University of Warwick

MA3B8 Complex Analysis

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INDEX

1. Differentiability and the Cauchy-Riemann Equations

Definitions. A *domain* $\mathcal{D} \subseteq \mathbb{C}$ is an open subset of the complex plane. I.e.,

$$\forall z_0 \in \mathcal{D} \;\; \exists \varepsilon > 0 \text{ such that } \left\{ z \in \mathbb{C} \;\middle|\; |z - z_0| < \varepsilon \right\} \subseteq \mathcal{D}.$$

We denote $\left\{ z \in \mathbb{C} \mid |z - z_0| < \varepsilon \right\}$ by $B_\varepsilon(z_0)$ and call it the ε*-ball* centred at z_0.

Definitions. A function $f : \mathcal{D} \to \mathbb{C}$ is called *differentiable* (or *holomorphic*) at $z_0 \in \mathcal{D}$ if

$$\lim_{\delta \to 0} \tfrac{f(z_0+\delta) - f(z_0)}{\delta}$$

exists, in which case the limit is called the *derivative* of f at z_0, denoted $f'(z_0)$.

Exercises. Show that if $f(z) = u(z) + iv(z) = u(x,y) + iv(x,y)$
(1)

$$\lim_{h \to 0 \text{ in } \mathbb{R}} \tfrac{f(z_0+h) - f(z_0)}{h} = \tfrac{\partial u}{\partial x}(x_0, y_0) + i\,\tfrac{\partial v}{\partial x}(x_0, y_0)$$

(2)

$$\lim_{k \to 0 \text{ in } \mathbb{R}} \tfrac{f(z_0+ik) - f(z_0)}{ik} = \tfrac{\partial v}{\partial y}(x_0, y_0) - i\,\tfrac{\partial u}{\partial y}(x_0, y_0)$$

and hence that the *Cauchy-Riemann equations* hold at z_0:

$$\tfrac{\partial u}{\partial x} = \tfrac{\partial v}{\partial y} \text{ and } \tfrac{\partial u}{\partial y} = -\tfrac{\partial v}{\partial x}$$

Theorem 1.1. *Let $f = u + iv : \mathcal{D} \to \mathbb{C}$. Suppose that*
(1) $\tfrac{\partial u}{\partial x}, \tfrac{\partial u}{\partial y}, \tfrac{\partial v}{\partial x}, \tfrac{\partial v}{\partial y}$ all exist in some neighborhood of $(x_0, y_0) \in \mathcal{D}$ and are continuous at (x_0, y_0);
(2) u, v satisfy the Cauchy-Riemann equations.
Then f is differentiable at (x_0, y_0).

Later we will see that if is (once) differentiable then it is differentiable infinitely many times.

Proposition 1.2. *If $f = u + iv$ is holomorphic then u, v are harmonic functions.*

Proof.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial^2 v}{\partial x \partial y}$$
$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = -\frac{\partial^2 v}{\partial y \partial x}$$

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The equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ is *Laplace's equation* for $f : \mathbb{R}^2 \to \mathbb{R}$.

We can perform an identification of $\mathbb{C}$ with $\mathbb{R}^2$ so that

$$f : \underset{\mathbb{R}^2}{\underset{\cup}{\mathcal{D}}} \to \underset{\mathbb{R}^2}{\underset{\parallel}{\mathbb{C}}}$$

Recall. Let $F : \mathcal{D} \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ and $(x_0, y_0) \in \mathcal{D}$. We say F is *differentiable* at (x_0, y_0) is there exists a linear function $L_{(x_0, y_0)} : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\lim_{(h,k)\to 0} \frac{\left\| F(x_0 + h, y_0 + k) - F(x_0, y_0) - L_{(x_0, y_0)}(h,k) \right\|}{\left\| (h,k) \right\|} = 0.$$

$L_{(x_0, y_0)}$ is called the *derivative* of F at (x_0, y_0), $dF_{(x_0, y_0)}$.

If $F = (F_1, F_2)$ is differentiable at (x_0, y_0) then $\frac{\partial F_1}{\partial x}, \frac{\partial F_1}{\partial y}, \frac{\partial F_2}{\partial x}, \frac{\partial F_2}{\partial y}$ exist at (x_0, y_0) and

$$L_{(x_0, y_0)} = \left. \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix} \right|_{(x_0, y_0)}$$

If for $F = (F_1, F_2) : \mathcal{D} \to \mathbb{R}^2$ the partial derivatives $\frac{\partial F_1}{\partial x}, \frac{\partial F_1}{\partial y}, \frac{\partial F_2}{\partial x}, \frac{\partial F_2}{\partial y}$ exist in a neighbourhood of (x_0, y_0) and are continuous at (x_0, y_0) then F is differentiable at (x_0, y_0).

If $f : \mathcal{D} \to \mathbb{C}$, $f = u + iv$, is holomorphic then consider the corresponding $\mathbb{R}^2$-valued function; it derivative matrix will be of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Given $a + ib$ we have a map $\mathbb{C} \to \mathbb{C} : z \mapsto (a + ib)z$ and a linear map $\mathbb{R}^2 \to \mathbb{R}^2$: $(x, y) \mapsto (ax - by, bx + ay)$.

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Examples. Power series such as

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \,,\ a_n \in \mathbb{C}\,,$$
$$g(z) = \sum_{n=1}^{\infty} b_n (z - z_0)^n \,,\ z_0, b_n \in \mathbb{C}\,,$$

are holomorphic functions.

Theorem 1.3. *For any power series* $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ $\exists R \in [0, \infty]$ *called the* radius of convergence *such that*

(1) $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ *converges absolutely for* $|z - z_0| < R$*;*

(2) $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ *diverges for* $|z - z_0| > R$*.*

$$\begin{aligned} R &= 1 \Big/ \limsup_{n \to \infty} |a_n|^{1/n} \\ &= 1 \Big/ \lim_{n \to \infty} \left(|a_{n+1}| / |a_n| \right) \end{aligned}$$

if the limit exists.

Theorem 1.4. *Let* $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ *have radius of convergence* $R > 0$*. Then* f *is holomorphic on* $B_R(z_0)$ *with derivative* $f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$*, and this derived series has radius of convergence* R*.*

Example. The real function $f(x) = e^x$ has the Taylor expansion

$$e^x = 1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \ldots + \tfrac{x^n}{n!} + \ldots$$

and $R = \infty$, so we define

$$e^z = 1 + z + \tfrac{z^2}{2!} + \tfrac{z^3}{3!} + \ldots + \tfrac{z^n}{n!} + \ldots$$

Note that for $y \in \mathbb{R}$,

$$\begin{aligned} e^{iy} &= 1 + (iy) + \tfrac{(iy)^2}{2!} + \tfrac{(iy)^3}{3!} + \dots \\ &= \left(1 - \tfrac{y^2}{2!} + \tfrac{y^4}{4!} - \dots\right) + i\left(y - \tfrac{y^3}{3!} + \tfrac{y^5}{5!} - \dots\right) \\ &= \cos y + i \sin y \end{aligned}$$

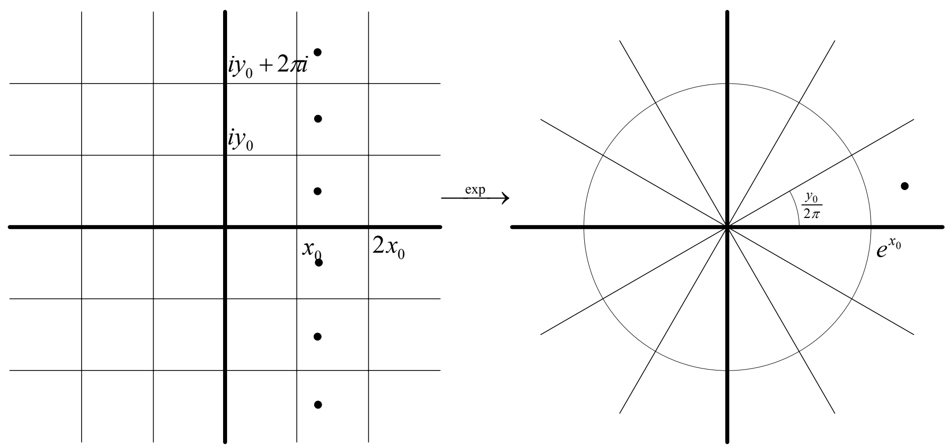
Also,

$$e^{z+w} = e^z e^w,$$
$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

We would like to define $z \mapsto \log z$ as the inverse of $z \mapsto e^z$. However, there is a problem, since $e^{z+2\pi in} = e^z$ for $n \in \mathbb{Z}$. $e^z = e^w \Rightarrow e^{z-w} = 1 \Rightarrow z - w = 2\pi in$ for some $n \in \mathbb{Z}$.

Definition. The multi-valued function $h : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ that assigns to each w the values $h(w)$ such that $e^{h(w)} = w$ is the *logarithm* function, log. It is well-defined up to multiples of $2\pi i$

How does the exponential function transform the complex plane?



We can find, for each $w \in \mathbb{C} \setminus \{0\}$, a z such that $e^z = w$ and $y_0 \le \operatorname{Im} z < y_0 + 2\pi$. This is called *choosing a branch of* $\log$; the usual choice is $-\pi$ to π.

The function on $\mathbb{C} \setminus \{ x \in \mathbb{R} \mid x \le 0 \}$ that gives the unique $z \in \mathbb{C}$ such that $e^z = w$ and $-\pi < \operatorname{Im} z < \pi$ is called the *principal branch of* $\log$ and is denoted Log.

If $w = re^{i\theta}$ where $-\pi < \theta < \pi$ then $\operatorname{Log} w = \log r + i\theta$, where $\log r$ is the usual real logarithm of r.

Exercise. Show $\operatorname{Log}:\mathbb{C}\setminus\{x\in\mathbb{R}\mid x\le 0\}\to\mathbb{C}$ is holomorphic. (Hint: use the relations $\operatorname{Re}\operatorname{Log}(x+iy)=\log\sqrt{x^2+y^2}$, $\operatorname{Im}\operatorname{Log}(x+iy)=\arctan\left(\frac{y}{x}\right)$.)

The problem of finding a maximal domain on which a given holomorphic function can be defined leads to the study of *Riemann surfaces*.

A distinguishing property of a domain is whether or not it has a "hole".

Examples. These domains have "holes":
(1) $\mathbb{C}\setminus 0$;
(2) $A_{r_1,r_2}(z)=\{w\in\mathbb{C}\mid r_1<|z-w|<r_2\}$;
(3) $B_2(0)\setminus\overline{B_{1/2}(i)}$.

Examples. These domains have no "holes":
(1) $D=B_1(0)$;
(2) $\mathbb{C}$.

Definition. An open set $\mathcal{D}$ is called *simply connected* if it has no "holes" and *multiply connected* if it has at least one hole. (More precisely, $\mathcal{D}$ is simply connected if $\pi_1(\mathcal{D},z)=\{0\}$ for all $z\in\mathcal{D}$ – see *MA3F1 Introduction to Topology* for an explanation of the fundamental group.)

Definition. $\mathcal{D}\subseteq\mathbb{C}$ is *connected* if it cannot be expressed as the disjoint union of two non-empty open proper subsets.

Definition. $\mathcal{D}\subseteq\mathbb{C}$ is *path connected* if any two points $z,z'\in\mathcal{D}$ can be joined by a path in $\mathcal{D}$, a continuous $\gamma:[a,b]\to\mathcal{D}$ such that $\gamma(a)=z$, $\gamma(b)=z'$.

Definition. $\mathcal{D}\subseteq\mathbb{C}$ is *step path connected* if any two points $z,z'\in\mathcal{D}$ can be joined by a step path in $\mathcal{D}$, a path consisting of a finite number of pieces, each of which is parallel to either the real or imaginary axis.

Proposition 1.5. *A domain $\mathcal{D}\subseteq\mathbb{C}$ is path connected if and only if it is step path connected.*

Proof. ($\Leftarrow$) A step path is a path, so this direction is trivial.

($\Rightarrow$) Suppose $\mathcal{D}$ is path connected and let $z,z'\in\mathcal{D}$. We require a step path from z to z'. Let $\gamma:[0,1]\to\mathcal{D}$ be a path from z to z'. For each $t\in[0,1]$ choose a ball $B_{\varepsilon(t)}(\gamma(t))\subseteq\mathcal{D}$. $\bigcup_{t\in[0,1]}B_{\varepsilon(t)}(\gamma(t))$ is a cover of the curve in $\mathcal{D}$. $[0,1]$ is compact, so $\gamma([0,1])$ is compact, so this cover admits a finite subcover $\bigcup_{k=1}^{n}B_{\varepsilon(t_k)}(\gamma(t_k))$. We can choose a sequence of points (s_i),

$$t_1 = s_1 < s_2 < \ldots < s_{n-1} < s_n = t_n$$

such that $\gamma([s_i, s_{i+1}]) \subseteq B_{\varepsilon(t_i)}(\gamma(t_i))$. We can replace $\gamma([s_i, s_{i+1}])$ with a step path in $B_{\varepsilon(t_i)}(\gamma(t_i))$ (since balls are clearly step path connected) to obtain a step path in $\mathcal{D}$.

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2. COMPLEX CONTOUR INTEGRATION

Definition. Let $f:\mathcal{D}\to\mathbb{C}$ be a continuous function and let $\gamma:[a,b]\to\mathcal{D}$ be a (piecewise) C^1 path. Then we define the *contour integral* of f over γ by

$$\int_\gamma f=\int_\gamma f(z)dz=\int_a^b f(\gamma(t))\gamma'(t)dt$$

where

$$\gamma(t)=x(t)+iy(t)\Rightarrow\gamma'(t)=x'(t)+iy'(t),$$
$$\int_a^b u(t)+iv(t)dt=\int_a^b u(t)dt+i\int_a^b v(t)dt\,.$$

Examples. (1) $f(z)=\frac{1}{z}$, $\gamma:[0,2\pi]\to\mathbb{C}:t\mapsto r_0e^{it}$.

$$\begin{aligned}\int_\gamma f&=\int_\gamma \tfrac{1}{z}dz\\&=\int_0^{2\pi}\tfrac{\gamma'(t)}{\gamma(t)}dt\\&=\int_0^{2\pi}\tfrac{ir_0e^{it}}{r_0e^{it}}dt\\&=2\pi i\end{aligned}$$

(2) $f(z)=|z|^2$, $\gamma_1(t)=(1+i)t$, $t\in[0,1]$.

$$\begin{aligned}\int_{\gamma_1} f&=\int_0^1|(1+i)t|^2(1+i)dt\\&=|1+i|^2(1+i)\int_0^1 t^2\,dt\\&=\tfrac{2}{3}(1+i)\end{aligned}$$

$\gamma_2(t)=t+it^2$, $t\in[0,1]$.

$$\begin{aligned}\int_{\gamma_2} f&=\int_0^1(t^2+t^4)(1+2ti)dt\\&=\int_0^1 t^2+t^4\,dt+i\int_0^1 2t^3+2t^5\,dt\\&=\left(\tfrac{1}{3}+\tfrac{1}{5}\right)+i\left(\tfrac{2}{4}+\tfrac{2}{6}\right)\\&=\tfrac{8}{15}+i\tfrac{5}{6}\end{aligned}$$

Definition. If γ is a "sum of curves" $\gamma=\gamma_1+\gamma_2+\ldots+\gamma_n$ then

$$\int_\gamma f = \sum_{k=1}^n \int_{\gamma_k} f .$$

Theorem 2.1. (The Fundamental Theorem of Contour Integrals) *Let $f:\mathcal{D}\to\mathbb{C}$ be continuous. Then the following are equivalent:*
(1) f has an anti-derivative on $\mathcal{D}$, i.e. $\exists F:\mathcal{D}\to\mathbb{C}$ such that $F'=f$;
(2) $\int_\gamma f$ is dependent only on the endpoints of γ.

Proof. $(1)\Rightarrow(2)$: Suppose $F'=f$. Choose $z_0, z_1 \in \mathcal{D}$ and $\gamma:[a,b]\to\mathcal{D}$ a path joining them.

$$\begin{aligned}\int_\gamma f &= \int_a^b f(\gamma(t))\gamma'(t)dt \\ &= \int_a^b F'(\gamma(t))\gamma'(t)dt\end{aligned}$$

By the Chain Rule,

$$\begin{aligned}\int_\gamma f &= \int_a^b \tfrac{d}{dt} F(\gamma(t))dt \\ &= F(\gamma(b)) - F(\gamma(a)) \\ &= F(z_1) - f(z_0)\end{aligned}$$

and this depends only on z_0 and z_1, not γ.

$(2)\Rightarrow(1)$: Choose $z_0\in\mathcal{D}$. Define $F(z)=\int_{z_0}^{z} f(w)dw$. Choose δ_0 such that $B_{\delta_0}(z)\subseteq\mathcal{D}$. Then for $\delta<\delta_0$, take $\gamma(t)=z+\delta t$, $t\in[0,1]$:

$$\begin{aligned}\tfrac{F(z+\delta)-F(z)}{\delta} &= \tfrac{1}{\delta}\left(\int_{z_0}^{z} f + \int_{z}^{z+\delta} f - \int_{z_0}^{z} f\right) \\ &= \tfrac{1}{\delta}\int_z^{z+\delta} f \\ &= \tfrac{1}{\delta}\int_\gamma f(\gamma(t))\gamma'(t)dt \\ &= \tfrac{1}{\delta}\int_0^1 f(z+\delta t)\delta\, dt \\ &= \int_0^1 f(z+\delta t)dt\end{aligned}$$

Now use continuity to obtain

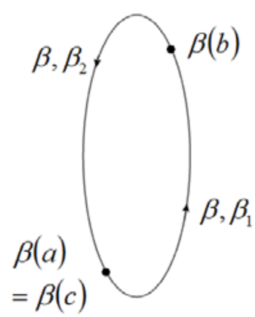
$$\lim_{\delta\to 0}\frac{F(z+\delta)-F(z)}{\delta}=\int_0^1 f(z)\,dt$$
$$=f(z)$$

■

Corollary 2.2. $f(z)=\frac{1}{z}$ *does not have an anti-derivative on* $\mathbb{C}\setminus\{0\}$.

Proposition 2.3. *Let* $f:\mathcal{D}\to\mathbb{C}$ *be continuous. Then the following are equivalent:*
(1) $\int_\beta f=0$ *for every closed curve (contour)* β*;*
(2) $\int_\gamma f$ *depends only upon the endpoints of* γ, γ *a path.*

Proof. $(2)\Rightarrow(1)$: Let $\beta:[a,c]\to\mathcal{D}$ be a closed curve. We require $\int_\beta f=0$. Let $b\in(a,c)$. $\beta(b)$



Let $\beta_1=\beta|_{[a,b]}$ and $\beta_2=\beta|_{[b,c]}$ be the two resulting paths.

$$\int_\beta f=\int_{\beta_1}f+\int_{\beta_2}f$$

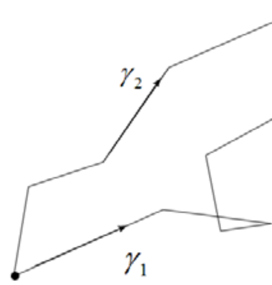
Let $-\beta_2$ be the curve β_2 traversed in the reverse direction: $-\beta_2:[0,1]\to\mathcal{D}$, $-\beta_2(t)=\beta_2(c+t(b-c))$.

$$\int_{-\beta_2}f=\int_0^1 f(\beta_2(c+t(b-c)))\beta_2'(c+t(b-c))(b-c)\,dt$$
$$=\int_0^1 f(\beta_2(u))\beta_2'(u)\frac{du}{dt}\,dt$$
$$=\int_c^b f(\beta_2(u))\beta_2'(u)\,du$$
$$=-\int_b^c f(\beta_2(u))\beta_2'(u)\,du$$
$$=-\int_{\beta_2}f$$

Hence,

$$\begin{aligned}\int_\beta f &= \int_{\beta_1} f + \int_{\beta_2} f \\ &= \int_{\beta_1} f - \int_{-\beta_2} f \\ &= 0\end{aligned}$$

$(1) \Rightarrow (2)$: Suppose γ_1, γ_2 are two paths in $\mathcal{D}$ starting and finishing at the same points respectively.



Let β be the path γ_1 then $-\gamma_2$.

$$0 = \int_\beta f = \int_{\gamma_1} f + \int_{-\gamma_2} f = \int_{\gamma_1} f - \int_{\gamma_2} f .$$

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Look at $f(z) = \frac{1}{z}$; this has no anti-derivative on $\mathbb{C} \setminus \{0\}$. Let $F(z) = \operatorname{Log} z$; F is defined on $\mathbb{C} \setminus \{ x \in \mathbb{R} \mid x \le 0 \}$.

Exercise. Show $\operatorname{Log}'(z) = \frac{1}{z}$.

The difference lies in the existence of holes as opposed to simple connectedness.

Ultimately, we will prove Cauchy's Theorem:

Cauchy's Theorem. *If $f : \mathcal{U} \to \mathbb{C}$ is holomorphic on a simply connected domain $\mathcal{U}$ then $\int_\gamma f = 0$ for every closed curve γ in $\mathcal{U}$.*

Definition. Let $\gamma : [a,b] \to \mathbb{C}$ be a closed curve not passing through $0 \in \mathbb{C}$. Then the *winding number* of γ around 0 is the (integer) number of times that γ winds around 0 in the counter-clockwise direction.

$$w(\gamma,0)=\frac{1}{2\pi i}\int_\gamma \frac{1}{z}\,dz\ .$$

Definition. Let $\gamma:[a,b]\to\mathbb{C}$ be a closed curve not passing through $z_0\in\mathbb{C}$. Then the *winding number* of γ around z_0 is

$$w(\gamma,z_0)=\frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_0}\,dz\ .$$

We could write $\gamma(t)=z_0+r(t)e^{i\theta(t)}$, where $\theta(t)$ is a continuous choice of argument. Then

$$\begin{aligned}
\frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_0} &= \frac{1}{2\pi i}\int_a^b \frac{\left(r(t)e^{i\theta(t)}\right)'}{r(t)e^{i\theta(t)}}\,dt \\
&= \frac{1}{2\pi i}\int_a^b \frac{r'(t)e^{i\theta(t)}}{r(t)e^{i\theta(t)}} + i\,\frac{r(t)\theta'(t)e^{i\theta(t)}}{r(t)e^{i\theta(t)}}\,dt \\
&= \frac{1}{2\pi i}\int_a^b \frac{r'(t)}{r(t)} + i\theta'(t)\,dt \\
&= \frac{1}{2\pi i}\left(\log r(b)-\log r(a)+i\theta(b)-i\theta(a)\right) \\
&= \frac{1}{2\pi}\left(\theta(b)-\theta(a)\right) \\
&= \text{change in angle}/2\pi
\end{aligned}$$

Definition. Let $\mathcal{U}\subseteq\mathbb{C}$ be connected. $\mathcal{U}$ is *simply connected* if $w(\gamma,z_0)=0$ for all closed curves γ in $\mathcal{U}$ and $z_0\notin\mathcal{U}$.

Theorem 2.4. *Suppose $\mathcal{U}\subseteq\mathbb{C}$ is simply connected and γ is a triangular path in $\mathcal{U}$. If $f:\mathcal{U}\to\mathbb{C}$ is holomorphic then $\int_\gamma f=0$.*

Lemma 2.5. *Let $f:\mathcal{D}\to\mathbb{C}$ be continuous and γ a path in $\mathcal{D}$. If $|f(\gamma(t))|\le M$ then $\left|\int_\gamma f\right|\le M\ell(\gamma)$, where $\ell(\gamma)$ is the length of the path γ.*

Proof of 2.5.

$$\begin{aligned}
\left|\int_\gamma f\right| &= \left|\int_a^b f(\gamma(t))\gamma'(t)\,dt\right| \\
&\le \int_a^b |f(\gamma(t))||\gamma'(t)|\,dt \\
&\le M\int_a^b |\gamma'(t)|\,dt \\
&= M\ell(\gamma)
\end{aligned}$$

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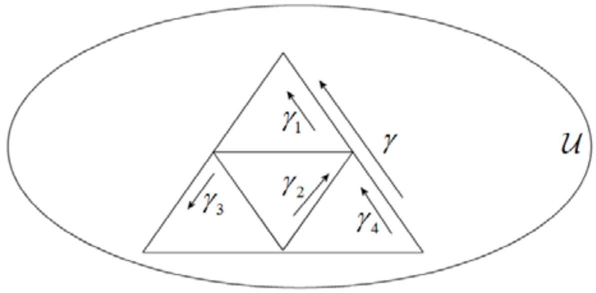
Proof of 2.4. By taking midpoints of the edges we have four new paths:



$$\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$$
$$\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f + \int_{\gamma_3} f + \int_{\gamma_4} f$$

We show $\int_\gamma f = 0$ by showing $\left|\int_\gamma f\right| = 0$.

$$\left|\int_\gamma f\right| \le \left|\int_{\gamma_1} f\right| + \left|\int_{\gamma_2} f\right| + \left|\int_{\gamma_3} f\right| + \left|\int_{\gamma_4} f\right|$$

Choose k such that $\left|\int_{\gamma_k} f\right| \ge \frac{1}{4}\left|\int_\gamma f\right|$. Label $\gamma_k = \gamma^1$. Note that $\ell(\gamma^1) = \frac{1}{2}\ell(\gamma)$. Continue the subdivision process to find a sequence of triangular paths (γ^n) such that

$$\left|\int_{\gamma^n} f\right| \ge \left(\tfrac{1}{4}\right)^n \left|\int_\gamma f\right|,$$
$$\ell(\gamma^n) = \left(\tfrac{1}{2}\right)^n \ell(\gamma).$$

Let T^n be the triangle bounded by γ^n. Since $\ell(\gamma^n) \to 0$ we have $\bigcap_{n=1}^\infty T^n = \{z_0\}$ by Baire's Theorem. Now since f is holomorphic (at z_0), given $\varepsilon > 0$, $\exists \delta > 0$ such that for $|z - z_0| < \delta$, $\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \varepsilon$, i.e.

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| < \varepsilon |z - z_0|.$$

Since the T^n are shrinking to $\{z_0\}$, $\exists N \in \mathbb{N}$ such that $n > N \Rightarrow T^n \subseteq B_\delta(z_0)$. Hence, for $n > N$ and $z \in \gamma^n$,

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| < \varepsilon \ell(\gamma^n).$$

Thus, by Lemma 2.5,

$$\left|\int_{\gamma^n} f(z) - f(z_0) - f'(z_0)(z - z_0)\,dz\right| \leq \varepsilon \ell(\gamma^n)\ell(\gamma^n) = \varepsilon\left(\tfrac{1}{4}\right)^n \ell(\gamma)^2 .$$

Observe that $\int_{\gamma^n} - f(z_0) - f'(z_0)(z - z_0)\,dz = 0$, so

$$\left|\int_{\gamma^n} f\right| \leq \varepsilon\left(\tfrac{1}{4}\right)^n \ell(\gamma)^2$$

$$\Rightarrow \qquad \left(\tfrac{1}{4}\right)^n \left|\int_{\gamma} f\right| \leq \varepsilon\left(\tfrac{1}{4}\right)^n \ell(\gamma)^2$$

$$\Rightarrow \qquad \left|\int_{\gamma} f\right| \leq \varepsilon \ell(\gamma)^2$$

$$\Rightarrow \qquad \left|\int_{\gamma} f\right| = 0$$

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Corollary 2.6. *Let* $f : B_r(z_0) \to \mathbb{C}$ *be holomorphic. Then* f *has an anti-derivative on* $B_r(z_0)$.

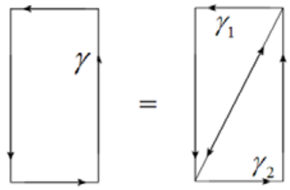
Proof. Let γ_z be the radial path in $B_r(z_0)$ from z_0 to z and define $F(z) = \int_{\gamma_z} f$. We wish to show that $F'(z) = f(z)$.

$$\begin{aligned}
\tfrac{F(z+\delta)-F(z)}{\delta} &= \tfrac{1}{\delta}\left(\int_{\gamma_{z+\delta}} f - \int_{\gamma_z} f\right) \\
&= \tfrac{1}{\delta}\left(\int_{\gamma_{z+\delta}} f + \int_{-\gamma_z} f\right) \\
&= \tfrac{1}{\delta}\left(\int_{\nabla} f + \int_{z \leftarrow z+\delta} f\right) \\
&= \tfrac{1}{\delta}\int_{z \leftarrow z+\delta} f \\
&= \tfrac{1}{\delta}\int_0^1 f(z + t\delta)\delta\, dt \\
&= \int_0^1 f(z + t\delta)\,dt \\
&\underset{\delta \to 0}{\longrightarrow} f(z)
\end{aligned}$$

■

Corollary 2.7. *If* $f : \mathcal{U} \to \mathbb{C}$ *is holomorphic on a simply connected domain* $\mathcal{U}$ *then for every rectangular path* γ *in* $\mathcal{U}$, $\int_{\gamma} f = 0$.

Proof.



$$\int_{\gamma} f = \int_{\gamma_1} f + \int_{\gamma_2} f = 0$$

■

Theorem 2.8. (Cauchy's Theorem) *If $f : \mathcal{U} \to \mathbb{C}$ is holomorphic on a simply connected domain $\mathcal{U}$ then $\int_{\gamma} f = 0$ for every closed curve γ in $\mathcal{U}$.*

Proof. Let $\gamma : [a,b] \to \mathcal{U}$ be a closed curve. Cover $\gamma([a,b])$ with discs D_i; choose a finite subcover $D_0, \ldots, D_n$ such that $\gamma([t_i, t_{i+1}]) \subseteq D_i$, where

$$a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b .$$

In D_i, let β_i be a step path such that $\beta_i(t_i) = \gamma(t_i)$, $\beta_i(t_{i+1}) = \gamma(t_{i+1})$. By Proposition 2.3, $\int_{\gamma|_{[t_i, t_{i+1}]}} f = \int_{\beta_i} f$. Let β be the path given by the β_i in order; $\beta = \beta_0 + \ldots + \beta_n$. Then

$$\int_{\beta} f = \sum_{i=0}^{n} \int_{\beta_i} f .$$

We now show that β can be written as a sum of rectangular paths. To do this, extend all horizontal and vertical segments of β to lines, thus breaking the plane up into a finite number of rectangles R_j, some of which may be infinite. In the interior of each R_j choose a point z_j and let

$$\nu_j = w(\beta, z_j).$$

Collect all R_j with $\nu_j \neq 0$ and, after re-indexing, say these are $R_1, \ldots, R_k$. Let ∂R_j be the rectangular path in the boundary of R_j traversed in the anti-clockwise direction. Define the path $\widetilde{\beta}$ by

$$\widetilde{\beta} = \sum_{j=1}^{k} \nu_j \partial R_j .$$

We claim that $\beta = \widetilde{\beta}$. Suppose not – then there is some line segment L in $\widetilde{\beta}$ and not in β. Suppose there are $\pm q$ copies of L ($L \subseteq \partial R_j$, say $+q$ copies if L is traversed in the direction coinciding with the direction of ∂R_j, $-q$ otherwise).

Now let $\alpha = \widetilde{\beta} - \beta - q\partial R_j$; the path α contains no copies of L. Hence,

$$w(\alpha, z_j) = w(\widetilde{\beta}, z_j) - w(\beta, z_j) - q = -q,$$
$$w(\alpha, z_{j'}) = w(\widetilde{\beta}, z_{j'}) - w(\beta, z_{j'}) = 0.$$

Since α contains no copies of L, $w(\alpha, z_j) = w(\alpha, z_{j'})$, so $q = 0$.

Now since $\mathcal{U}$ is simply connected and $f : \mathcal{U} \to \mathbb{C}$ is holomorphic, $\int_{\partial R_j} f = 0$ for all j, so $\int_\gamma f = 0$.

■

Theorem 2.9. *Let $f : \mathcal{D} \to \mathbb{C}$ be holomorphic and let γ be a closed curve in $\mathcal{D}$ that does not wind around any points outside of $\mathcal{D}$. Then $\int_\gamma f = 0$.*

Remark. This is a strengthening of Cauchy's Theorem.

Proof. First replace γ with a step path β as in Proposition 1.5 so that $\int_\beta f = \int_\gamma f$. Second, break up the plane into rectangles R_i and take $z_i \in R_i$. Collect the R_i for which $\nu_i = w(\beta, z_i) \neq 0$ and re-index as $R_1, \ldots, R_k$. As before, $\beta = \sum_{i=1}^k \nu_i \partial R_i$. $R_i \subseteq \mathcal{D}$ by the assumption that γ (and hence β) does not wind around any points not in $\mathcal{D}$, so $\int_{\partial R_i} f = 0$ by Corollary 2.7. Hence $\int_\gamma f = \int_\beta f = 0$.

■

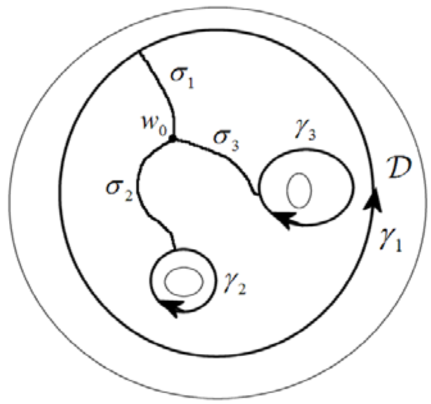
Corollary 2.10. *If $f : \mathcal{U} \to \mathbb{C}$ is holomorphic and $\mathcal{U}$ simply connected then f has an anti-derivative on $\mathcal{U}$.*

Theorem 2.11. (Generalized Cauchy Theorem) *Let $f : \mathcal{D} \to \mathbb{C}$ be holomorphic and let $\gamma = \gamma_1 + \ldots + \gamma_n$ be a sum of closed contours in $\mathcal{D}$ with*

$$w(\gamma, z) = w(\gamma_1, z) + \ldots + w(\gamma_n, z) = 0$$

for all $z \notin \mathcal{D}$. Then $\int_\gamma f = 0$.

Proof. Construct a new closed curve δ in $\mathcal{D}$ by adding segments $\pm\sigma_i$, $1 \le i \le n$. Choose $w_0 \in \mathcal{D}$ and let w_i be the starting / finishing point of γ_i. Let σ_i be a path in $\mathcal{D}$ from w_0 to w_i. Let $\delta = (\sigma_1 + \gamma_1 - \sigma_1) + \ldots + (\sigma_n + \gamma_n - \sigma_n)$, a closed curve.



For $z \notin \mathcal{D}$,

$$\begin{aligned} w(\delta, z) &= \sum_{i=1}^{n} w(\sigma_i + \gamma_i - \sigma_i, z) \\ &= \sum_{i=1}^{n} w(\gamma_i, z) \\ &= 0 \end{aligned}$$

By Cauchy's Theorem,

$$0 = \int_\delta f = \sum_{i=1}^{n} \int_{\sigma_i + \gamma_i - \sigma_i} f = \int_\gamma f .$$

■

Theorem 2.12. (Cauchy's Integral Formula) *Let* $f : \mathcal{D} \to \mathbb{C}$ *be holomorphic and* $z_0 \in D$. *Choose* $R > 0$ *such that* $\overline{B_R(z_0)} \subseteq \mathcal{D}$ *and let* γ *be a closed contour in* $B_R(z_0)$ *with* $w(\gamma, z_0) = 1$. *Then*

$$f(z_0) = \tfrac{1}{2\pi i} \int_\gamma \tfrac{f(z)}{z - z_0} dz .$$

Proof. $\frac{f(z)}{z-z_0}$ is holomorphic on $\mathcal{D} \setminus \{z_0\}$.

$$\tfrac{1}{2\pi i} \int_\gamma \tfrac{f(z)}{z - z_0} dz = \tfrac{1}{2\pi i} \int_{\partial B_r(z_0)} \tfrac{f(z)}{z - z_0} dz = f(z_0),$$

by Theorem 2.11, where $0 < r < R$.

■

Definition. A holomorphic function defined on all of $\mathbb{C}$ is called *entire*.

Theorem 2.13. (Liouville's Theorem) *Let $f:\mathbb{C}\to\mathbb{C}$ be a bounded entire function. Then f is constant.*

Proof. Suppose f is entire and bounded; let $M>0$ be such that $\forall z\in\mathbb{C}$, $|f(z)|\le M$. We show that $f(z)\equiv f(0)$ by Cauchy's Integral Formula:

$$f(z)=\tfrac{1}{2\pi i}\int_{\partial B_R(0)}\tfrac{f(w)}{w-z}\,dw$$
$$f(0)=\tfrac{1}{2\pi i}\int_{\partial B_R(0)}\tfrac{f(w)}{w}\,dw$$

So,

$$\begin{aligned}
f(z)-f(0)&=\tfrac{1}{2\pi i}\int_{\partial B_R(0)}\tfrac{f(w)}{w-z}-\tfrac{f(w)}{w}\,dw\\
&=\tfrac{1}{2\pi i}\int_{\partial B_R(0)}\tfrac{zf(w)}{(w-z)w}\,dw\\
|f(z)-f(0)|&\le\tfrac{1}{2\pi}\tfrac{|z|M}{R(R-|z|)}\ell(\partial B_R(0))\\
&=\tfrac{M|z|}{R-|z|}\\
&\underset{R\to\infty}{\longrightarrow}0
\end{aligned}$$

So $f(z)\equiv f(0)$.

■

Theorem 2.14. (The Fundamental Theorem of Algebra) *Let $p:\mathbb{C}\to\mathbb{C}$ be a non-constant polynomial with coefficients in $\mathbb{C}$. Then $\exists z_0\in\mathbb{C}$ such that $p(z_0)=0$.*

Proof. Suppose $\forall z\in\mathbb{C}$, $p(z)\ne 0$. Then $\frac{1}{p(z)}$ is entire. $|p(z)|\underset{|z|\to\infty}{\longrightarrow}\infty$, so $\exists K>0$ such that $|z|>K\Rightarrow\left|\frac{1}{p(z)}\right|<1$. Moreover, since $\frac{1}{p}$ is continuous on $\overline{B_K(0)}$, $\exists M>0$ such that $\left|\frac{1}{p(z)}\right|\le M$ for $z\in\overline{B_K(0)}$. So $\frac{1}{p}$ is bounded on $\mathbb{C}$. Liouville's Theorem implies that $\frac{1}{p}$ is a constant, so p is a constant, a contradiction.

■

Theorem 2.15. (Gauss' Mean Value Theorem) *Suppose $f:B_R(z_0)\to\mathbb{C}$ is holomorphic. Then for $0<r<R$,*

$$f(z_0) = \tfrac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + re^{it}\right)dt\,.$$

Proof. This follows from Cauchy's Integral Formula:

$$\begin{aligned} f(z_0) &= \tfrac{1}{2\pi i}\int_{\partial B_r(z_0)} \tfrac{f(z)}{z-z_0}\,dz \\ &= \tfrac{1}{2\pi i}\int_0^{2\pi} \tfrac{f\left(z_0+re^{it}\right)}{re^{it}}\,ire^{it}\,dt \\ &= \tfrac{1}{2\pi}\int_0^{2\pi} f\left(z_0+re^{it}\right)dt \end{aligned}$$

■

Theorem 2.16. (Maximum Modulus Principle) *Let $f : \mathcal{D} \to \mathbb{C}$ be holomorphic and non-constant on a connected domain $\mathcal{D}$. Then $|f|$ cannot attain a maximum in (the interior of) $\mathcal{D}$.*

Proof. Suppose not – suppose $|f|$ attains a maximum at $z_0 \in \mathcal{D}$, i.e. $|f(z_0)| \geq |f(z)|$ for all $z \in \mathcal{D}$. Choose $R > 0$ such that $\overline{B_R(z_0)} \subseteq \mathcal{D}$. By Gauss' Mean Value Theorem, for all $0 < r < R$,

$$f(z_0) = \tfrac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + re^{it}\right)dt$$

$$\Rightarrow \qquad |f(z_0)| \leq \tfrac{1}{2\pi}\int_0^{2\pi} \left|f\left(z_0 + re^{it}\right)\right|dt \leq |f(z_0)|$$

$$\Rightarrow \qquad |f(z_0)| = \tfrac{1}{2\pi}\int_0^{2\pi} \left|f\left(z_0 + re^{it}\right)\right|dt$$

It follows that $\forall t \in [0, 2\pi], r \in (0, R),\ |f(z_0)| = \left|f\left(z_0 + re^{it}\right)\right|$.

Hence, if $|f|$ is maximized at z_0 $\exists B_r(z_0) \subseteq \mathcal{D}$ such that $|f|$ is maximized on $B_r(z_0)$. Hence $\mathcal{D}' = \left\{ z \in \mathcal{D} \mid |f(z)| = |f(z_0)| \right\}$ is an open set, but since $|f|$ is continuous it is also closed. Hence, by connectedness, $\mathcal{D}' = \mathcal{D}$.

■

Theorem 2.17. *Let $f : B_R(z_0) \to \mathbb{C}$ be holomorphic. Then f has a power series expansion on $B_R(z_0)$:*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for $z \in B_R(z_0)$, and furthermore

$$a_n = \frac{1}{2\pi i}\int_{\partial B_r(z_0)} \frac{f(z)}{(z-z_0)^{n+1}}\,dz\,,$$

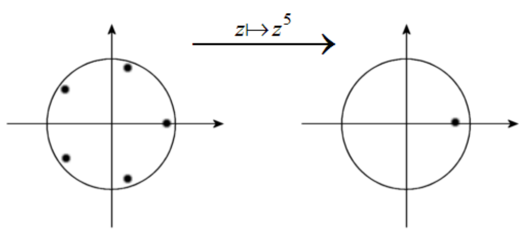
where $0 < r < R$.

Proof. (Sketch Proof.) We work from the formula $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw$. Expand $\frac{1}{w-z}$ in a power series centred at z_0:

$$\frac{1}{w-z} = \sum_{n=0}^{\infty}\left(\frac{1}{w-z_0}\right)^n (z-z_0)^n$$

Substitute into the integrand. It then remains to show that we can integrate term-by-term. ■

With this theorem in hand we can describe the local behaviour of a holomorphic map. Consider $f(z) = z^k$, $k \geq 2$.



Consider $D = B_R(0)$; f maps D to $D' = B_{R'}(0)$. For any $w \in D' \setminus \{0\}$, $f^{-1}(w)$ consists of k distinct points in D. Moreover, there is a neighbourhood U_w of w such that $f^{-1}(U_w) = \coprod_{i=1}^{k} U_i$, where the U_i are disjoint neighbourhoods of the k pre-images of w, and $f|_{U_i} : U_i \to U_w$ is a homeomorphism. We say that $f(z) = z^k$ is a k*-fold branched covering*, branched over $0 \in D$.

In fact, any holomorphic map is a k-fold branched covering:

$$f(z) - f(z_0) = \sum_{n=1}^{\infty} a_n (z-z_0)^n$$

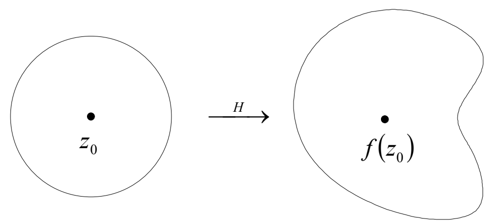
If f is non-constant $\exists n \in \mathbb{N}$ such that $a_n \neq 0$; let k be the smallest such n.

$$\begin{aligned} f(z) - f(z_0) &= a_k (z-z_0)^k + a_{k+1}(z-z_0)^{k+1} + \ldots \\ &= (z-z_0)^k \left(a_k + a_{k+1}(z-z_0) + \ldots\right) \\ &= (z-z_0)^k g(z) \end{aligned}$$

Since $g(z_0) \neq 0$, $\exists r > 0$ such that $g(z) \neq 0$ for all $z \in B_r(z_0)$. Now by Question 12 on Question Sheet 1 there is a holomorphic $h : B_r(z_0) \to \mathbb{C}$ such that $h(z)^k = g(z)$, so

$$f(z) - f(z_0) = (\underbrace{(z - z_0)h(z)}_{H(z)})^k$$

Observe that $H'(z_0) = h(z_0) \neq 0$.



By the Inverse Function Theorem, H is invertible, i.e. is a change of coordinates. So $f(z) - f(z_0)$ is a k-fold branched convering.

Theorem 2.18. (Open Mapping Theorem) *If $f : \mathcal{D} \to \mathbb{C}$ is a non-constant holomorphic map then f is open, i.e. $\forall z_0 \in \mathcal{D}$ there is an open neighbourhood U of $f(z_0)$ such that $U \subseteq f(\mathcal{D})$.*

Proof. This follows from the above local model of holomorphic maps.

■

Theorem 2.19. *The zeroes of a non-constant holomorphic function $f : \mathcal{D} \to \mathbb{C}$ are isolated, i.e. if $f(z_0) = 0$ $\exists B_R(z_0) \subseteq \mathcal{D}$ such that $f(z) \neq 0$ for $z \in B_R(z_0) \setminus \{z_0\}$.*

Proof. By the local behaviour of f we know that $\exists B_R(z_0) \subseteq \mathcal{D}$ such that

$$f|_{B_R(z_0)} : B_R(z_0) \to f(B_R(z_0))$$

is a k-fold branched covering branched over z_0, so z_0 is the only zero of f in the disc.

■

Theorem 2.20. *Let $f, g : \mathcal{D} \to \mathbb{C}$ be holomorphic. Suppose $f(z_n) = g(z_n)$ for a convergent sequence of distinct points in $\mathcal{D}$, $z_n \to \hat{z} \in \mathcal{D}$. Then $f = g$ on $\mathcal{D}$.*

Proof. Let $h = f - g$. h has a non-isolated zero at $\hat{z} \in \mathcal{D}$, and so $h = 0$ on $\mathcal{D}$. Hence $f = g$ on $\mathcal{D}$

■

Example. $\sin^2 z + \cos^2 z = 1$ for $z \in \mathbb{R}$, so $\sin^2 z + \cos^2 z = 1$ for $z \in \mathbb{C}$.

3. POLES, RESIDUES AND INTEGRALS

Definition. If $f : B_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ is holomorphic then f is said to have an *isolated singularity* at z_0.

Examples.

$$f(z) = \tfrac{1}{z}\sin z$$
$$g(z) = \tfrac{1}{z}$$
$$h(z) = e^{-1/z}$$

Theorem 3.1. (Laurent's Theorem) *Suppose f is holomorphic on the open annulus $A_{R_1,R_2}(z_0) = \{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}$, where $R_2 > R_1 \geq 0$. Then f has a Laurent series expansion on $\mathcal{A} = A_{R_1,R_2}(z_0)$:*

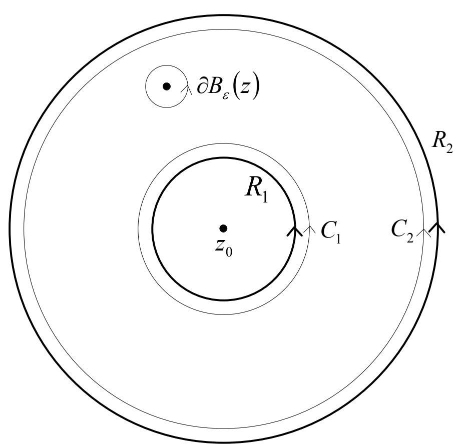
$$f(z) = \sum_{n\in\mathbb{Z}} a_n (z - z_0)^n$$

with

$$a_n = \tfrac{1}{2\pi i}\int_{\partial B_r(z_0)} \tfrac{f(w)}{(w-z_0)^{n+1}}\, dw,$$
$$R_1 < r < R_2.$$

Proof. (Sketch proof.) Use Cauchy's Integral Formula:

$$f(z) = \tfrac{1}{2\pi i}\int_{\partial B_\varepsilon(z)} \tfrac{f(w)}{w-z}\, dw$$



$$C_1(t) = z_0 + R_1' e^{it},\ t \in [0, 2\pi]$$
$$C_2(t) = z_0 + R_2' e^{it},\ t \in [0, 2\pi]$$

Observe that $w(C_2 - C_1 - \partial B_\varepsilon(z), z') = 0$ for all $z' \notin \mathcal{A} \setminus \{z\}$. By the Generalized Cauchy Theorem,

$$f(z) = \tfrac{1}{2\pi i}\int_{\partial B_\varepsilon(z)} \tfrac{f(w)}{w-z}\,dw = \tfrac{1}{2\pi i}\left(\int_{C_2} \tfrac{f(w)}{w-z}\,dw - \int_{C_1} \tfrac{f(w)}{w-z}\,dw\right)$$

For $\int_{C_2} \frac{f(w)}{w-z}\,dw$ expand $\frac{1}{w-z}$ as a power series centered about z_0:

$$\tfrac{1}{w-z} = \sum_{n=0}^{\infty} \tfrac{1}{(w-z_0)^{n+1}} (z - z_0)^n$$

since $\left|\frac{z-z_0}{w-z_0}\right| < 1$. For $\int_{C_1} \frac{f(w)}{w-z}\,dw$ expand:

$$\tfrac{1}{w-z} = \sum_{n=0}^{\infty} (w - z_0)^n \tfrac{1}{(z-z_0)^{n+1}}$$

since $\left|\frac{z-z_0}{w-z_0}\right| > 1$. Integrate term-by-term to obtain the claimed Laurent series expansion.

■

Examples. (1) $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ for $|z| < 1$. Hence,

$$\tfrac{1/z}{1/z-1} = -\tfrac{1}{z}\tfrac{1}{1-1/z} = -\tfrac{1}{z}\sum_{n=0}^{\infty}\left(\tfrac{1}{z}\right)^n \text{ for } |z| > 1 .$$

(2)

$$\begin{aligned}\tfrac{\sin z}{z} &= \tfrac{1}{z}\left(z - \tfrac{z^3}{3!} + \tfrac{z^5}{5!} - \dots\right)\\ &= 1 - \tfrac{z^2}{3!} + \tfrac{z^4}{5!} - \dots\end{aligned}$$

(3) $\frac{1}{z^k}$ is its own Laurent series for $k \geq 1$.

(4) For $|z| > 0$,

$$\begin{aligned}e^{-1/z} &= 1 + \left(-\tfrac{1}{z}\right) + \left(-\tfrac{1}{z}\right)^2/2! + \left(-\tfrac{1}{z}\right)^3/3! + \dots\\ &= \sum_{n=0}^{\infty} \tfrac{(-1)^n}{n!}\left(\tfrac{1}{z}\right)^n\end{aligned}$$

Definition. A holomorphic function $f : B_R(z_0) \setminus \{z_0\} \to \mathbb{C}$ is said to have a *removable singularity* at z_0 if f can be extended to a holomorphic function on $B_R(z_0)$.

Proposition 3.2. *Let* $f : B_R(z_0) \setminus \{z_0\} \to \mathbb{C}$ *be holomorphic. Then the singularity at* z_0 *is removable if and only if* f *is bounded on* $B_r(z_0) \setminus \{z_0\}$ *for some* $0 < r \le R$.

Proof. ($\Rightarrow$) Trivial.

($\Leftarrow$) Assume $|f|$ is bounded by M on $B_r(z_0) \setminus \{z_0\}$. We will show that in the Laurent expansion $f(z) = \sum_{n \in \mathbb{Z}} a_n (z - z_0)^n$, $a_n = 0$ for $n < 0$.

$$\begin{aligned} a_n &= \tfrac{1}{2\pi i} \int_{\partial B_r(z_0)} \tfrac{f(z)}{(z-z_0)^{n+1}} \, dz \\ &= \tfrac{1}{2\pi i} \int_{\partial B_r(z_0)} f(z)(z - z_0)^{-n-1} \, dz \end{aligned}$$

Hence, by the Estimation Lemma,

$$\begin{aligned} |a_n| &\le \tfrac{1}{2\pi} M r^{-n-1} 2\pi r \\ &= M r^{-n} \end{aligned}$$

$M r^{-n} \underset{r \to 0}{\to} 0$ so $a_n = 0$ for all negative n. So the Laurent series is, in fact, a power series,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for $z \in B_R(z_0) \setminus \{z_0\}$. By defining $f(z_0) = a_0$ we have a holomorphic extension of f to $B_R(z_0)$.

■

Proposition 3.3. *Let* $f : B_R(z_0) \setminus \{z_0\} \to \mathbb{C}$ *be holomorphic. Suppose* $\lim_{z \to z_0} f(z) = \infty$. *Then* $\exists N \in \mathbb{N}$ *such that* $n > N \Rightarrow a_{-n} = 0$.

We say that f has a *pole of order* N at z_0.

Proof. $g(z) = \frac{1}{f(z)}$ is holomorphic on $B_{R'}(z_0) \setminus \{z_0\}$ for some $0 < R' \le R$. Observe that $\lim_{z \to z_0} g(z) = 0$, hence g is bounded on $B_{R''}(z_0) \setminus \{z_0\}$, $0 < R'' \le R'$. So

$$g(z) = (z - z_0)^N \left(b_N + b_{N+1}(z - z_0) + b_{N+2}(z - z_0)^2 + \ldots \right), \; b_N \ne 0$$

Take $h(z) = b_N + b_{N+1}(z - z_0) + b_{N+2}(z - z_0)^2 + \ldots$, so

$$g(z) = (z - z_0)^N h(z).$$

Moreover, $h(z) \neq 0$ for $z \in B_{R''}(z_0)$, so

$$\begin{aligned} f(z) &= \tfrac{1}{(z-z_0)^N} \tfrac{1}{h(z)} \\ &= \tfrac{1}{(z-z_0)^N} \sum_{n=0}^{\infty} a_n (z-z_0)^n \end{aligned}$$

Note that $a_0 \neq 0$.

■

Corollary 3.4. *g has a pole of order N at z_0 if and only if $\frac{1}{g}$ has a zero of order N at z_0.*

Definition. If f has an isolated singularity at z_0 that is neither removable nor a pole then f is said to have an *essential singularity* at z_0.

Corollary 3.5. *f has an essential singularity at z_0 if and only if in the Laurent expansion of f about z_0 there are infinitely many $n \in \mathbb{N}$ such that $a_{-n} \neq 0$.*

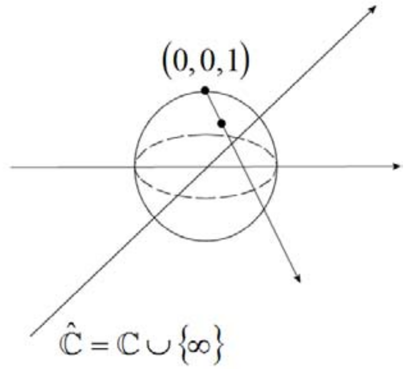
Proposition 3.6. *Suppose $f : B_R(z_0) \setminus \{z_0\} \to \mathbb{C}$ has an essential singularity at z_0. Then in any neighbourhood of z_0, f takes values arbitrarily close to any $\alpha \in \mathbb{C}$.*

Theorem 3.7. (Picard's Theorem) *Let f have an essential singularity at $z_0 \in \mathbb{C}$. Then on any small neighbourhood of z_0, f takes every value in $\mathbb{C}$, with possibly one exception.*

Although essential singularities are wildly behaved, in some sense poles are no worse than zeroes.

Idea. A holomorphic function $f : B_R(z_0) \setminus \{z_0\} \to \mathbb{C}$ whose singularity at z_0 is not essential can be extended to a holomorphic function $f : B_R(z_0) \to \mathbb{C} \cup \{\infty\}$.

We call $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the *Riemann sphere*. Topologically, $\hat{\mathbb{C}}$ is the usual two-dimensional sphere, the one-point compactification of $\mathbb{R}^2$. Complex analytically we can represent it as the unit sphere in $\mathbb{R}^3$, $S^2 = \{(u,v,w) \in \mathbb{R}^3 \mid u^2 + v^2 + w^2 = 1\}$, whose points are identified with $\hat{\mathbb{C}}$ via stereographic projection $\pi : S^2 \to \hat{\mathbb{C}}$.



For $(u,v,w) \in S^2 \setminus \{(0,0,1)\}$, $\pi(u,v,w) =$ the point of intersection of the line through $(0,0,1)$ and (u,v,w) with the (u,v)-plane. Define $\pi(0,0,1) = \infty$.

The line through $(0,0,1)$ and (u,v,w) is parameterized by $t(0,0,1) + (1-t)(u,v,w)$. The value of t corresponding to $\pi(u,v,w)$ is where $t + (1-t)w = 0$, so $t = \frac{w}{w-1}$. Hence,

$$\begin{aligned}\pi(u,v,w) &= \tfrac{w}{w-1}(0,0,1) + \tfrac{1}{1-w}(u,v,w) \\ &= \left(\tfrac{u}{1-w}, \tfrac{v}{1-w}, 0\right) \\ &= \tfrac{u}{1-w} + i\tfrac{v}{1-w}\end{aligned}$$

Consider $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $f(z) = \frac{1}{z}$. This can be considered as a map $\hat{f} : S^2 \to S^2$, where $\hat{f} = \pi^{-1} \circ f \circ \pi$. z is a coordinate on $S^2 \setminus \{(0,0,1)\}$ and $\frac{1}{z}$ is a coordinate on $S^2 \setminus \{(0,0,-1)\}$. Hence, S^2 (or $\hat{\mathbb{C}}$) can be considered as two copies of $\mathbb{C}$ with variables z and w respectively, under the identification $\frac{\mathbb{C} \amalg \mathbb{C}}{\sim}$, where $z \sim w \Leftrightarrow z, w \neq 0, z = \frac{1}{w}$.

Suppose f has a pole at z_0:

$$B_R(z_0) \xrightarrow{f} \hat{\mathbb{C}} \xrightarrow{z \mapsto \frac{1}{z}} \hat{\mathbb{C}}$$
$$z \mapsto \tfrac{1}{f(z)}, \text{ holomorphic near } z_0$$

We can consider the behaviour of a holomorphic function near ∞:

$$f : \{z \in \mathbb{C} \mid |z| > R\} \to \mathbb{C}$$
$$g(w) = f\left(\tfrac{1}{w}\right) \text{ on } \left\{w \in \mathbb{C} \mid 0 < |w| < \tfrac{1}{R}\right\}$$

We will say that f has a zero / removable singularity / pole / essential singularity at ∞ if g has a zero / removable singularity / pole / essential singularity at 0.

Examples. (1) $f(z)=\frac{1}{1-z}$; $g(w)=f\left(\frac{1}{w}\right)=\frac{1}{1-1/w}=\frac{w}{w-1}$. f has a zero of order 1 at ∞.
(2) $f(z)=z^k+1$; $g(w)=f\left(\frac{1}{w}\right)=\frac{1}{w^k}+1$; $k\in\mathbb{N}$. f has a pole of order k at ∞.
(3) $f(z)=e^z$; $g(w)=e^{1/w}$. f has an essential singularity at ∞.

Definition. Let $\mathcal{D}\subseteq\hat{\mathbb{C}}$ be a domain. A function $f:\mathcal{D}\to\hat{\mathbb{C}}$ that is holomorphic except at a finite number of poles is called *meromorphic*.

Theorem 3.8. *A meromorphic function $f:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ is a rational function, i.e. a quotient of two polynomials, $f(z)=\frac{p(z)}{q(z)}$, $p,q\in\mathbb{C}[z]$.*

Proof. Let $\{z_i\}_{i=1}^k$ be the set of poles of f in $\mathbb{C}$; say the order of the pole at z_i in n_i.

$$f(z)\prod_{i=1}^k (z-z_i)^{n_i}=g(z)$$

g is entire; we require that g be a polynomial.

Now, at ∞, since f has at worst a pole at ∞, say of order m:

$$f\left(\tfrac{1}{w}\right)\prod_{i=1}^k \left(\tfrac{1}{w}-z_i\right)^{n_i}=g\left(\tfrac{1}{w}\right)\text{ near } w=0$$

$g\left(\frac{1}{w}\right)$ has at worst a pole of order $n_1+\ldots+n_k+m$ at $w=0$.

$$g(z)=\sum_{n=0}^{\infty}a_n z^n,\ z\in\mathbb{C}$$
$$g(w)=\sum_{n=0}^{\infty}a_n\left(\tfrac{1}{w}\right)^n$$

$a_n=0$ for $n>n_1+\ldots+n_k+m$, so g is a polynomial.

$$f(z)=\frac{g(z)}{\prod_{i=1}^k (z-z_i)^{n_i}}$$

■

In the next section we consider functions $f:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ of the form

$$f(z)=\frac{\alpha z+\beta}{\gamma z+\delta},$$

the linear fractional transformations.

Definition. If $f : B_R(z_0)\setminus\{z_0\}\to\mathbb{C}$ has an isolated singularity at z_0 we define the *residue* of f at z_0 to be

$$\text{Res}(f, z_0) = \tfrac{1}{2\pi i}\int_{\partial B_r(z_o)} f(z)dz = a_{-1},\ 0 < r < R .$$

Theorem 3.9. (Cauchy's Residue Theorem) *Let* $f : \mathcal{D}\to\mathbb{C}$ *be holomorphic except possibly at isolated singularities. Let* γ *be a simple (i.e. non-self-intersecting) closed curve in* $\mathcal{D}$ *such that all points inside* γ *are contained in* $\mathcal{D}$ *and* $w(\gamma, z) = 1$ *for all* z *inside* γ*. Then*

$$\int_\gamma f = 2\pi i\sum_{j=1}^{k}\text{Res}(f, z_j),$$

where $z_1,\dots,z_k$ *are the singularities of* f *inside* γ*.*

In order to make sense of the term "inside γ" we need the Jordan Curve Theorem, which we shall not prove:

Theorem 3.10. (Jordan Curve Theorem) *If* γ *is a simple closed curve in* $\mathbb{C}$ *then* $\mathbb{C}\setminus\gamma$ *consists of two disjoint domains: a bounded domain (the "inside of* γ*") and an unbounded domain (the "outside of* γ*"), and the curve* γ *is the boundary of each of these two domains.*

Proof of 3.9. Let Γ denote the inside of γ. Choose small circular paths $\gamma_1,\dots,\gamma_k$ centred on $z_1,\dots,z_k$ respectively such that $\gamma_i\subseteq\Gamma$ and $w(\gamma_i, z_j) = \delta_{ij}$. We claim

$$\int_\gamma f = \int_{\gamma_1} f + \ldots + \int_{\gamma_k} f .$$

This follows from the Generalized Cauchy Theorem, which guarantees that

$$\int_\gamma f + \int_{-\gamma_1} f + \ldots + \int_{-\gamma_k} f = 0 .$$

for if $\mathcal{D}' =$ neighbourhood of $\gamma\cup$ neighbourhood of $\Gamma\setminus\{z_1,\dots,z_k\}$ and $z\notin\mathcal{D}'$,

$$w(\gamma-\gamma_1-\ldots-\gamma_k, z) = w(\gamma, z) - (w(\gamma_1, z) + \ldots + w(\gamma_k, z)) = 0 .$$

Hence,

$$\int_\gamma f = \sum_{j=1}^k \int_{\gamma_j} f$$
$$= 2\pi i \sum_{j=1}^k \operatorname{Res}(f, z_j)$$

■

Examples. Let $f : B_R(z_0) \setminus \{z_0\} \to \mathbb{C}$.
(1) A simple pole (a pole of order 1):

$$f(z) = \tfrac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

Write $f(z) = \frac{g(z)}{h(z)}$; at a singularity / pole, $g(z) \neq 0$, $h(z) = 0$. Look at $h'(z_0)$; if $h'(z_0) \neq 0$ then f has a simple pole at z_0.

$$\lim_{z \to z_0} (z - z_0) f(z) = a_{-1}$$

If $f = \frac{g}{h}$ then

$$\lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} \tfrac{(z - z_0) g(z)}{h(z)} = \tfrac{g(z_0)}{h'(z_0)}$$

by l'Hôpital's Rule.

(2) A pole of order k:

$$f(z) = \tfrac{a_{-k}}{(z - z_0)^k} + \dots + \tfrac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

Multiply through by $(z - z_0)^k$:

$$(z - z_0)^k f(z) = a_{-k} + a_{-(k-1)}(z - z_0) + \dots + a_{-1}(z - z_0)^{k-1} + a_0(z - z_0)^k + \dots$$

Differentiate $k - 1$ times (using Leibniz's Rule) and evaluate at $z = z_0$:

$$\tfrac{d^{k-1}}{dz^{k-1}} (z - z_0)^k f(z) = (k-1)!\, a_{-1}$$

$$\Rightarrow \qquad a_{-1} = \tfrac{1}{(k-1)!} \tfrac{d^{k-1}}{dz^{k-1}} (z - z_0)^k f(z) \Big|_{z = z_0}$$

So, for example,

$$\begin{aligned}\operatorname{Res}\left(\tfrac{(z+1)^2}{(z-1)^2},1\right) &= \tfrac{d}{dz}(z+1)^2\Big|_{z=1}\\ &= 2(z+1)\big|_{z=1}\\ &= 4\end{aligned}$$

(3) If $f(z)=\sin\left(\frac{1}{z}\right)=\frac{1}{z}-\frac{1}{3!}\left(\frac{1}{z}\right)^3+\frac{1}{5!}\left(\frac{1}{z}\right)^5-\ldots$ then we see that $\operatorname{Res}\left(\sin\frac{1}{z},0\right)=1$.

We can evaluate real integrals by means of residues.

Examples. (1) $\int_{-\infty}^{+\infty}\frac{x^2}{1+x^4}\,dx=\frac{\pi}{\sqrt{2}}$.

Consider $f(z)=\frac{z^2}{1+z^4}$. Let γ_R be the straight path $[-R,R]$ and the upper half of $\partial B_R(0)$. By the Residue Theorem, if $p_1=e^{\pi i/4}$, $p_2=e^{3\pi i/4}$

$$\int_{\gamma_R} f(z)\,dz = 2\pi i\left(\operatorname{Res}(f,p_1)+\operatorname{Res}(f,p_2)\right)$$

Write $f(z)=\frac{p(z)}{q(z)}$: $\operatorname{Res}(f,p_j)=\frac{p(p_j)}{q'(p_j)}=\frac{p_j^2}{4p_j^3}=\frac{1}{4p_j}$. So

$$\begin{aligned}\int_{\gamma_R} f(z)\,dz &= \tfrac{2\pi i}{4}\left(e^{-\pi i/4}+e^{-3\pi i/4}\right)\\ &= \tfrac{\pi}{\sqrt{2}}\end{aligned}$$

We now wish to show that the integral over the semicircular portion C_R of the path goes to zero as $R\to\infty$:

$$\begin{aligned}\left|\int_{C_R} f\right| &= \left|\int_0^{\pi} f\left(Re^{it}\right)iRe^{it}\,dt\right|\\ &= \int_0^{\pi}\left|f\left(Re^{it}\right)\right|R\,dt\end{aligned}$$

Since $\left|\frac{z^2}{1+z^4}\right|\le\frac{|z|^2}{|z|^4-1}=\frac{R^2}{R^4-1}$ on C_R,

$$\int_0^{\pi}\left|f\left(Re^{it}\right)\right|R\,dt\le\tfrac{R^3}{R^4-1}\pi\underset{R\to\infty}{\longrightarrow}0$$

So

$$\tfrac{\pi}{\sqrt{2}}=\lim_{R\to\infty}\int_{\gamma_R} f=\int_{-\infty}^{+\infty}\tfrac{x^2}{1+x^4}\,dx.$$

(2) $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

Consider the curve $\gamma_{\delta,R}$: the straight path $[-R,-\delta]$, the upper half of $\partial B_\delta(0)$, the straight path $[\delta, R]$, the upper half of $\partial B_R(0)$.

By Cauchy's Residue Theorem $\int_{\gamma_{\delta,R}} \frac{e^{iz}}{z}\,dz = 0$. First,

$$\begin{aligned}\int_{-R}^{-\delta} \frac{e^{ix}}{x}\,dx + \int_\delta^R \frac{e^{ix}}{x}\,dx &= \int_R^\delta \frac{e^{-iu}}{-u}(-du) + \int_\delta^R \frac{e^{ix}}{x}\,dx \\ &= \int_\delta^R \frac{e^{ix}-e^{-ix}}{x}\,dx \\ &= 2i\int_\delta^R \frac{\sin x}{x}\,dx \\ &\underset{\substack{\delta\to 0\\ R\to\infty}}{\to} \int_0^\infty \frac{\sin x}{x}\,dx\end{aligned}$$

Secondly, $\int_{C_R} \frac{e^{iz}}{z}\,dz \underset{R\to\infty}{\to} 0$:

$$\begin{aligned}\left|\int_{C_R} \frac{e^{iz}}{z}\,dz\right| &\le \int_0^\pi \left|\frac{e^{iRe^{it}}}{Re^{it}} iRe^{it}\right| dt \\ &= \int_0^\pi \left|ie^{iRe^{it}}\right| dt \\ &= \int_0^\pi \left|e^{iR\cos t - R\sin t}\right| dt \\ &= \int_0^\pi e^{-R\sin t}\,dt \\ &= 2\int_0^{\pi/2} e^{-R\sin t}\,dt\end{aligned}$$

Observe that $\sin t \ge \frac{2}{\pi}t$, so $-R\sin t \le -R\frac{2}{\pi}t$:

$$\begin{aligned}\left|\int_{C_R} \frac{e^{iz}}{z}\,dz\right| &\le 2\int_0^{\pi/2} e^{-2Rt/\pi}\,dt \\ &= 2\left(-\frac{\pi}{2R}\right)e^{-2Rt/\pi}\Big|_0^{\pi/2} \\ &= \frac{\pi}{R}\left(1-e^{-R}\right) \\ &\underset{R\to\infty}{\to} 0\end{aligned}$$

Thirdly,

$$
\begin{aligned}
\int_{C_\delta} \tfrac{e^{iz}}{z}\,dz &= -\int_0^\pi \tfrac{e^{i\delta e^{it}}}{\delta e^{it}}\, i\delta e^{it}\,dt \\
&= -i\int_0^\pi e^{i\delta e^{it}}\,dt \\
&\underset{\delta\to 0}{\to} -i\int_0^\pi 1\,dt \\
&= -i\pi
\end{aligned}
$$

So

$$
0 = \lim_{\substack{R\to\infty \\ \delta\to\infty}} \int_{\gamma_{\delta,R}} \tfrac{e^{iz}}{z}\,dz = 2i\int_0^\infty \tfrac{\sin x}{x}\,dx - i\pi
$$

So $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$, as claimed.

(3) $\int_0^{2\pi} \frac{1}{a+\cos\theta}\,d\theta$ for $a > 1$.

If $z = e^{i\theta}$, $dz = ie^{i\theta}\,d\theta = iz\,d\theta$ and $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$.

$$
\int_{S^1} \tfrac{1}{a+\frac{1}{2}\left(z+z^{-1}\right)}\,\tfrac{1}{iz}\,dz = \tfrac{2}{i}\int_{S^1} \tfrac{1}{z^2+2az+1}\,dz
$$

The zeroes of $z^2 + 2az + 1$ are $z = -a \pm \sqrt{a^2+1}$. $\alpha = -a + \sqrt{a^2+1}$ is inside S^1 and $\beta = -a - \sqrt{a^2-1}$ is outside S^1.

$$
\begin{aligned}
\operatorname{Res}\left(\tfrac{1}{z^2+2az+1}, \alpha\right) &= \operatorname{Res}\left(\tfrac{1}{(z-\alpha)(z-\beta)}, \alpha\right) \\
&= \tfrac{1}{\alpha-\beta} \\
&= \tfrac{1}{2\sqrt{a^2-1}}
\end{aligned}
$$

So $\int_0^{2\pi} \frac{1}{a+\cos\theta}\,d\theta = \frac{2}{i}\,2\pi i \operatorname{Res}\left(\frac{1}{z^2+2az+1}, \alpha\right) = \frac{2\pi}{\sqrt{a^2-1}}$.

(4) $\int_0^\infty \frac{\log x}{1+x^2}\,dx = 0$.

Take a branch of $\log$ such that the polar angle of z is in the interval $\theta \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$. For $z = re^{i\theta}$,

$$
\begin{aligned}
\operatorname{Re}\log z &= \log r \\
\operatorname{Im}\log z &= \theta
\end{aligned}
$$

This is possible for $z \in \mathbb{C} \setminus \{iy \mid y \leq 0\}$. Take $f(z) = \frac{\log z}{1+z^2}$ and consider $\int_{\gamma_{R,\delta}} f(z)\,dz$.

$$\int_{-R\to-\delta} f(z)\,dz = \int_{-R}^{-\delta} \frac{\log|x| + \pi i}{1+x^2}\,dx$$

$$\int_{\delta\to R} f(z)\,dz = \int_{\delta}^{R} \frac{\log x}{1+x^2}\,dx$$

Hence,

$$\int_{\substack{-R\to-\delta \\ \delta\to R}} f(z)\,dz = 2\int_{\delta}^{R} \frac{\log x}{1+x^2}\,dx + \pi i \int_{\delta}^{R} \frac{1}{1+x^2}\,dx$$

$$\underset{\substack{\delta\to 0 \\ R\to\infty}}{\longrightarrow} 2\int_{0}^{\infty} \frac{\log x}{1+x^2}\,dx + \pi i \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= 2\int_{0}^{\infty} \frac{\log x}{1+x^2}\,dx + \pi i\left(\tfrac{\pi}{2}\right)$$

We can show that $\int_{C_\delta} f(z)\,dz \underset{\delta\to 0}{\longrightarrow} 0$ and $\int_{C_R} f(z)\,dz \underset{R\to\infty}{\longrightarrow} 0$. Therefore, by Cauchy's Residue Theorem,

$$\int_{\gamma_{\delta,R}} f(z)\,dz = 2\pi i \operatorname{Res}(f, i)$$

$$= 2\pi i \lim_{z\to i} \frac{(z-i)\log z}{1+z^2}$$

$$= \frac{\log i}{i+i} 2\pi i$$

$$= \frac{\pi^2 i}{2}$$

$$\underset{\substack{\delta\to 0 \\ R\to\infty}}{\longrightarrow} 2\int_{0}^{\infty} \frac{\log x}{1+x^2}\,dx + \frac{\pi^2 i}{2}$$

So $\int_{0}^{\infty} \frac{\log x}{1+x^2}\,dx = 0$, as required.

We can also use the Residue Theorem to find sums of series.

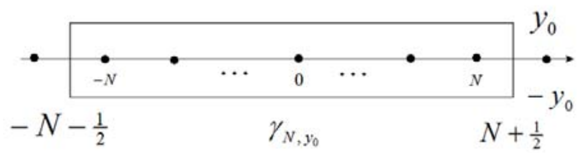
Example. $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Find a function with zeroes at integer points, e.g. $\sin \pi z$; $\frac{1}{\sin \pi z}$ has a simple pole at each $n \in \mathbb{Z}$.

$$\operatorname{Res}\left(\tfrac{1}{\sin \pi z}, n\right) = \lim_{z \to n} \tfrac{z-n}{\sin \pi z} = \tfrac{1}{\pi \cos \pi n} = \tfrac{(-1)^n}{\pi}$$

Consider $\frac{\cos \pi z}{\sin \pi z}$; this has residue $\frac{1}{\pi}$ at $z = n$. Take a holomorphic function f.

$$\operatorname{Res}\left(f(z)\tfrac{\cos \pi z}{\sin \pi z}, n\right) = \tfrac{f(z)}{\pi}$$



$$\begin{aligned}
\left|\tfrac{\cos \pi z}{\sin \pi z}\right| &= \left|\tfrac{\cos \pi(x+iy)}{\sin \pi(x+iy)}\right| \\
&= \left|\tfrac{e^{i\pi x-\pi y}+e^{-i\pi x+\pi y}}{e^{i\pi x-\pi y}-e^{-i\pi x+\pi y}}\right| \\
&= \left|\tfrac{\cos \pi x\left(e^{\pi y}+e^{-\pi y}\right)-i\sin \pi x\left(e^{\pi y}-e^{-\pi y}\right)}{-\cos \pi x\left(e^{\pi y}-e^{-\pi y}\right)+i\sin \pi x\left(e^{\pi y}+e^{-\pi y}\right)}\right| \\
&= \sqrt{\tfrac{e^{2\pi y}+e^{-2\pi y}+2\cos 2\pi x}{e^{2\pi y}+e^{-2\pi y}-2\cos 2\pi x}}
\end{aligned}$$

On vertical sides $\left|\frac{\cos \pi z}{\sin \pi z}\right| \le \sqrt{\frac{e^{2\pi y}+e^{-2\pi y}+2}{e^{2\pi y}+e^{-2\pi y}-2}} \le M$ for $y \ge y_0 > 0$. If $|f(z)| \le \frac{M'}{|z|^2}$ for some M' then

$$\left|\int_{\gamma_{N,y_0}} f(z)\cot \pi z \, dz\right| \le M \tfrac{M'}{(N+1/2)^2+y_0^2} 2(2N+1+y_0) \underset{N\to\infty}{\longrightarrow} 0 \text{ with } y_0 = N + \tfrac{1}{2}.$$

For $f(z) = z^{-2}$, since

$$\operatorname{Res}\left(z^{-2}\cot \pi z, n\right) = \begin{cases} \frac{1}{\pi n^2} & n \ne 0 \\ -\frac{\pi}{3} & n = 0 \end{cases}$$

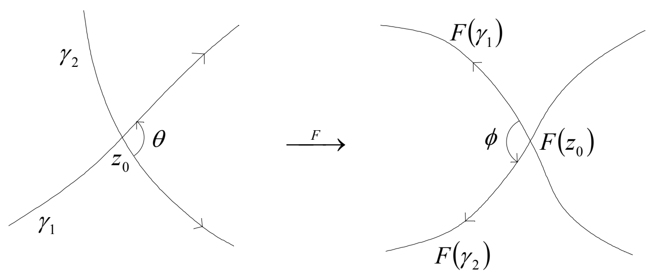
we have that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, as claimed.

4. CONFORMAL MAPS

Recall. $F:\mathbb{R}^2\to\mathbb{R}^2$ is an isometry if $\forall x,y\in\mathbb{R}^2$, $\|F(x)-F(y)\|=\|x-y\|$. The isometries of $\mathbb{R}^2$ are the rigid motions: rotations about points, reflections about lines, translations along lines, and composites of these.

We can relax the isometry condition and require only that maps preserve angles – what does this mean?

Let $F:\mathbb{R}^2\to\mathbb{R}^2$ be differentiable.



Let $\gamma_1,\gamma_2:(-\varepsilon,\varepsilon)\to\mathbb{R}^2$ be two small paths about $z_0\in\mathbb{R}^2$, so $\gamma_1(0)=\gamma_2(0)=z_0$. The directed angle θ between γ_1 and γ_2 is given by $\cos(\theta)=\gamma_1'(0)\cdot\gamma_2'(0)$. Similarly,

$$\begin{aligned}\cos(\phi)&=(F\circ\gamma_1)'(0)\cdot(F\circ\gamma_2)'(0)\\&=(dF\circ\gamma_1')(0)\cdot(dF\circ\gamma_2')(0)\end{aligned}$$

F preserves angles at z_0 if given paths γ_1,γ_2 through z_0 the angle from $F\circ\gamma_1$ to $F\circ\gamma_2$ equals the angle from γ_1 to γ_2. In the diagrams above this corresponds to having $\theta=\phi$.

Definition. $F:\mathcal{U}\subseteq\mathbb{R}^2\to\mathcal{V}\subseteq\mathbb{R}^2$ is *conformal* if it preserves angles at all points of $\mathcal{U}$.

Obviously, any isometry of $\mathbb{R}^2$ is also a conformal map (if one agrees to ignore the orientation-reversing properties of reflections).

Theorem 4.1. *Let $F:\mathcal{U}\subseteq\mathbb{R}^2\to\mathcal{V}\subseteq\mathbb{R}^2$, $F=(u,v)$, be differentiable with continuous derivatives. Then F is conformal if and only if $f=u+iv$ is holomorphic with $f'(z)\neq 0$ for all points $z\in\mathcal{U}$.*

Proof. ($\Leftarrow$) Suppose f is holomorphic at $z_0\in\mathcal{U}$ and that $f'(z_0)\neq 0$. If $\gamma:(-\varepsilon,\varepsilon)\to\mathcal{U}$ is a path through z_0 the tangent vector to $f\circ\gamma$ at $t=0$ is

$$\tfrac{d}{dt}f(\gamma(t))\Big|_{t=0}=f'(\gamma(0))\gamma'(0)=f'(z_0)\gamma'(0)$$

which is simply $\gamma'(0)$ expanded by $|f'(z_0)|$ and rotated by $\arg f'(z_0)$.

($\Rightarrow$) Suppose F is conformal;

$$dF = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

Consider

$$\begin{aligned} \mathbf{v}_\theta &= dF\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \\ &= (\cos\theta)\begin{pmatrix} u_x \\ v_x \end{pmatrix} + (\sin\theta)\begin{pmatrix} u_y \\ v_y \end{pmatrix} \\ &= (\cos\theta)\mathbf{v}_0 + (\sin\theta)\mathbf{v}_{\pi/2} \end{aligned}$$

Since F is conformal, the angle from $\mathbf{v}_0$ to $\mathbf{v}_\theta$ is θ.

$$\Rightarrow \qquad \cos\theta = \frac{\mathbf{v}_0\cdot\mathbf{v}_\theta}{\|\mathbf{v}_0\|\|\mathbf{v}_\theta\|} = \frac{(\cos\theta)\|\mathbf{v}_0\|^2}{\|\mathbf{v}_0\|\|\mathbf{v}_\theta\|} = (\cos\theta)\frac{\|\mathbf{v}_0\|}{\|\mathbf{v}_\theta\|}$$

$$\Rightarrow \qquad \|\mathbf{v}_\theta\| = \|\mathbf{v}_0\| \quad \forall\theta \neq \tfrac{\pi}{2}, \tfrac{3\pi}{2}$$

Also, the angle from $\mathbf{v}_\theta$ to $\mathbf{v}_{\pi/2}$ is $\frac{\pi}{2}-\theta$.

$$\Rightarrow \qquad \sin\theta = \frac{\mathbf{v}_{\pi/2}\cdot\mathbf{v}_\theta}{\|\mathbf{v}_{\pi/2}\|\|\mathbf{v}_\theta\|} = \frac{(\sin\theta)\|\mathbf{v}_{\pi/2}\|^2}{\|\mathbf{v}_{\pi/2}\|\|\mathbf{v}_\theta\|} = (\cos\theta)\frac{\|\mathbf{v}_{\pi/2}\|}{\|\mathbf{v}_\theta\|}$$

$$\Rightarrow \qquad \|\mathbf{v}_\theta\| = \|\mathbf{v}_{\pi/2}\| \quad \forall\theta \neq 0, \pi$$

Hence $\|\mathbf{v}_\theta\| = \|\mathbf{v}_0\|$ for all θ.

The angle from $\begin{pmatrix} u_x \\ v_x \end{pmatrix}$ to $\begin{pmatrix} u_y \\ v_y \end{pmatrix}$ is $\frac{\pi}{2}$, and by the above the two vectors have the same length.

$$u_y + iv_y = i(u_x + iv_x) = -v_x + iu_x$$

By the Cauchy-Riemann Equations $u+iv$ is holomorphic and $|f'(z_0)|^2 = u_x^2 + u_y^2 \neq 0$.

■

Definition. A map $f:\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a *linear fractional transformation* if it is of the form

$$f(z) = \frac{\alpha z + \beta}{\gamma z + \delta}, \ \alpha\delta - \beta\gamma \neq 0 .$$

Since

$$f'(z) = \frac{\alpha(\gamma z + \delta) - \gamma(\alpha z + \beta)}{(\gamma z + \delta)^2} = \frac{\alpha\delta - \beta\gamma}{(\gamma z + \delta)^2} \neq 0 ,$$

linear fractional transformations are conformal.

Examples. $f(z) = z + \beta$; $g(z) = \alpha z$; $h(z) = \frac{1}{z}$.

Proposition 4.2. *The linear fractional transformations of* $\hat{\mathbb{C}}$ *form a group* G *under composition of maps.*

Proof. Routine, with

$$f(z) = \frac{\alpha z + \beta}{\gamma z + \delta} \Rightarrow f^{-1}(z) = \frac{\delta z - \beta}{-\gamma z + \alpha} .$$

■

Proposition 4.3. *The group* G *is triply transitive. I.e., if* (z_1, z_2, z_3) *and* (w_1, w_2, w_3) *are triples of distinct points in* $\hat{\mathbb{C}}$ *then there is an* $f \in G$ *such that* $f(z_i) = w_i$ *for* $i = 1,2,3$.

Proof. We first show $\exists f \in G$ such that $(z_1, z_2, z_3) \mapsto (0,1,\infty)$:

$$f(z) = \frac{z - z_1}{z - z_3} \frac{z_2 - z_3}{z_2 - z_1} .$$

Then $g(z) = \frac{z - w_1}{z - w_3} \frac{w_2 - w_3}{w_2 - w_1}$ takes (w_1, w_2, w_3) to $(0,1,\infty)$. Hence, $h = g^{-1} \circ f$ takes (z_1, z_2, z_3) to (w_1, w_2, w_3).

■

Proposition 4.4. *A linear fractional transformation is determined by its values at three distinct points.*

Proof. For $i = 1,2,3$ let $z_i \in \hat{\mathbb{C}}$, $f(z_i) = w_i$. Let h be as in the proof of Proposition 4.3 above such that $h(z_i) = w_i$. Consider $(h^{-1} \circ f) : z_i \mapsto z_i$. Let $k = h^{-1} \circ f$, $k(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$, which fixes three distinct points.

$\frac{\alpha z + \beta}{\gamma z + \delta} = z$ is a quadratic equation in z, which cannot have three distinct roots unless $\gamma = \beta = 0$, $\alpha = \delta$, in which case $k = \mathrm{id}_{\hat{\mathbb{C}}}$ and $h = f$.

■

Theorem 4.5. *A linear fractional transformation maps circles / lines to circles / lines. (On the Riemann sphere $\hat{\mathbb{C}}$, lines and circles are the same.)*

Proof. Let f be a linear fractional transformation and let C be a circle. We wish to show that $f(C)$ is also a circle. Note that a circle C is determined by any three points through which it passes. Choose $z_1, z_2, z_3 \in C$ distinct. From the proof of Proposition 4.3 there is a unique linear fractional transformation g such that $g:(z_1, z_2, z_3) \mapsto (0,1,\infty)$. In fact, g takes C to $\mathbb{R}$.

Claim. If h is a linear fractional transformation then $h^{-1}(\mathbb{R})$ is a circle.

Proof. Write $h(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$ and find a z such that $h(z) \in \mathbb{R}$. If $h(z) \in \mathbb{R}$ then $\frac{\alpha z+\beta}{\gamma z+\delta} = \frac{\overline{\alpha z}+\overline{\beta}}{\overline{\gamma z}+\overline{\delta}}$. Hence $(\alpha z + \beta)(\overline{\gamma z} + \overline{\delta}) = (\overline{\alpha z} + \overline{\beta})(\gamma z + \delta)$ and so

$$(\alpha\overline{\gamma} - \overline{\alpha}\gamma)|z|^2 + (\alpha\overline{\delta} - \overline{\beta}\gamma)z - (\overline{\alpha}\delta - \beta\overline{\gamma})\overline{z} + \beta\overline{\delta} - \overline{\beta}\delta = 0 .$$

Case 1: if $\alpha\overline{\gamma} - \overline{\alpha}\gamma = 0$ then we obtain a line since $\alpha\delta - \beta\gamma \neq 0$.

Case 2: if $\alpha\overline{\gamma} - \overline{\alpha}\gamma \neq 0$ then the equation becomes $A|z|^2 + Bz + \overline{B}\overline{z} + C = 0$, $A = \alpha\overline{\gamma} - \overline{\alpha}\gamma$, $B = \alpha\overline{\delta} - \overline{\beta}\gamma$, $C = \beta\overline{\delta} - \overline{\beta}\delta$. Since $\alpha\delta - \beta\gamma \neq 0$, this is the equation of a circle, and the claim is proved.

By the claim, $g(C) = \mathbb{R}$. Hence $k = g^{-1} \circ f$ takes $\mathbb{R}$ to $f(C)$, which is a circle by the claim.

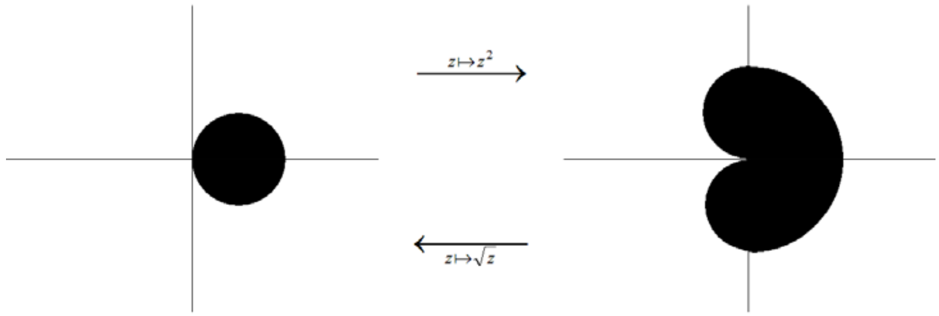
■

Although the one-to-one conformal maps from $\hat{\mathbb{C}}$ to itself are quite rigid (they are precisely the linear fractional transformations), on subdomains $\mathcal{D} \subset \mathbb{C}$ they are quite flexible. Ultimately we shall prove the Riemann Mapping Theorem:

Riemann Mapping Theorem. *If $\mathcal{U} \subset \mathbb{C}$ is a simply connected domain there exists a one-to-one conformal map $f:\mathcal{U} \to D$ onto the open unit disc. Moreover, if we fix $z_0 \in \mathcal{U}$ we can find such an $f:\mathcal{U} \to D$ such that $f(z_0) = 0$ and $f'(z_0)$ is real and strictly positive, and this uniquely determines f.*

In most cases we cannot extend f to $\overline{\mathcal{U}} = \mathcal{U} \cup \partial\mathcal{U}$.

Example.



Definition. Two domains $\mathcal{D},\mathcal{D}'\subset\mathbb{C}$ are called *conformally equivalent* if there exists a conformal bijection $\mathcal{D}\to\mathcal{D}'$.

So the Riemann Mapping Theorem tells us that any simply connected domain is conformally equivalent to the unit disc D.

Question. What are the conformal bijections $D\to D$?

Theorem 4.6. (Schwarz Lemma) *Suppose $f:D\to D$ is holomorphic and that $f(0)=0$. Then*

$$|f(z)|\le|z| \qquad (*)$$

and

$$|f'(0)|\le 1 \qquad (**)$$

If equality is achieved for some $z_0\in D$ then $f(z)=e^{i\theta_0}z$ for some fixed $\theta_0\in[0,2\pi]$.

Proof. Consider the function $g:z\mapsto\frac{f(z)}{z}$. This has a removable singularity at $z=0$. We show that $|g(z)|\le 1$.

Choose $z_0\in D$ and $|z_0|<r<1$. Now consider $|g(z_0)|$:

$$|g(z_0)|\le\max_{|z|\le r}|g(z)|=\max_{z\in\partial B_r(0)}|g(z)|$$

by the Maximum Modulus Principle. Since $|g(z)|=\left|\frac{f(z)}{z}\right|$ on $\partial B_r(0)$ we have $|g(z)|=\frac{|f(z)|}{r}\le\frac{1}{r}$, which is true for all $|z_0|<r<1$; so $|g(z)|\le 1=\lim_{r\to 1}\frac{1}{r}$. Hence $|f(z)|\le|z|$ $\forall z\in D$.

Now since $|f'(0)|=|g(0)|\le 1$ we have $(**)$.

If $|f(z_0)| = |z_0|$ for some $z_0 \in D$ then $|g(z_0)| = 1$. Since by the Maximum Modulus Principle g cannot attain maximum modulus on D unless $|g|$ is constant, we have $|g(z_0)| = 1 \Rightarrow g(z) = e^{i\theta_0}$ for some real θ_0.

Similarly, if $|f'(z_0)| = 1$, $|g(0)| = 1$, so $g(z) = e^{i\theta_0}$, so $f(z) = e^{i\theta_0} z$.

■

Theorem 4.7. *Let $f : D \to D$ be a conformal bijection. Then f is a linear fractional transformation of the form*

$$f(z) = e^{i\theta_0} \tfrac{z - z_0}{1 - \bar{z}_0 z}$$

for some $z_0 \in D$, $\theta_0 \in [0, 2\pi]$.

Proof. First check that a linear fractional transformation of the form $f(z) = e^{i\theta_0} \tfrac{z - z_0}{1 - \bar{z}_0 z}$ sends D to itself.

Consider $g = f^{-1} : D \to D$, which is also one-to-one conformal; $(g \circ f)(z) = z$, so $g'(f(z)) f'(z) = 1$. Let $z_0 \in D$ be such that $f(z_0) = 0$, so $g'(0) f'(z_0) = 1$. Estimate $|g'(0)|$: we know $g(0) = z_0$, so put

$$\tilde{g}(z) = \tfrac{z - z_0}{1 - \bar{z}_0 z}.$$

Then $\tilde{g} \circ g : D \to D$ is such that $(\tilde{g} \circ g)(0) = \tilde{g}(z_0) = 0$. So $|\tilde{g}'(z_0) g'(z_0)| \le 1$.

$$\tilde{g}'(z) = \tfrac{1 - |z_0|^2}{(1 - \bar{z}_0 z)^2}$$

$$\Rightarrow \qquad \tilde{g}'(z_0) = \tfrac{1}{1 - |z_0|^2}$$

$$\Rightarrow \qquad |g'(0)| \le 1 - |z_0|^2$$

Similarly, put $\tilde{f}$ to be the inverse of $z \mapsto \tfrac{z - z_0}{1 - \bar{z}_0 z}$,

$$\tilde{f}(z) = \tfrac{z + z_0}{1 + \bar{z}_0 z}.$$

So $(f \circ \tilde{f})(0) = 0$. Again by Schwarz's Lemma, $|f'(z_0) \tilde{f}'(0)| \le 1$. Since $\tilde{f}'(0) = 1 - |z_0|^2$, $|f'(z_0)| \le \tfrac{1}{1 - |z_0|^2}$.

Since $f'(z_0)g'(0)=1$, $|g'(0)|=\frac{1}{1-|z_0|^2}=|f'(z_0)|$,

$$\Rightarrow \qquad |(f\circ\tilde{f})'(0)|=1$$

$$\Rightarrow \qquad f\circ\tilde{f}\equiv e^{i\theta_0}$$

$$\Rightarrow \qquad f(z)=e^{i\theta_0}\,\tfrac{z-z_0}{1-\bar{z}_0 z}$$

■

The uniqueness part of the Riemann Mapping Theorem follows easily from Theorem 4.7. In the proof it is stated that if $f:D\to D$ is a conformal bijection with $f(z_0)=0$ then $f(z)=e^{i\theta_0}\,\tfrac{z-z_0}{1-\bar{z}_0 z}$ for some $\theta_0\in[0,2\pi]$. If $g:\mathcal{U}\to D$ is another conformal bijection such that $g(z_0)=0$ and $g'(z_0)>0$ then $g\circ f^{-1}:D\to D$ is a conformal bijection fixing 0. So $(g\circ f^{-1})(z)=e^{i\theta_0}z$ for some θ_0, so $g(z)=e^{i\theta_0}f(z)$. Furthermore, since $g'(z_0)=e^{i\theta_0}f'(z_0)$, $e^{i\theta_0}=1$, so $f=g$.

The existence part of the Riemann Mapping Theorem will be shown by considering the collection of functions

$$\mathcal{F}=\{\,f:\mathcal{U}\to D \mid f \text{ holomorphic}, 1\text{-}1, f(z_0)=0, f'(z_0)>0\,\}.$$

We need to show that $\exists f\in\mathcal{F}$ such that $f(\mathcal{U})=D$. To do this we need to study certain facts about spaces of continuous functions.

Let $C(\mathcal{D},\mathbb{C})$ be the collection of all continuous functions from an open domain $\mathcal{D}\subseteq\mathbb{C}$ to $\mathbb{C}$. (We can replace $\mathbb{C}$ by any complete metric space.) What does it mean for a sequence of functions f_n to converge to some other function f in $C(\mathcal{D},\mathbb{C})$? What does it mean to say that $f,g\in C(\mathcal{D},\mathbb{C})$ are close together?

We say that $f_n\to f$ *pointwise* as $n\to\infty$ if for each $z\in\mathcal{D}$,

$$f_n(z)\underset{n\to\infty}{\to} f(z).$$

However, this is not good enough as continuity is not necessarily preserved by taking a pointwise limit.

We say that $f_n\to f$ *uniformly on compact subsets* as $n\to\infty$ if $\forall\varepsilon>0$ and compact $K\subseteq\mathcal{D}$ $\exists N\in\mathbb{N}$ such that $n>N, z\in K\Rightarrow |f_n(z)-f(z)|<\varepsilon$.

If $f_n \to f$ uniformly on compact subsets and the f_n are each continuous then f is continuous.

Two functions $f, g \in C(\mathcal{D}, \mathbb{C})$ are close if they differ by a small amount on each compact $K \subseteq \mathcal{D}$.

Definition. Given a compact $K \subseteq \mathcal{D}$, $\varepsilon > 0$, $f \in C(\mathcal{D}, \mathbb{C})$ let

$$B_K(f, \varepsilon) = \left\{ g \in C(\mathcal{D}, \mathbb{C}) \;\middle|\; \sup_{z \in K} |f(z) - g(z)| < \varepsilon \right\}.$$

The sets $B_K(f, \varepsilon)$ form a basis for a topology on $C(\mathcal{D}, \mathbb{C})$, called the *topology of compact convergence*, equivalent to the compact-open topology on $C(\mathcal{D}, \mathbb{C})$.

Proposition 4.8. *$\{f_n\} \subseteq C(\mathcal{D}, \mathbb{C})$ converge to $f \in C(\mathcal{D}, \mathbb{C})$ in the topology of compact convergence if and only if $f_n \to f$ uniformly on compact subsets.*

Theorem 4.9. *Suppose $f_n : \mathcal{D} \to \mathbb{C}$ are holomorphic functions converging uniformly to some f on compact subsets. Then f is holomorphic. Moreover, $f_n' \to f'$ uniformly on compact subsets.*

Proof. We show that f is holomorphic by Morera's Theorem. Let $\overline{B_R(z_0)} \subseteq \mathcal{D}$ and let γ be a closed curve in $B_R(z_0)$. Since $\overline{B_R(z_0)}$ is compact and γ is compact, and $f_n \to f$ uniformly on compact subsets, $\int_\gamma f_n \to \int_\gamma f$. By Cauchy's Theorem, $\int_\gamma f_n \equiv 0$, so $\int_\gamma f = 0$. By Morera's Theorem, f is holomorphic on $B_R(z_0)$, so f is holomorphic on $\mathcal{D}$.

Choose $z_0 \in \mathcal{D}$ and $R > 0$ such that $\overline{B_R(z_0)} \subseteq \mathcal{D}$. Let $z \in B_R(z_0)$.

$$\begin{aligned} f_n'(z) &= \tfrac{1}{2\pi i} \int_{\partial B_R(z_0)} \tfrac{f_n(w)}{(w-z)^2}\, dw \\ &\to \tfrac{1}{2\pi i} \int_{\partial B_R(z_0)} \tfrac{f(w)}{(w-z)^2}\, dw \\ &= f'(z) \end{aligned}$$

This implies that $f_n' \to f'$ uniformly on compact subsets.

■

Theorem 4.10. (Hurwitz's Theorem) *Suppose* $f_n : \mathcal{D} \to \mathbb{C}$ *are holomorphic functions converging uniformly to some* f *on compact subsets. If each* f_n *has no zeroes in* $\mathcal{D}$ *then either* f *has no zeroes in* $\mathcal{D}$ *or* f *is identically zero on* $\mathcal{D}$.

Proof. Suppose f is not identically zero. Suppose $\exists z_0 \in \mathcal{D}$ such that $f(z_0) = 0$. Since the zeroes of a holomorphic function are isolated we can choose $\overline{B_R(z_0)} \subseteq \mathcal{D}$ such that f has no other zeroes in $B_R(z_0)$. From Question Sheet 2 the number of zeroes of f in $B_R(z_0)$ (counted with multiplicity) is $\frac{1}{2\pi i}\int_{\partial B_R(z_0)} \frac{f'}{f}$. Since $f_n \to f$ uniformly on compact subsets,

$$\frac{1}{2\pi i}\int_{\partial B_R(z_0)} \frac{f_n'}{f_n} \to \frac{1}{2\pi i}\int_{\partial B_R(z_0)} \frac{f'}{f}.$$

But $\frac{1}{2\pi i}\int_{\partial B_R(z_0)} \frac{f_n'}{f_n} = 0$ for each $n \in \mathbb{N}$ since f_n has no zeroes anywhere in $\mathcal{D}$. So either f has no zeroes in $\mathcal{D}$, or it is everywhere zero.

■

Definition. A family $\mathcal{F} \subseteq C(\mathcal{D}, \mathbb{C})$ is said to be a *normal family* if every sequence (f_n) in $\mathcal{F}$ contains a subsequence (f_{n_k}) that converges to some f uniformly on compact subsets. (It is not necessary that $f \in \mathcal{F}$.)

Remark. In fact, $C(\mathcal{D}, \mathbb{C})$ is a (complete) metric space. A normal family $\mathcal{F}$ has a (sequentially) compact closure.

Definition. A family $\mathcal{F} \subseteq C(\mathcal{D}, \mathbb{C})$ is said to be *equicontinuous* on $E \subseteq \mathcal{D}$ if for all $\varepsilon > 0$ there is a $\delta > 0$ such that $|z - w| < \delta \Rightarrow |f(z) - f(w)| < \varepsilon$ for all $f \in \mathcal{F}$ and $z, w \in \mathcal{D}$.

Theorem 4.11. (Arzela-Ascoli) *A family of functions* $\mathcal{F} \subseteq C(\mathcal{D}, \mathbb{C})$ *is normal if and only if both*
(1) $\mathcal{F}$ *is equicontinuous on every compact subset* $K \subseteq \mathcal{D}$*; and*
(2) for each $z \in \mathcal{D}$, $\{ f(z) \mid f \in \mathcal{F} \}$ *is bounded in* $\mathbb{C}$.

Proof. ($\Rightarrow$) Suppose $\mathcal{F}$ is normal. We first show that $\{ f(z) \mid f \in \mathcal{F} \}$ is bounded.

Suppose not. Then there is a sequence (f_n) in $\mathcal{F}$ such that $|f_n(z)| > n$. Since $\mathcal{F}$ is normal there is a subsequence (f_{n_k}) such that $f_{n_k} \to f$ uniformly on compact subsets. If so, $|f_{n_k}(z) - f(z)| < 1$ for large k; then $|f_{n_k}(z)| < |f(z)| + 1$ for large k, a contradiction, since a continuous function is bounded on a compact set.

We now show that if $\mathcal{F}$ is normal then $\mathcal{F}$ is equicontinuous on each compact $K \subseteq \mathcal{D}$.

Suppose not. Then there is a compact $K \subseteq \mathcal{D}$ such that $\exists \varepsilon_0 > 0$ such that $\forall n \in \mathbb{N}$ $\exists z_n, w_n \in K$ such that $|z_n - w_n| < \frac{1}{n}$ and $|f_n(z_n) - f_n(w_n)| \geq \varepsilon_0$ for $f_n \in \mathcal{F}$. Since $\mathcal{F}$ is normal, there is a subsequence (f_{n_k}) of (f_n) such that $f_{n_k} \to f$ uniformly on compact subsets. I.e., for any $\varepsilon > 0$ and compact E, $|f_{n_k}(z) - f(z)| < \varepsilon$ for all $z \in E$ and sufficiently large k. For our compact set K there exists N such that $k > N \Rightarrow$ $|f_{n_k}(z) - f(z)| < \frac{\varepsilon_0}{3}$ for all $z \in K$. Since f is continuous on $\mathcal{D}$ it is uniformly continuous on K. Hence, there is a $\delta > 0$ such that $|z - w| < \delta \Rightarrow |f(z) - f(w)| < \frac{\varepsilon_0}{3}$ for all $z, w \in K$. Then

$$\begin{aligned} \left|f_{n_k}(z_{n_k}) - f_{n_k}(w_{n_k})\right| &\leq \left|f_{n_k}(z_{n_k}) - f(z_{n_k})\right| + \left|f(z_{n_k}) - f(w_{n_k})\right| + \left|f(w_{n_k}) - f_{n_k}(w_{n_k})\right| \\ &< \tfrac{\varepsilon_0}{3} + \tfrac{\varepsilon_0}{3} + \tfrac{\varepsilon_0}{3} \\ &= \varepsilon_0, \end{aligned}$$

which contradicts $|f_n(z_n) - f_n(w_n)| \geq \varepsilon_0$ above.

($\Leftarrow$) Now suppose $\mathcal{F}$ satisfies (1) and (2); let (f_n) be a sequence in $\mathcal{F}$.

We show that (f_n) has a subsequence that converges on a dense set of points $\{z_m \mid m \in \mathbb{N}\} \subseteq \mathcal{D}$. We can always find such a countable collection, e.g. an enumeration of $(\mathbb{Q} + i\mathbb{Q}) \cap \mathcal{D}$.

For $z = z_1$, by (2), $\{f_n(z_1) \mid n \in \mathbb{N}\}$ is bounded. So there is a subsequence $(f_{n_{1l}})$ such that $f_{n_{1l}}(z_1)$ converges as $l \to \infty$. Now take a subsequence $(f_{n_{2l}})$ of $(f_{n_{1l}})$, by (2), such that $f_{n_{2l}}(z_2)$ converges as $l \to \infty$. Continue inductively, obtaining rows of subscripts

$$\begin{gathered} n_{11} < n_{12} < n_{13} < \dots \\ n_{21} < n_{22} < n_{23} < \dots \\ \vdots \\ n_{k1} < n_{k2} < n_{k3} < \dots \\ \vdots \end{gathered}$$

such that each row is a subset of the one above it, and such that $f_{n_{kl}}(z_m)$ converges as $l \to \infty$ for $m = 1, \dots, k$. Take the diagonal sequence $f_{n_l} = f_{n_{ll}}$, $l \in \mathbb{N}$. It is easy to see that for any m, $f_{n_l}(z_m)$ converges as $l \to \infty$.

We now show that for any compact $K \subseteq \mathcal{D}$ f_{n_l} is uniformly Cauchy on K, i.e. $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $i, j > N, z \in K \Rightarrow \left| f_{n_i}(z) - f_{n_j}(z) \right| < \varepsilon$. (Exercise: Check that being uniformly Cauchy on compact subsets implies uniform convergence on compact subsets.) Let $\varepsilon > 0$. Since $\mathcal{F}$ is equicontinuous on K, $\exists \delta > 0$ such that $\forall z, w \in K, f \in \mathcal{F}$, $|z - w| < \delta \Rightarrow |f(z) - f(w)| < \frac{\varepsilon}{3}$. Since K is compact we can cover K with a finite number of balls of radius $\leq \frac{\delta}{2}$. Since $\{ z_m \mid m \in \mathbb{N} \}$ is dense, we can choose a z_m in each ball, say $z_1, \ldots, z_M$. Let N be such that $i, j > N \Rightarrow \left| f_{n_i}(z_m) - f_{n_j}(z_m) \right| < \varepsilon$ for $m = 1, \ldots, M$. Then for $z \in K$,

$$\begin{aligned}
\left| f_{n_i}(z) - f_{n_j}(z) \right| &\leq \left| f_{n_i}(z) - f_{n_i}(z_m) \right| + \left| f_{n_i}(z_m) - f_{n_j}(z_m) \right| + \left| f_{n_j}(z_m) - f_{n_j}(z) \right| \\
&< \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} \\
&= \varepsilon
\end{aligned}$$

So, given (1) and (2), $\mathcal{F}$ is normal.

■

Definition. $\mathcal{F} \subseteq C(\mathcal{D}, \mathbb{C})$ is *locally uniformly bounded* on $\mathcal{D}$ if $\forall z_o \in \mathcal{D}$, $\exists B_R(z_0) \subseteq \mathcal{D}$ such that $\exists M > 0$ such that $\forall z \in B_R(z_0), f \in \mathcal{F}$, $|f(z)| < M$.

Theorem 4.12. (Montel's Theorem) *If $\mathcal{F} \subseteq C(\mathcal{D}, \mathbb{C})$ is a locally uniformly bounded family of holomorphic functions then $\mathcal{F}$ is a normal family.*

Proof. By Arzela-Ascoli we need to check
(1) $\forall z \in \mathcal{D}$, $\{ f(z) \mid f \in \mathcal{F} \}$ is bounded;
(2) $\forall$ compact $K \subseteq \mathcal{D}$, $\mathcal{F}$ is equicontinuous on K.

(1) is clear, since $\mathcal{F}$ is locally uniformly bounded.

(2) We first show that given $z_0 \in \mathcal{D}$ $\exists \overline{B_R(z_0)} \subseteq \mathcal{D}$ such that $\mathcal{F}$ is equicontinuous on $\overline{B_R(z_0)}$. Since $\mathcal{F}$ is locally uniformly bounded $\exists \overline{B_r(z_0)} \subseteq \mathcal{D}$ and $M > 0$ such that $|f| < M$ for all $f \in \mathcal{F}$. Let $\varepsilon > 0$, $w, z \in \overline{B_{r/2}(z_0)}$ and $f \in \mathcal{F}$.

$$\begin{aligned}
f(z) - f(w) &= \tfrac{1}{2\pi i} \int_{\partial B_{r/2}(z_0)} \tfrac{f(\xi)}{(\xi - z)} d\xi - \tfrac{1}{2\pi i} \int_{\partial B_{r/2}(z_0)} \tfrac{f(\xi)}{(\xi - w)} d\xi \\
&= \tfrac{1}{2\pi i} \int_{\partial B_{r/2}(z_0)} \tfrac{f(\xi)(z - w)}{(\xi - z)(\xi - w)} d\xi
\end{aligned}$$

Use the Estimation Lemma with $|\xi - z|, |\xi - w| \geq \frac{r}{2}$:

$$\left|f(z)-f(w)\right| \leq \tfrac{1}{2\pi} M \tfrac{|z-w|}{(r/2)^2} 2\pi r = \tfrac{4M}{r}|z-w|$$

Let $\delta = \frac{r\varepsilon}{4M}$. Then we have that $\mathcal{F}$ is equicontinuous on $B_{r/2}(z_0)$. Any compact $K \subseteq \mathcal{D}$ can be covered by such discs, and this cover has a finite subcover. So $\mathcal{F}$ is equicontinuous on K.

■

Theorem 4.13. (Riemann Mapping Theorem) *Given any simply connected domain $\mathcal{U} \subset \mathbb{C}$ and $z_0 \in \mathcal{U}$ there is a unique conformal bijection $f : \mathcal{U} \to D$ such that $f(z_0) = 0$ and $f'(z_0) > 0$.*

Proof. Uniqueness has already been shown. Consider

$$\mathcal{F} = \{\, f : \mathcal{U} \to D \mid f \text{ holomorphic}, 1\text{-}1, f(z_0) = 0, f'(z_0) > 0 \,\}$$

and proceed to show
(1) $\mathcal{F} \neq \varnothing$;
(2) $\exists f \in \mathcal{F}$ with maximal derivative at z_0;
(3) the f in (2) maps onto D.

Step 1. Let $a \in \mathbb{C} \setminus \mathcal{U}$. $g(z) = z - a$ is never zero on $\mathcal{U}$. g has a square root; we can find a branch of the multivalued function $\sqrt{z-a}$ that is holomorphic on $\mathcal{U}$. Set $h(z) = \sqrt{z-a}$. h is one-to-one on $\mathcal{U}$: if $\sqrt{z_1 - a} = \sqrt{z_2 - a}$ then $z_1 = z_2$.

If $h(z) = w$ for some $z \in \mathcal{U}$ then h never takes the value $-w$ in $\mathcal{U}$. Since $h : \mathcal{U} \to \mathbb{C}$ is holomorphic, it is open, by the Open Mapping Theorem. So there is a $B_R(h(z_0)) \subseteq h(\mathcal{U})$. By the above, $-B_R(h(z_0)) = B_R(-h(z_0))$ is such that $-B_R(h(z_0)) \cap h(\mathcal{U}) = \varnothing$. So $h(\mathcal{U})$ lies in the interior of $B_R(-h(z_0))$.

We can construct a linear fractional transformation k that takes $\mathbb{C} \setminus B_R(-h(z_0))$ onto $\overline{D}$. We can further compose with a linear fractional transformation of the form

$$j(z) = e^{i\theta_0} \tfrac{z-(k\circ h)(z_0)}{1-\overline{(k\circ h)(z_0)}z}.$$

Then $(j \circ k \circ h)(z_0) = 0$ and $(j \circ k \circ h)'(z_0) > 0$.

Step 2. Let $M = \sup_{f \in \mathcal{F}} f'(z_0)$. Let $f_n \in \mathcal{F}$ be such that $f_n'(z_0) \underset{n\to\infty}{\to} M$. Observe that since $\mathcal{F}$ is (locally) uniformly bounded (D is bounded), $M < \infty$ by Cauchy's Integral Formula.

By Montel's Theorem, $\mathcal{F}$ is a normal family. Hence, (f_n) has a subsequence (f_{n_k}) converging to some f uniformly on compact subsets $K \subseteq \mathcal{U}$. $f : \mathcal{U} \to \mathbb{C}$ is holomorphic, $f(z_0) = 0$ and $f'(z_0) = M$. Furthermore, since $|f_{n_k}| < 1$, $|f| < 1$, i.e. $f(\mathcal{U}) \subseteq D$.

We now show f is one-to-one. Choose $z_1 \in \mathcal{U}$; show $f(z) \neq f(z_1)$ for $z \in \mathcal{U} \setminus \{z_1\}$. Put $g_{n_k}(z) = f_{n_k}(z) - f_{n_k}(z_1)$. g_{n_k} converges uniformly to $f(z) - f(z_1)$ on compact subsets of $\mathcal{U}$, and on every compact subset of $\mathcal{U} \setminus \{z_1\}$. Since g_{n_k} is never zero on $\mathcal{U} \setminus \{z_1\}$, by Hurwitz's Theorem, either $f(z) - f(z_1)$ is never zero on $\mathcal{U} \setminus \{z_1\}$ or it is everywhere zero. But f is non-constant since $f'(z_0) = M > 0$.

Step 3. We proceed by contradiction: suppose $\exists w_0 \in D \setminus f(\mathcal{U})$. Then the linear fractional transformation $z \mapsto \frac{f(z) - w_0}{1 - \overline{w}_0 f(z)}$ is $\phi \circ f$ where $\phi : D \to D : z \mapsto \frac{z - w_0}{1 - \overline{w}_0 z}$ and $\phi \circ f$ is never zero on $\mathcal{U}$. Therefore, there exists a branch of the square root

$$F(z) = \sqrt{\frac{f(z) - w_0}{1 - \overline{w}_0 f(z)}} .$$

Note that $F(\mathcal{U}) \subseteq D$ and F is one-to-one. Further compose F with the linear fractional transformation ψ,

$$\psi(z) = \frac{z - F(z_0)}{1 - \overline{F(z_0)} z} \frac{|F'(z_0)|}{F'(z_0)} .$$

Again note that $\psi \circ F : \mathcal{U} \to D$ is one-to-one and $(\psi \circ F)(z_0) = 0$,

$$\begin{aligned}(\psi \circ F)'(z_0) &= \frac{|F'(z_0)|}{F'(z_0)} \frac{1}{1 + |F(z_0)|^2} F'(z_0) \\ &= \frac{|F'(z_0)|}{1 + |F(z_0)|^2}\end{aligned}$$

$$F'(z_0) = \frac{1 - |w_0|^2}{2\sqrt{|w_0|}} f'(z_0)$$

$$\Rightarrow \qquad |F'(z_0)| = \frac{1 - |w_0|^2}{2\sqrt{|w_0|}} M$$

$$1 - |F(z_0)|^2 = 1 - \left|\sqrt{-w_0}\right|^2 = 1 - |w_0|$$

$$\begin{aligned}\Rightarrow \qquad (\psi \circ F)'(z_0) &= \frac{1 - |w_0|^2}{2\sqrt{|w_0|}} \frac{1}{1 - |w_0|} M \\ &= \frac{1 + |w_0|}{2\sqrt{|w_0|}} M \\ &> M\end{aligned}$$

So there is no such w_0, so $f(\mathcal{U}) = D$.

■

5. HARMONIC MAPS

Let $\Omega \subseteq \mathbb{R}^n$ be some open set. A function $u : \Omega \to \mathbb{R}$ satisfies *Laplace's equation* if

$$u_{x_1x_1} + \ldots + u_{x_nx_n} = 0 .$$

Such a function, where the first and second partial derivatives exist and are continuous, is called *harmonic*.

Dirichlet Problem. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $f : \partial\Omega \to \mathbb{R}$ be continuous. Find a solution to the boundary value problem

$$\Delta u = u_{x_1x_1} + \ldots + u_{x_nx_n} = 0 \text{ on } \Omega ,$$
$$u = f \text{ on } \partial\Omega .$$

When $n = 2$ we can use complex analysis to solve the Dirichlet Problem. Recall that if $f = u + iv : \mathcal{D} \to \mathbb{C}$ is holomorphic then u, v are harmonic on $\mathcal{D}$.

First note that if, for example, $u(x, y) = \log(x^2 + y^2)^{1/2}$, $(x, y) \neq 0$, is harmonic on $\mathbb{C} \setminus \{0\}$ there is no holomorphic $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ for which $u = \operatorname{Re} f$.

Theorem 5.1. *Given a harmonic function $u : \mathcal{U} \to \mathbb{R}$ on a simply connected domain $\mathcal{U} \subseteq \mathbb{C}$ there is a holomorphic $f : \mathcal{U} \to \mathbb{C}$ such that $u = \operatorname{Re} f$.*

Proof. If there were such an f we would have $f' = u_x + iv_x$ and $f' = u_x + iv_y$. Consider $g = u_x + iv_y$. We see that g is holomorphic since

$$(u_x)_x = u_{xx} = -u_{yy} = -(u_y)_y ,$$
$$(u_x)_y = u_{xy} = u_{yx} = -(-u_y)_x .$$

Then by Cauchy's Theorem, $\int_\gamma g = 0$ for every closed curve γ in $\mathcal{U}$. By Theorem 2.1, g has an anti-derivative on $\mathcal{U}$. Let f be an anti-derivative of g on $\mathcal{U}$. $\operatorname{Re} f = u + c$ for some constant c . $f - c$ is a holomorphic function with $\operatorname{Re}(f - c) = u$.

■

Corollary 5.2. *Let $\mathcal{U} \subseteq \mathbb{C}$ be a simply connected domain and $f : \mathcal{U} \to \mathbb{C}$ holomorphic. If $u : f(\mathcal{U}) \to \mathbb{R}$ is harmonic then $u \circ f$ is harmonic.*

Proof. Let $g : f(\mathcal{U}) \to \mathbb{C}$ be holomorphic such that $u = \operatorname{Re} g$. Then $u \circ f = \operatorname{Re}(g \circ f)$ is harmonic.

■

Hence, if we can solve the Dirichlet Problem on D and can find a bijective conformal map $g:\mathcal{U}\to D$ such that $g:\overline{U}\to\overline{D}$ is continuous, we can solve the Dirichlet Problem on $\mathcal{U}$.

Theorem 5.3. *Let $u:\mathcal{D}\to\mathbb{R}$ be harmonic. Then it satisfies the Mean Value Property that $\forall z_0\in\mathcal{D}$ and $\overline{B_r(z_0)}\subseteq\mathcal{D}$,*

$$u(z_0)=\tfrac{1}{2\pi}\int_0^{2\pi}u\left(z_0+re^{it}\right)dt.$$

Proof. Let $f:\overline{B_r(z_0)}\to\mathbb{C}$ be holomorphic such that $u=\operatorname{Re}f$. By Gauss' Mean Value Theorem,

$$f(z_0)=\tfrac{1}{2\pi}\int_0^{2\pi}f\left(z_0+re^{it}\right)dt.$$

Now take real parts.

■

Theorem 5.4. (Maximum Principle) *Suppose $u:\mathcal{D}\to\mathbb{R}$ is harmonic and non-constant on a connected domain $\mathcal{D}$. The u cannot attain a maximum on (the interior of) $\mathcal{D}$.*

Proof. This follows directly from the Mean Value Property: suppose there were a maximum at $z_0\in\mathcal{D}$. Since

$$u(z_0)=\tfrac{1}{2\pi}\int_0^{2\pi}u\left(z_0+re^{it}\right)dt\le u(z_0),$$

$u(z_0)=u\left(z_0+re^{it}\right)$ for all t,r such that $\overline{B_r(z_0)}\subseteq\mathcal{D}$. So u is constant, a contradiction.

■

We return to the Dirichlet Problem for the disc. Let $F:\partial D\to\mathbb{R}$ be continuous. Find the harmonic function $u:D\to\mathbb{R}$ such that u is continuous on D and $\lim_{z\to e^{i\theta}}u(z)=F\left(e^{i\theta}\right)$.

Boundary values determine the function on the interior for holomorphic functions by Cauchy's Integral Formula

$$f(z)=\tfrac{1}{2\pi i}\int_{\partial B_r(z_0)}\tfrac{f(\xi)}{\xi-z}d\xi.$$

Theorem 5.5. (Poisson's Integral Formula) *Let u be harmonic on a domain containing $\overline{B_R(0)}$. Then for $z=re^{i\theta}\in B_R(0)$,*

$$u\left(re^{i\theta}\right)=\frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2-2Rr\cos(\theta-t)+r^2}u\left(Re^{it}\right)dt\,.$$

$P(t)=P(r,\theta,R,t)=\frac{R^2-r^2}{R^2-2Rr\cos(\theta-t)+r^2}$ is called the *Poisson kernel*.

Proof. Let $f:\overline{B_R(0)}\to\mathbb{C}$ be holomorphic with $u=\operatorname{Re} f$. Use a "reflection trick" to show

$$f\left(re^{i\theta}\right)=\frac{1}{2\pi}\int_0^{2\pi}P(t)f\left(Re^{it}\right)dt\,.$$

This will immediately yield the desired result by taking real parts. We can assume $re^{i\theta}\neq 0$ since the result is immediate from the Mean Value Theorem otherwise. Set $z^*=\frac{R^2}{\bar{z}}=\frac{R^2}{r}e^{i\theta}$. Then

$$\frac{1}{2\pi i}\int_{\partial B_R(0)}\frac{f(\xi)}{\xi-z^*}d\xi=0$$

by Cauchy's Theorem. So

$$\begin{aligned}f(z)&=\frac{1}{2\pi i}\int_{\partial B_R(0)}\frac{f(\xi)}{\xi-z}d\xi-\frac{1}{2\pi i}\int_{\partial B_R(0)}\frac{f(\xi)}{\xi-z^*}d\xi\\&=\frac{1}{2\pi i}\int_{\partial B_R(0)}\left(\frac{1}{\xi-z}-\frac{1}{\xi-z^*}\right)f(\xi)d\xi\\&=\frac{1}{2\pi i}\int_{\partial B_R(0)}\frac{R^2-r^2}{(\xi-z)(R^2-\xi\bar{z})}f(\xi)d\xi\end{aligned}$$

Substitute in $\xi=Re^{it}$, $z=re^{i\theta}$:

$$\begin{aligned}\frac{d\xi}{(\xi-z)(R^2-\xi\bar{z})}&=\frac{iRe^{it}\,dt}{\left(Re^{it}-re^{i\theta}\right)\left(R^2-rRe^{i(t-\theta)}\right)}\\&=\frac{i\,dt}{\left(Re^{it}-re^{i\theta}\right)\left(Re^{-it}-re^{-i\theta}\right)}\\&=\frac{i\,dt}{R^2-2rR\cos(\theta-t)+r^2}\end{aligned}$$

Hence,

$$f(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2-2Rr\cos(\theta-t)+r^2}f\left(Re^{it}\right)dt\,.$$

■

Theorem 5.6. *Let $u:B_R(0)\to\mathbb{R}$ be harmonic; suppose u is continuous on $\overline{B_R(0)}$. For $z=re^{i\theta}\in B_R(0)$,*

$$u\left(re^{i\theta}\right)=\tfrac{1}{2\pi}\int_0^{2\pi}P(t)u\left(\mathrm{Re}^{it}\right)dt\ .$$

Proof. Let $t_n \uparrow 1$ and set $u_n(z)=u(t_n z)$. u_n is harmonic on $\overline{B_R(0)}$. By Theorem 5.5,

$$\begin{aligned}\tfrac{1}{2\pi}\int_0^{2\pi}P(t)u_n\left(Re^{it}\right)dt &= u_n\left(re^{i\theta}\right)\\ &= u\left(t_n re^{i\theta}\right)\\ &\underset{n\to\infty}{\to} u\left(re^{i\theta}\right)\end{aligned}$$

We show

$$\tfrac{1}{2\pi}\int_0^{2\pi}P(t)u_n\left(Re^{it}\right)dt \underset{n\to\infty}{\to} \tfrac{1}{2\pi}\int_0^{2\pi}P(t)u\left(Re^{it}\right)dt\ .$$

Consider the quantity

$$I_n=\tfrac{1}{2\pi}\int_0^{2\pi}P(t)\left|u_n\left(Re^{it}\right)-u\left(Re^{it}\right)\right|dt\ .$$

Given $\varepsilon>0$, $\left|u_n\left(Re^{it}\right)-u\left(Re^{it}\right)\right|<\varepsilon$ for sufficiently large n, since u is uniformly continuous on $\overline{B_R(0)}$. So

$$I_n<\tfrac{1}{2\pi}\varepsilon\int_0^{2\pi}P(t)dt=\tfrac{\varepsilon}{2\pi}\underset{n\to\infty}{\to}0\ .$$

■

Theorem 5.7. *Let $F:\partial B_R(0)\to\mathbb{R}$ be continuous. Define*

$$u\left(re^{i\theta}\right)=\tfrac{1}{2\pi}\int_0^{2\pi}\tfrac{R^2-r^2}{R^2-2Rr\cos(\theta-t)=r^2}F\left(Re^{it}\right)dt\ .$$

Then (1) u is harmonic on $B_R(0)$; and
(2) $\lim_{r\to R}u\left(re^{i\theta}\right)=F\left(Re^{i\theta}\right)$.

Proof. (1) Key fact: if $\xi=Re^{it}$, $z=re^{i\theta}$, then $P(r,\theta,R,t)=\mathrm{Re}\left(\frac{\xi+z}{\xi-z}\right)$. Hence, if $u=\mathrm{Re}\,g$ for some holomorphic g,

$$u(z)=\tfrac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\left(\tfrac{\xi+z}{\xi-z}\right)F\left(Re^{it}\right)dt$$

$$\Rightarrow\qquad g(z)=\tfrac{1}{2\pi}\int_0^{2\pi}\tfrac{\xi+z}{\xi-z}F\left(Re^{it}\right)dt$$

g is holomorphic in z:

$$\frac{g(z+h)-g(z)}{h} = \frac{1}{2\pi}\int_0^{2\pi} \frac{2\xi}{(\xi-z)(\xi-(z+h))} F\left(Re^{it}\right)dt$$
$$\underset{h\to 0}{\to} \frac{1}{2\pi}\int_0^{2\pi} \frac{2\xi}{(\xi-z)^2} F\left(Re^{it}\right)dt$$

Since $u = \operatorname{Re} g$, u is harmonic.

(2) We show $\left|u\left(re^{i\theta}\right) - F\left(Re^{i\theta}\right)\right| \underset{r\uparrow R}{\to} 0$. Trick: $\int_0^{2\pi} P(r,\theta,R,t)dt = 2\pi$.

$$u\left(re^{i\theta}\right) - F\left(Re^{i\theta}\right) = \frac{1}{2\pi}\int_0^{2\pi} P(r,\theta,R,t)\left(F\left(Re^{it}\right) - F\left(Re^{i\theta}\right)\right)dt$$

Note that if $t = \theta$ then $P(r,\theta,R,t) = \frac{R+r}{R-r}$. Choose $\varepsilon > 0$. We can find $\delta > 0$ such that $|t-\theta| < \delta \Rightarrow \left|F\left(Re^{it}\right) - F\left(Re^{i\theta}\right)\right| < \varepsilon$. Then

$$\begin{aligned}\frac{1}{2\pi}\int_{\theta-\delta}^{\theta+\delta} P(t)\left|F\left(Re^{it}\right) - F\left(Re^{i\theta}\right)\right|dt &< \varepsilon \frac{1}{2\pi}\int_{\theta-\delta}^{\theta+\delta} P(t)dt \\ &< \varepsilon \frac{1}{2\pi}\int_0^{2\pi} P(t)dt \\ &= \varepsilon\end{aligned}$$

Consider $\frac{1}{2\pi}\int_{\theta-\delta}^{\theta+\delta} P(t)\left|F\left(Re^{it}\right) - F\left(Re^{i\theta}\right)\right|dt$. When $t \in [0,\theta-\delta] \cup [\theta+\delta, 2\pi]$,

$$\begin{aligned}P(t) &\le \frac{R^2-r^2}{R^2 - 2Rr\cos\delta + r^2} \\ &\le \frac{R^2-r^2}{R^2 - 2R^2r^2\cos^2\delta + (R\cos\delta)^2} \\ &\le \frac{R^2-r^2}{R^2\sin^2\delta} \\ &\le \frac{(R+r)(R-r)}{R^2\sin^2\delta} \\ &\le \frac{2(R-r)}{R\sin^2\delta}\end{aligned}$$

We can choose $\delta' > 0$ such that $|R-r| < \delta' \Rightarrow |P(t)| < \varepsilon$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} P(t)\left|F\left(Re^{it}\right) - F\left(Re^{i\theta}\right)\right|dt < \varepsilon M .$$

■

Remark. We can modify the proof to show that $\lim_{z \to Re^{i\theta}} u(z) = F(Re^{i\theta})$ and show that u is continuous on $\overline{B_R(0)}$.

Theorem 5.8. *Suppose $u : \mathcal{D} \to \mathbb{R}$ is continuous and satisfies the Mean Value Property. Then u is harmonic.*

Proof. Choose $\overline{B_R(z_0)} \subseteq \mathcal{D}$. By Theorem 5.7 there exists a harmonic function $\tilde{u}$ on $\overline{B_R(z_0)}$ such that $\tilde{u}|_{\partial B_R(z_0)} = u|_{\partial B_R(z_0)}$. Then $\tilde{u} - u$ satisfies the Mean Value Property and $\tilde{u} - u = 0$ on $\partial B_R(z_0)$. Hence $\tilde{u} = u$ on $\overline{B_R(z_0)}$.

■