomorphic to a submodule of M. Also

$$M'' \cong \frac{M}{\ker g} = \frac{M}{\operatorname{im} f}.$$

Given modules A and B we can construct the short exact sequence

$$0 \longrightarrow B \xrightarrow{i} A \xrightarrow{\pi} \frac{A}{B} \longrightarrow 0,$$

where i is the inclusion map and π the natural projection.

Proposition 8.3.2. *Given a short exact sequence*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

of right R-modules the following are equivalent:

(i) $\operatorname{im}\alpha$ *is a direct summand of* B*;*

(ii) $\exists$ *an* R*-homomorphism* $\gamma : C \to B$ *with* $\beta\gamma = \mathrm{id}_C$*;*

(iii) $\exists$ *an* R*-homomorphism* $\delta : B \to A$ *with* $\delta\alpha = \mathrm{id}_A$*.*

Proof. (i)$\Rightarrow$(ii). Let $B = \operatorname{im}\alpha \oplus B_1$, B_1 a submodule of B. So $B = \ker\beta \oplus B_1$. Let $\beta_1 := \beta|_{B_1}$. We have

$$C = \beta(B) = \beta(\ker\beta \oplus B_1) = \beta B_1 = \beta_1 B_1,$$

so β_1 is an epimorphism. Also $\ker\beta_1 \subseteq \ker\beta \cap B_1 = 0$. Thus β_1 is an isomorphism of B_1 onto C. Define $\gamma := \beta_1^{-1} : C \to B$. Then $\beta\gamma = \mathrm{id}_C$.

(ii)$\Rightarrow$(i). We shall show that $B = \ker\beta \oplus \gamma\beta(B)$. If $b \in B$, $b = (b - \gamma\beta b) + \gamma\beta(b)$. $b - \gamma\beta(b) \in \ker\beta$ since

$$\beta(b - \gamma\beta(b)) = \beta(b) - \beta\gamma\beta(b) = \beta(b) - \mathrm{id}_C\,\beta(b) = \beta(b) - \beta(b) = 0,$$

and if $z \in \ker\beta \cap \gamma\beta(B)$ then $z = \gamma\beta(b)$ for some $b \in B$, and $\beta(z) = 0$. Thus

$$0 = \beta(z) = \beta\gamma\beta(b) = \beta(b),$$

so $z = 0$, $B = \ker\beta \oplus \gamma\beta(B) = \operatorname{im}\alpha \oplus \gamma\beta(B)$.

Similarly, we can show the equivalence of (i) and (iii). □

Definition 8.3.3. We say that the short exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

splits if any one (and hence all) of the above conditions holds.

Note that if the above sequence splits then $B = \operatorname{im}\alpha \oplus B_1$, $B_1 \cong C$; i.e., $B \cong A \oplus C$, an external direct sum.

8.4 Projective Modules

Definition 8.4.1. A right R-module P is said to be *projective* if given any diagram of the form

$$\begin{array}{ccccc} & & P & & \\ & & \big\downarrow{\scriptstyle \mu} & & \\ A & \xrightarrow{\pi} & B & \longrightarrow & 0 \text{ exact} \end{array}$$

there is an R-homomorphism $\bar{\mu} : P \to A$ such that $\mu = \pi\bar{\mu}$, i.e. $\mu(x) = \pi(\bar{\mu}(x))$ for all $x \in P$.

$$\begin{array}{ccccc} & & P & & \\ & \overset{\bar{\mu}}{\swarrow} & \downarrow{\scriptstyle \mu} & & \\ A & \xrightarrow{\pi} & B & \longrightarrow & 0 \text{ exact} \end{array}$$

Lemma 8.4.2. *A free module is projective.*

Proof. Let F be a free module with a free basis $\{e_\alpha\}$. Consider

$$\begin{array}{ccccc} & & F & & \\ & \overset{\bar{\mu}}{\swarrow} & \downarrow{\scriptstyle \mu} & & \\ A & \xrightarrow{\pi} & B & \longrightarrow & 0 \text{ exact} \end{array}$$

Let $b_\alpha := \mu(e_\alpha)$. As π is an epimorphism we can choose $a_\alpha \in A$ such that $b_\alpha = \pi(a_\alpha)$. Define $\bar{\mu} : F \to A$ by $\bar{\mu}\left(\sum_\alpha e_\alpha r_\alpha\right) := \sum_\alpha a_\alpha r_\alpha$, $r_\alpha \in R$. Then $\bar{\mu}$ is an R-homomorphism $F \to A$ and

$$\begin{aligned} \pi\bar{\mu}\left(\sum_\alpha e_\alpha r_\alpha\right) &= \pi\left(\sum_\alpha a_\alpha r_\alpha\right) \\ &= \sum_\alpha \pi(a_\alpha) r_\alpha \\ &= \sum_\alpha b_\alpha r_\alpha \\ &= \sum_\alpha \mu(e_\alpha) r_\alpha \\ &= \mu\left(\sum_\alpha e_\alpha r_\alpha\right). \end{aligned}$$

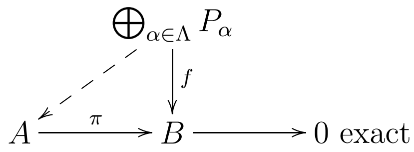
So $\pi\bar{\mu} = \mu$. □

We shall see that a projective module need not be free.

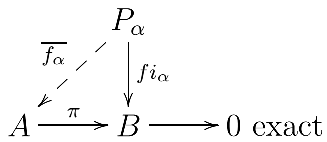
Lemma 8.4.3. *Let P_α, $\alpha \in \Lambda$, be right R-modules. Then $\bigoplus_{\alpha\in\Lambda} P_\alpha$ is projective if and only if all P_α are projective.*

Proof. Let i_β be the inclusion map $P_\beta \hookrightarrow \bigoplus_{\alpha\in\Lambda} P_\alpha$; let p_β be the projection map $\bigoplus_{\alpha\in\Lambda} P_\alpha \to P_\beta$.

($\Leftarrow$) Consider the diagram

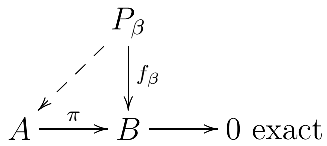


This gives rise to diagrams

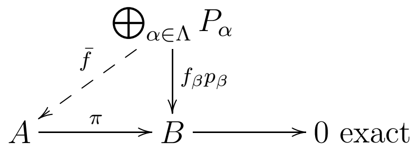


Since each P_α is projective there are R-homomorphisms $\overline{f_\alpha} : P_\alpha \to A$ such that $fi_\alpha = \pi\overline{f_\alpha}$. Define $\bar{f} : \bigoplus_{\alpha\in\Lambda} P_\alpha \to A$ by $f = \sum_{\alpha\in\Lambda} \overline{f_\alpha} p_\alpha$. (This makes sense: for any $z \in \bigoplus_{\alpha\in\Lambda} P_\alpha$ we have $p_\alpha(z) = 0$ for all except a finite number of α's.) We have $\pi\bar{f} = \sum_{\alpha\in\Lambda} \pi\overline{f_\alpha}p_\alpha = \sum_{\alpha\in\Lambda} fi_\alpha\pi_\alpha = f$ as required.

($\Rightarrow$) For any $\beta \in \Lambda$ consider



This gives rise to



Since $\bigoplus_{\alpha\in\Lambda} P_\alpha$ is projective, there is an R-homomorphism $\bar{f} : \bigoplus_{\alpha\in\Lambda} P_\alpha \to A$ such that $\pi\bar{f} = f_\beta p_\beta$. So $\pi\bar{f}i_\beta = f_\beta p_\beta i_\beta = f_\beta$ and $\bar{f}i_\beta$ maps $P_\beta \to A$. Thus P_β is projective. ☐

Proposition 8.4.4. *The following are equivalent:*

(i) P is a projective module;

(ii) P is a direct summand of a free module;

(iii) every short exact sequence $0 \to M' \to M \to P \to 0$ splits.

Proof. (iii)⇒(ii). Consider the short exact sequence

$$0 \longrightarrow K_P \longrightarrow F_P \longrightarrow P \to 0,$$

where K_P is the kernel of the canonical map $F_P \to P$. Since this short exact sequence splits we have $F_P \cong P \oplus K_P$.

(ii)⇒(i). Follows from Lemma 8.4.2 and Lemma 8.4.3.

(i)⇒(iii). Consider

$$\begin{array}{ccccccccc} & & & & & & P & & \\ & & & & & \overset{\bar{\mu}}{\swarrow} & \downarrow \mathrm{id}_P & & \\ 0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & P & \longrightarrow & 0 \text{ exact} \end{array}$$

Since P is projective, there is an R-homomorphism $\bar{\mu} : P \to M$ such that $g\bar{\mu} = \mathrm{id}_P$. Thus the given short exact sequence splits. □

Theorem 8.4.5. *The following are equivalent:*

(i) R is semi-simple Artinian;

(ii) every unital right R-module is projective.

Proof. (i)⇒(ii). Let M be a right R-module. By Theorem 6.6.2, $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$, each M_λ irreducible. Proposition 6.6.1 implies that each M_λ is isomorphic to a right ideal of R. A right ideal of R is a direct summand of R since R_R is completely reducible. So, by Lemma 8.4.2 and Lemma 8.4.3, right ideals of R are projective. So M is projective by Lemma 8.4.3.

(ii)⇒(i). Let $I \trianglelefteq_r R$. Consider the short exact sequence

$$0 \longrightarrow I \xrightarrow{i} R \xrightarrow{\pi} \frac{R}{I} \longrightarrow 0.$$

This short exact sequence splits since $\frac{R}{I}$ is a projective R-module. So $R = I \oplus K$, $K \trianglelefteq R$. Thus I is a direct summand of R. So R_R is completely reducible and thus R is semi-simple Artinian. $\square$

If R is a ring with 1, then all right R-modules are free if and only if R is a division ring (Exercise Sheet 5, Question 8).

Example 8.4.6. Projective $\not\Rightarrow$ free. Let $R = \frac{\mathbb{Z}}{6\mathbb{Z}}$, $A = \frac{2\mathbb{Z}}{6\mathbb{Z}}$, $B = \frac{3\mathbb{Z}}{6\mathbb{Z}}$. Then $R = A \oplus B$, and A and B are projective by Lemma 8.4.2 and Lemma 8.4.3. A, B cannot be free since they have fewer elements than R.