MA3F4 Linear Analysis
Revision Notes

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1 Normed Spaces

1.1 Normed Spaces

Definitions 1.1.1. Let X be a vector space over a field $\mathbb{F}$ ($=\mathbb{R}$ or $\mathbb{C}$). A map $\|\cdot\| : X \to [0,\infty)$ is a *norm* on X if

(i) homogeneity: $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{F}$, $x \in X$;

(ii) triangle inequality: $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$;

(iii) definiteness: $\|x\| = 0 \Leftrightarrow x = 0$.

A map satisfying only (i) and (ii) is called a *semi-norm*. $(X, \|\cdot\|)$ is called a *(semi-)normed space* if X is a vector space and $\|\cdot\|$ is a (semi-)norm on X.

Definitions 1.1.2. Let $(X, \|\cdot\|)$ be a (semi-)normed space and $(x_n) \subseteq X$ a sequence.

(i) x_n *converges* to $x \in X$, written as $x_n \to x$ or $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} \|x_n - x\| = 0$;

(ii) (x_n) is *Cauchy* if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $n, m \geq N \Rightarrow \|x_n - x_m\| < \varepsilon$.

Definitions 1.1.3. A (semi-)normed space $(X, \|\cdot\|)$ is *complete* if every Cauchy sequence in X converges to some point in X. A complete normed space is called a *Banach space*.

Proposition 1.1.4. *Let $\ell^\infty(T)$ denote the vector space of bounded $\mathbb{F}$-valued functions on T, with $\|\cdot\|_\infty$ the supremum norm. Then $(\ell^\infty(T), \|\cdot\|_\infty)$ is a Banach space.*

Lemma 1.1.5. *If X is a Banach space and U a closed subspace of X then U is complete.*

Definition 1.1.6. $C(T)$ denotes the space of continuous function $T \to \mathbb{F}$.

Corollary 1.1.7. *$C_b(T)$, the space of continuous bounded functions $T \to \mathbb{F}$, is a Banach space with respect to $\|\cdot\|_\infty$.*

Theorem 1.1.8. (Hölder inequality for sequences.) *Let $1 \leq p \leq \infty$ and q be such that $1/p + 1/q = 1$ (if $p = 1$, $q = \infty$ and vice versa). Let $x \in \ell^p$, $y \in \ell^q$. Then*

$$\|xy\|_1 \leq \|x\|_p \|y\|_q.$$

Corollary 1.1.9. (Minkowski inequality.) *Let $x, y \in \ell^p$, $1 \leq p \leq \infty$. Then*

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p,$$

Example 1.1.10. L^p spaces. Let (T, Σ, μ) be a measure space, $1 \leq p < \infty$, and define

$$\mathcal{L}^p(T,\mu) := \{f : T \to \mathbb{F} | f \text{ is } \Sigma\text{-measurable and } \|f\|_{L^p} < \infty\}$$

where

$$\|f\|_{L^p} := \left(\int_T |f|^p \, d\mu\right)^{1/p}.$$

Lemma 1.1.11. *Let $(X, \|\cdot\|')$ be a (semi-)normed space. Then*

(i) $N := \{x \in X | \|x\|' = 0\}$ *is a subspace of* X*;*

(ii) $\|[x]\| := \|x\|'$ *defines a norm on* X/N*;*

(iii) if X *is complete then* $(X/N, \|\cdot\|)$ *is complete.*

Theorem 1.1.12. $L^p(T, \mu) := \mathcal{L}^p(T, \mu)/N$ *with the induced norm is a Banach space.*

Remark 1.1.13. We treat elements of L^p as functions, not equivalence classes, and write $f \in L^p$ rather than $[f] \in L^p$.

Theorem 1.1.14. (Hölder inequality for L^p.) *Let* $1 \leq p \leq \infty$ *and* q *be such that* $1/p + 1/q = 1$*. Let* $f \in L^p(T)$*,* $y \in L^q(T)$*. Then* $fg \in L^1(T)$ *and*

$$\|xy\|_1 \leq \|x\|_p \|y\|_q.$$

Example 1.1.15. Spaces of measures. Let (T, Σ) be a measurable space. The space of *signed measures* ($\mathbb{F} = \mathbb{R}$) or *complex measures* ($\mathbb{F} = \mathbb{C}$):

$$M(T, \Sigma) := \{\mu : \Sigma \to \mathbb{F} | \mu \text{ is } \sigma\text{-additive}\}.$$

The *variation norm* is

$$\|\mu\| = |\mu|(T) := \sup \left\{ \sum_{j=1}^{n} |\mu(E_j)| \,\middle|\, T = \bigcup_{j=1}^{n} E_j, n \in \mathbb{N} \right\}.$$

1.2 Basic Properties of Normed Spaces

Proposition 1.2.1. *Let* X *be a normed space. Then*

(i) $x_n \to x$ *and* $y_n \to y \Rightarrow x_n + y_n \to x + y$*;*

(ii) $\lambda_n \to \lambda \in \mathbb{F}$*,* $x_n \to x \Rightarrow \lambda_n x_n \to \lambda x$*;*

(iii) $x_n \to x \Rightarrow \|x_n\| \to \|x\|$.

Corollary 1.2.2. *If* X *is a normed space and* U *a subspace then* $\bar{U}$ *is also a subspace.*

Definition 1.2.3. Two norms $\|\cdot\|$ and $\|\cdot\|'$ on X are *equivalent* if there are $0 \leq m \leq M$ such that for all $x \in X$,

$$m\|x\| \leq \|x\|' \leq M\|x\|$$

Theorem 1.2.4. *Let* $\|\cdot\|$ *and* $\|\cdot\|'$ *be two norms on* X *and* (x_n) *a sequence in* X*. The following are equivalent:*

(i) $\|\cdot\|$ *and* $\|\cdot\|'$ *are equivalent;*

(ii) $\|x_n - x\| \to 0 \Leftrightarrow \|x_n - x\|' \to 0$;

(iii) $\|x_n\| \to 0 \Leftrightarrow \|x_n\|' \to 0$.

Theorem 1.2.5. *On a finite-dimensional vector space all norms are equivalent.*

Theorem 1.2.6. *For a normed space X, the following are equivalent:*

(i) $\dim X < \infty$;

(ii) the closed unit ball $\{x \in X | \|x\| \leq 1\}$ *is compact;*

(iii) every bounded sequence in X has a convergent subsequence.

1.3 Quotients and Sums of Normed Spaces

Definition 1.3.1. Let X be a normed space and $A \subseteq X$. The *distance* between $x \in X$ and A is $d(x, A) := \inf_{y \in A} \|x - y\|$.

Theorem 1.3.2. *Let X be a normed space and U a subspace. For $x \in X$ let $[x] := x + U \in X/U$ denote the equivalence class. Then*

(i) $\|[x]\| := d(x, U)$ *defines a semi-norm on X/U with* $\|[x]\| \leq \|x\|$;

(ii) if U is closed, then $\| \cdot \|$ is a norm on X/U;

(iii) if X is complete and U is closed, then $(X/U, \| \cdot \|)$ is a Banach space.

Lemma 1.3.3. *In a semi-normed space X, the following are equivalent:*

(i) X is complete;

(ii) if (x_n) is a sequence in X with $\sum_{n=1}^{\infty} \|x_n\| < \infty$, then $\exists x \in X$ such that $\|\sum_{n=1}^{N} x_n - x\| \to 0$.

Theorem 1.3.4. *Let X, Y be normed spaces.*

(i) Let $1 \leq p \leq \infty$. Then

$$\|(x, y)\|_p := \begin{cases} (\|x\|^p + \|y\|^p)^{1/p} & p < \infty \\ \max\{\|x\|, \|y\|\} & p = \infty \end{cases}$$

defines a norm on $X \oplus Y$; let $X \oplus_p Y$ denote $X \oplus Y$ with this norm.

(ii) All these norms are equivalent and generate the product topoology on $X \times Y = X \oplus Y$.

(iii) If X, Y are Banach spaces then so is $X \oplus_p Y$.

1.4 Separability

Definition 1.4.1. A topological space is *separable* if it has a countable dense subset.

Lemma 1.4.2. *A normed space X is separable if and only if there is a countable set A such that $X = \overline{\operatorname{span} A}$.*

Examples 1.4.3. (i) If $\dim X < \infty$, $X = \operatorname{span}\{e_1, \ldots, e_n\}$, then X is separable.

(ii) ℓ^p is separable for $1 \leq p \leq \infty$.

Theorem 1.4.4. (Weierstrass.) *The space of polynomials*

$$P([a, b]) := \operatorname{span}\{X_n : t \mapsto t^n | n \in \mathbb{N}\}$$

is dense in $(C([a, b]), \|\cdot\|_\infty)$.

Corollary 1.4.5. $(C([a, b]), \|\cdot\|_\infty)$ *is separable.*

Corollary 1.4.6. $L^p([a, b])$ *is separable for* $1 \leq p \leq \infty$.

Corollary 1.4.7. $L^p(\mathbb{R})$ *is separable for* $1 \leq p \leq \infty$.

Definitions 1.4.8. Let $\mathcal{A} \subseteq C(T)$.

(i) $\mathcal{A}$ is an *algebra* if it is a vector space over $\mathbb{F}$ such that $f, g \in \mathcal{A} \Rightarrow fg \in \mathcal{A}$.

(ii) $\mathcal{A}$ *separates points* if $\forall s, t \in T$ with $s \neq t$, $\exists f \in \mathcal{A}$ such that $f(s) \neq f(t)$.

(iii) If $\mathbb{F} = \mathbb{C}$, $\mathcal{A}$ is *conjugate to itself* if $f \in \mathcal{A} \Rightarrow \bar{f} \in \mathcal{A}$.

Theorem 1.4.9. (Stone-Weierstrass.) *Let $\mathcal{A} \subseteq C(T)$ be an algebra, where T is a compact metric space. If $\mathcal{A}$ separates points, contains the constant functions, (and, in the case $\mathbb{F} = \mathbb{C}$, is conjugate to itself), then $\mathcal{A}$ is dense in $(C(T), \|\cdot\|_\infty)$.*

2 Functional Operators

2.1 Basic Properties of Linear Operators

Definitions 2.1.1. Let X, Y be normed spaces. A continuous linear map $A : X \to Y$ is called an *operator*, or, in the case $Y = \mathbb{F}$, a *functional.*

Theorem 2.1.2. *Let X, Y be normed spaces and $A : X \to Y$ linear. Then the following are equivalent:*

(i) A is continuous;

(ii) A is continuous at $0 \in X$;

(iii) $\exists M \geq 0$ *such that* $\forall x \in X$, $\|Ax\|_Y \leq M\|x\|_X$*;*

(iv) A *is uniformly continuous.*

Definitions 2.1.3. Let X, Y be normed spaces. Then

$$\mathcal{B}(X, Y) := \{A : X \to Y | A \text{ is linear and continuous}\}$$

denotes the vector space of bounded linear functions with *operator norm*

$$\|A\|_{\mathcal{B}(X,Y)} := \sup_{\|x\| \leq 1} \frac{\|Ax\|}{\|x\|}.$$

Theorem 2.1.4. *(i)* $\|\cdot\|_{\mathcal{B}(X,Y)}$ *defines a norm on* $\mathcal{B}(X, Y)$*.*

(ii) If Y *is complete then so is* $(\mathcal{B}(X, Y), \|\cdot\|_{\mathcal{B}(X,Y)})$*.*

Theorem 2.1.5. *Let* D *be a dense subspace of a normed space* X*,* Y *a Banach space, and* $A \in \mathcal{B}(D, Y)$*. Then* A *has a unique continuous continuation to* X*.*

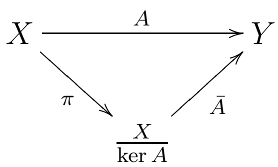
Lemma 2.1.6. *Let* X*,* Y*,* Z *be normed spaces,* $B \in \mathcal{B}(X, Y)$*,* $A \in \mathcal{B}(Y, Z)$*. Then* $AB \in \mathcal{B}(X, Z)$ *and* $\|AB\| \leq \|A\|\|B\|$*.*

Examples 2.1.7.

Definitions 2.1.8. A bounded linear operator $A : X \to Y$ of normed spaces is an *isomorphism* if A is bijective and A^{-1} is continuous. If $\|Ax\| = \|x\|$ for all $x \in X$, then A is *isometric*. If two normed spaces X, Y allow for an (isometric) isomorphism $A : X \to Y$, they are called (*isometrically*) *isomorphic* and we write $X \cong Y$. I.e., isomorphisms are linear surjections such that $\exists m, M \geq 0$ such that $\forall x \in X$, $m\|x\| \leq \|Ax\| \leq M\|x\|$.

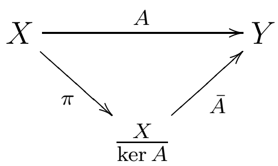
Definition 2.1.9. Let X, Y be normed spaces. A linear map $A : X \to Y$ is called a *quotient map* if A maps the open unit ball in X onto the open unit ball in Y.

Proposition 2.1.10. *If* X*,* Y *are normed spaces and* $A : X \to Y$ *is linear,*



Thus $\|\bar{A}\| = \|A\|$ *and* $\bar{A}$ *is injective.*

Theorem 2.1.11. *If* X*,* Y *are normed spaces and* $A \in \mathcal{B}(X, Y)$*,*



$\bar{A}$ *is an isometric isomorphism if and only is* A *is an isometric isomorphism.*

Theorem 2.1.12. (Neumann series.) *If X is a normed space, $A \in \mathcal{B}(X, X)$, and $\sum_{n=0}^{\infty} A^n$ coverges in $\mathcal{B}(X, X)$, then $(\mathrm{id} - A)^{-1} = \sum_{n=0}^{\infty} A^n \in \mathcal{B}(X, X)$.*

Remark 2.1.13. The assumptions of the above theorem are satisfied if X is complete and $\sum_{n=0}^{\infty} \|A^n\| < \infty$, in particular if $\|A\| \leq 1$.

2.2 Dual Spaces and the Representations

Definition 2.2.1. $X^* := \mathcal{B}(X, \mathbb{F})$ is the *dual space* of the normed space X.

Remark 2.2.2. Since $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ is complete, X^* is always a Banach space with norm $\|x'\| := \sup_{\|x\| \leq 1} |x'(x)|$ for $x' \in X^*$.

Theorem 2.2.3. (Duals of sequence spaces.) *Let $1 \leq p < \infty$ and $1/p + 1/q = 1$. Then the map $A : \ell^q \to (\ell^p)^*$ given by $(Ax)(y) := \sum_{n=1}^{\infty} s_n t_n$ where $x = (s_n) \in \ell^q$ and $y = (t_n) \in \ell^p$ is an isometic isomorphism.*

Remark 2.2.4. $(\ell^\infty)^*$ is not isomorphic to ℓ^1.

Theorem 2.2.5. (Duals of L^p spaces.) *Let $1 \leq p < \infty$, $1/p + 1/q = 1$ and (T, Σ, μ) a σ-finite measure space. Then $A : L^q(T) \to (L^p(T))^*$ given by $(Ag)(f) := \int_T fg \, d\mu$ defines an isometric isomorphism.*

3 The Hahn-Banach Theorem

3.1 Continuation of Linear Functionals

Definition 3.1.1. $p : X \to \mathbb{R}$ is *sublinear* if

(i) $p(\lambda x) = \lambda p(x)$ for all $\lambda \geq 0$, $x \in X$;

(ii) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.

Examples 3.1.2. (i) Semi-norms are sublinear.

(ii) Linear maps are sublinear.

(iii) In ℓ^∞, the map $(t_n) \mapsto \mathrm{Re}(\limsup_n t_n)$ is sublinear.

Theorem 3.1.3. (Hahn-Banach Theorem, real linear algebra.) *Let X be a real vector space and U a subspace. Let $p : X \to \mathbb{R}$ be sublinear and $\ell : U \to \mathbb{R}$ linear with $\ell(x) \leq p(x)$ for all $x \in U$. Then there is a linear continuation $L : X \to \mathbb{R}$ of ℓ, i.e. $L|_U = \ell$ with $L(x) \leq p(x)$ for all $x \in X$.*

Axiom 3.1.4. Zorn's Lemma. Let $(S, \leq)$ be a partially ordered set. if every chain in S has an upper bound, then S has at least one maximal element.

Remark 3.1.5. Zorn's Lemma is equivalent to the non-constructive Axiom of Choice.

Lemma 3.1.6. *Let X be a complex vector space.*

(i) If $\ell : X \to \mathbb{R}$ is a $\mathbb{R}$-linear functional then $\tilde{\ell}(x) := \ell(x) - i\ell(x)$ is a $\mathbb{C}$-linear functional and $\ell = \operatorname{Re}\tilde{\ell}$.

(ii) If $h : X \to \mathbb{C}$ is $\mathbb{C}$-linear and $\ell = \operatorname{Re} h$, and $\tilde{\ell}$ is as in (i), then ℓ is $\mathbb{R}$-linear and $h = \tilde{\ell}$.

(iii) If $p : X \to \mathbb{R}$ is a semi-norm and $\ell : X \to \mathbb{C}$ is $\mathbb{C}$-linear, then $|\ell(x)| \leq p(x)$ for all $x \in X$ if and only if $|\operatorname{Re}\ell(x)| \leq p(x)$ for all $x \in X$.

(iv) If X is a normed space and $\ell : X \to \mathbb{C}$ is $\mathbb{C}$-linear and continuous then $\|\ell\| = \|\operatorname{Re}\ell\|$.

Theorem 3.1.7. (Hahn-Banach Theorem, complex linear algebra.) *Let X be a complex vector space and U a subspace. Let $p : X \to \mathbb{R}$ be sublinear and $\ell : U \to \mathbb{R}$ linear with $\operatorname{Re}\ell(x) \leq p(x)$ for all $x \in U$. Then there is a $\mathbb{C}$-linear continuation $L : X \to \mathbb{C}$ of ℓ, i.e. $L|_U = \ell$ with $\operatorname{Re} L(x) \leq p(x)$ for all $x \in X$.*

Theorem 3.1.8. (Hahn-Banach Theorem, normed spaces.) *Let X be a normed space and U a subspace. For every $u' \in U^*$, $\exists x' \in X^*$ with $x'|_U = u'$ and $\|x'\| = \|u'\|$.*

Corollary 3.1.9. *If X is a normed space and $0 \neq x \in X$, then $\exists x' \in X^*$ such that $\|x'\| = 1$ and $x'(x) = \|x\|$.*

Corollary 3.1.10. *If X is a normed space and $x \in X$, then*

$$\|x\| = \sup_{\|x'\| \leq 1} |x'(x)|.$$

Remark 3.1.11. *Note the symmetry between this result and*

$$\|x'\| := \sup_{\|x'\| \leq 1} |x'(x)|$$

for $x' \in X^*$.

Corollary 3.1.12. *Let X be a normed space an U a closed proper subspace. Then, given $x \in X \setminus U$, $\exists x' \in X^*$ such that $x'|_U = 0$ and $x'(x) \neq 0$.*

Corollary 3.1.13. *Let X be a normed space and U a subspace. Then U is dense in X if and only if $x' \in X^*$ and $x'|_U = 0 \Rightarrow x' = 0$.*

Theorem 3.1.14. *Let X be a normed space. Then X^* separable $\Rightarrow$ X separable.*

Remark 3.1.15. The converse is not true: $(\ell^\infty)^* \ncong \ell^1$ because ℓ^1 is separable but ℓ^∞ is not.

Theorem 3.1.16. (Runge's Approximation Theorem.) *Let $K \subset \mathbb{C}$ be compact and let f be analytic (holomorphic) on a neighbourhood Ω of K. Let $P \subset \bar{\mathbb{C}} \setminus K$ contain at least one point from each connected component of $\bar{\mathbb{C}} \setminus K$. Then, given $\varepsilon > 0$, there is a rational function R with poles in the set P such that $\max_{z \in K} |f(z) - R(z)| < \varepsilon$.*

Remark 3.1.17. $\bar{\mathbb{C}} \setminus K$ has at most countably many components, i.e. P can be chosen as $P = \bigcup_{j \in \mathbb{N}} \{\alpha_j\}$. The point α_∞ in the unbounded neighbourhood of $\bar{\mathbb{C}} \setminus K$ can be chosen as $\alpha_\infty = \infty$.

3.2 Reflexivity

Definitions 3.2.1. $X^{**} := (X^*)^*$ is called the *bidual / second dual / double dual* of the normed space X. Given $x \in X$, define $i(x) : X^* \to \mathbb{F}$ by $i(x)(x') := x'(x)$. This is a bounded linear functional on X^*:

$$|i(x)(x')| = |x'(x)| \leq \|x'\|\|x\|,$$

so $\|i(x)\|_{X^{**}} \leq \|x\| < \infty$, so $i(x) \in X^{**}$.

Theorem 3.2.2. *This map $i : X \to X^{**}$ is a linear isometry, but is not generally surjective.*

Corollary 3.2.3. *Every normed space X is isometrically isomorphic to a dense subspace of a Banach space.*

Examples 3.2.4. This gives an elegant way of completing a normed space X.

(i) $X := C(T)$, $X^* = \ell^1$, $X^{**} = \ell^\infty$.

(ii) $X := \ell^p$, $X^* = \ell^q$, $X^{**} = \ell^p$ for $1 < p < \infty$, $1/p + 1/q = 1$.

(iii) $X := L^p$, $X^* = L^q$, $X^{**} = L^p$ for $1 < p < \infty$, $1/p + 1/q = 1$.

Definition 3.2.5. A Banach space X is called *reflexive* if $X^{**} \cong X$, i.e. i is surjective.

Remarks 3.2.6. A non-complete normed space cannot be reflexive. ℓ^p and L^p are reflexive for $1 < p < \infty$. $C(T)$ is not reflexive.

Theorem 3.2.7. *(i) Let U be a closed subspace of X. Then X reflexive $\Rightarrow U$ reflexive.*

(ii) Let X be a Banach space. Then X is reflexive $\Leftrightarrow X^*$ is reflexive.

(iii) Let X be reflexive. Then X is separable $\Leftrightarrow X^*$ is separable.

Remarks 3.2.8. (i) $C(T)$ is not reflexive, so $C(T)^* = \ell^1$ is not reflexive, so $(\ell^\infty)^* \supsetneq \ell^1$.

(ii) L^1 is separable, but $L^\infty = (L^1)^*$ is not separable, so neither L^1 nor L^∞ is reflexive. In particular, $(L^\infty)^* \supsetneq L^1$.

3.3 Weak and Weak* Convergence

Definitions 3.3.1. Let X be a normed space.

(i) A sequence $(x_n) \subseteq X$ *converges weakly* to $x \in X$ if $x'(x_n) \to x'(x)$ for all $x' \in X^*$. Write this as $x_n \rightharpoonup x$, $x_n \xrightarrow{\sigma} x$, or $x_n \xrightarrow{w} x$.

(ii) A sequence $(x'_n) \subseteq X$ *converges weakly** to $x' \in X^*$ if $x'_n(x) \to x'(x)$ for all $x \in X$ (i.e. pointwise convergence). Write this as $x_n \overset{*}{\rightharpoonup} x$, or $x_n \xrightarrow{\sigma^*} x$. Since X^* separates points in X, weak limits are unique.

Remarks 3.3.2. (i) $x_n \to x \Rightarrow x_n \rightharpoonup x$; $x'_n \to x' \Rightarrow x'_n \overset{*}{\rightharpoonup} x$.

(ii) The converse implications are generally false.

(iii) We will see later that weak and weak* convergent sequences are bounded.

Theorem 3.3.3. (Arzela-Ascoli.) *Let T be a separable metric space and $(x_n) \subseteq C(T)$. If*

(i) (x_n) is bounded pointwisem i.e. $\forall t \in T$, $\exists M(t) > 0$ such that $|x_n(t)| \leq M(t)$ for all $n \in \mathbb{N}$; and

(ii) (x_n) is equicontinuous, i.e. $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall n \in \mathbb{N}$, $d(s,t) < \delta \Rightarrow |x_n(s) - x_n(t)| < \varepsilon$

then there is an $x \in C(T)$ and a subsequence (x_{n_j}) such that $x_{n_j} \to x$ locally uniformly, i.e. uniformly on compact subsets of T.

Corollary 3.3.4. (Banach-Alaoglu.) *Let X be a separable normed space. Then every bounded sequence in X^* has a weakly* convergent subsequence.*

Corollary 3.3.5. *In a reflexive space, every bounded sequence has a weakly convergent subsequence.*

3.4 Adjoint Operators

Definition 3.4.1. Let X, Y be normed spaces and $A \in \mathcal{B}(X,Y)$. The *adjoint operator* $A^* : Y^* \to X^*$ is defined by $(A^*y')x = y'(Ax)$ for all $x \in X$, $y' \in Y^*$.

Theorem 3.4.2. *$A^* \in \mathcal{B}(Y^*, X^*)$ and $\|A^*\| = \|A\|$.*

Lemma 3.4.3. *Let $A \in \mathcal{B}(X,Y)$ and $B \in \mathcal{B}(Y,Z)$. Then $(BA)^* = A^*B^*$.*

Lemma 3.4.4. *The following diagram is commutative:*

$$\begin{array}{ccc} X & \xrightarrow{A} & Y \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle i_Y} \\ X^{**} & \xrightarrow[A^{**}]{} & Y^{**} \end{array}$$

and A^{**} is a continuation of A in $\mathcal{B}(X^{**}, Y^{**})$.

Definitions 3.4.5. Let $U \subseteq X$ and $V \subseteq X^*$ then the closed subspaces

$$U^\perp := \{x' \in X^* | x'(x) = 0 \text{ for all } x \in U\}$$
$$V_\perp := \{x \in X | x'(x) = 0 \text{ for all } x' \in V\}$$

are called the *annihilators* of U in X^* and V in X.

Theorem 3.4.6. *If $A \in \mathcal{B}(X,Y)$ then $\overline{\operatorname{im} A} = (\ker A^*)_\perp$.*

Corollary 3.4.7. *Let $A \in \mathcal{B}(X,Y)$ with $\operatorname{im} A$ closed. Then given $y \in Y$, the equation $Ax = y$ has a solution $x \in X$ if and only if $A^* y' = 0 \Rightarrow y'(y) = 0$. Thus, it is soluble for all $y \in Y$ if* $\ker A^* = 0$.

Remarks 3.4.8. (i) The Hahn-Banach Theorem tells us that X^* is rich enough to encode many properties of X.

(ii) $\|x\| = \sup_{\|x'\|<1} |x'(x)|$.

(iii) X^* separable $\Rightarrow X$ separable.

(iv) X^* reflexive $\Rightarrow X$ reflexive.

(v) Weak limits are unique $\Rightarrow$ weak convergence is useful.

(vi) $\|A\| = \|A^*\|$ is useful.

4 Baire's Theorem and Consequences

4.1 Baire's Category Theorem

Definitions 4.1.1. Let X be a topological space and $M \subseteq X$.

(i) M is *nowhere dense* if $\bar{M}$ has no interior points, i.e. $X \setminus \bar{M}$ is dense in X.

(ii) M is *meagre* (or *of the first category*) if it is a countable union of nowhere dense sets.

(iii) M is *fat* (or *of the second category*) if it is not meagre.

Remark 4.1.2. Meagre sets are the topological equivalent of null sets in measure theory. Countable unions of meagre sets are meagre.

Example 4.1.3. Let X be a normed space and U a closed proper subspace. Then U is meagre.

Theorem 4.1.4. (Baire's Category Theorem.) *Let X be a complete metric space.*

(i) $M \subseteq X$ is meagre $\Rightarrow X \setminus M$ is dense in X.

(ii) X is of the second category.

Theorem 4.1.5. (Equivalent to Baire's Theorem.) *Let X be a complete metric space and (U_n) a sequence of open dense subsets. Then $\bigcup_{n\in\mathbb{N}} U_n$ is dense in X.*

Theorem 4.1.6. *The set of nowhere differentiable continuous functions on $[0,1]$ is dense in $(C([0,1],\mathbb{R}), \|\cdot\|_\infty)$.*

4.2 The Principle of Uniform Boundedness

Theorem 4.2.1. (The Banach-Steinhaus Theorem.) *Let X be a Banach space and Y a normed space, I a dense set and $A_i \in \mathcal{B}(X,Y)$ for $i \in I$. If $\sup_{i\in I} \|A_i x\|_Y < \infty$ for all $x \in X$, then $\sup_{i\in I} \|A_i\|_{\mathcal{B}(X,Y)} < \infty$.*

Corollary 4.2.2. *For a subset M of a normed space X, M is bounded if and only if $x'(M) \subseteq \mathbb{F}$ is bounded for all $x' \in X^*$.*

Corollary 4.2.3. *Weakly convergent sequences are bounded.*

Corollary 4.2.4. *Let X be a normed space and $M \subseteq X^*$, then M is bounded if and only if the set $\{x'(x)|x' \in M\} \subseteq \mathbb{F}$ is bounded for all $x \in X$.*

Corollary 4.2.5. *Let X be a Banach space, Y a normed space, and $(A_n) \subseteq \mathcal{B}(X,Y)$. If $Ax := \lim_{n\to\infty} A_n x$ exists for all $x \in X$ then $A \in \mathcal{B}(X,Y)$.*

4.3 The Open Mapping Theorem

Definition 4.3.1. A map between topological spaces is *open* if it maps open sets to open sets.

Lemma 4.3.2. *For a linear map $A : X \to Y$ of normed spaces, the following are equivalent:*

(i) *A is open;*

(ii) *A maps open balls around 0 onto open neighbourhoods of 0: if $U_r = \{x \in X | \|x\| < r\}$, $V_\varepsilon = \{y \in Y | \|y\| < \varepsilon\}$ it holds that $\forall r > 0$, $\exists \varepsilon > 0$ such that $V_\varepsilon \subseteq A(U_r)$;*

(iii) *$\exists \varepsilon > 0$ such that $V_\varepsilon \subseteq A(U_1)$.*

Theorem 4.3.3. (Open Mapping Theorem.) *Let X, Y be Banach spaces and let $A \in \mathcal{B}(X,Y)$ be surjective. Then A is open.*

Corollary 4.3.4. *Let X, Y be Banach spaces and $A \in \mathcal{B}(X,Y)$ bijective. Then A^{-1} is continuous.*

Corollary 4.3.5. *Let $\|\cdot\|$ and $\|\cdot\|'$ be norms on X such that both $(X, \|\cdot\|)$ and $(X, \|\cdot\|')$ are Banach spaces. If $\exists M < \infty$ such that $\|x\| \leq M\|x\|'$ for all $x \in X$, then $\|\cdot\|$ and $\|\cdot\|'$ are equivalent.*

4.4 The Closed Graph Theorem

Definitions 4.4.1. Let X, Y be normed spaces, $D \subseteq X$ a subspace, and $A : D \to Y$ linear. Then A is closed if, for $(x_n) \subseteq D$,

$$\left.\begin{matrix} x_n \to x \in X \\ Ax_n \to y \in Y \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} x \in D \\ Ax = y \end{matrix}\right.$$

For linear operators A defines on some domain $D \subseteq X$, we write $\mathrm{dom}(A) = D$ or $A : X \supseteq \mathrm{dom}(A) \to Y$.

Remark 4.4.2. Note how closedness and continuity are different in the case $D = X$:

(i) $x_n \to x$;

(ii) $Ax_n \to y$;

(iii) $Ax = y$;

A is continuous if (i) $\Rightarrow$ (ii) and (iii). A is closed if (i) and (ii) $\Rightarrow$ (iii).

Definition 4.4.3. For a linear map $A : X \supseteq D \to Y$, D a subspace, the *graph* of A is $\mathrm{gr}(A) := \{(x, Ax) \in X \times Y | x \in D\}$.

Lemma 4.4.4. *Let X, Y, D be as before. Then A is closed if and only if* $\mathrm{gr}(A)$ *is closed in $X \subseteq Y$.*

Lemma 4.4.5. *Let X, Y be Banach spaces, $A : X \supseteq D \to Y$ closed. Then*

(i) D equipped with the graph norm $|||x||| := \|x\|_X + \|Ax\|_Y$ is a Banach space;

(ii) $A : (D, ||| \cdot |||) \to Y$ is continuous.

Theorem 4.4.6. *Let X, Y be Banach spaces, $A : X \supseteq D \to Y$ closed and surjective. Then A is open. If A is also injective, then A^{-1} is continuous.*

Theorem 4.4.7. (Closed Graph Theorem.) *Let X, Y be Banach spaces and $A : X \to Y$ linear and closed. Then A is continuous.*

Corollary 4.4.8. (Hellinger-Töplitz Theorem.) *Let $A : H \to H$ be linear and everywhere defined on a Hilbert space H with $\langle x, Ay\rangle = \langle Ax, y\rangle$ for all $x, y \in H$. Then A is continuous.*

5 Fréchet Spaces

5.1 Fréchet Spaces

Definitions 5.1.1. A family of semi-norms $(\| \cdot \|_\alpha)_{\alpha \in A}$ on a vector space X *separates points* if $\|x\|_\alpha = 0 \ \forall \alpha \in A \Rightarrow x = 0$. A vector space X with a family of norms that separates points is called a *locally convex space.*

Example 5.1.2. $C^\infty(\mathbb{R})$ with $\|x\|_j := \|x^{(j)}\|_\infty$.

Remarks 5.1.3. The natural topology on a locally convex space is the weakest topology in which all the semi-norms $\| \cdot \|_\alpha$ are continuous. The topology is metrizable if and only if A is countable.

Proposition 5.1.4. *Let X be a vector space and $(\|\cdot\|_j)_{j\in\mathbb{N}_0}$ a family of semi-norms that separate points. Then*

$$d(x,y) := \sum_{j=0}^{\infty} 2^{-j} \frac{\|x-y\|_j}{1+\|x-y\|_j}$$

defines a metric on X.

Definition 5.1.5. A complete metric space X with the metric given by a countable family $(\|\cdot\|_j)_{j\in\mathbb{N}_0}$ of semi-norms that separate points is called a *Fréchet space*.

Since Fréchet spaces are complete, Baire's Theorem applies, and so one can prove results analagous to those of Section 4, such as Banach-Steinhaus and

Theorem 5.1.6. *If X, Y are Fréchet spaces and $A : X \to Y$ is a continuous linear surjection, then A is open.*

5.2 Schwartz Functions and Tempered Distributions

Definition 5.2.1. Given a multi-index $\alpha \in \mathbb{N}_0^d$, define $|\alpha| := \alpha_1 + \cdots + \alpha_d$ and the *mixed partial derivative*

$$\partial_t^\alpha x := \frac{\partial^{|\alpha|} x}{\partial^{\alpha_1} t_1 \ldots \partial^{\alpha_d} t_d}.$$

Definition 5.2.2. The set *functions of rapid decrease* or *Schwartz functions*, $\mathcal{S}(\mathbb{R}^d)$, is the collection of $\phi \in C^\infty(\mathbb{R}^d)$ for which

$$\|\phi\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^d} \left| x^\alpha \partial_x^\beta \phi(x) \right| < \infty$$

for all $\alpha, \beta \in \mathbb{N}_0^d$.

Theorem 5.2.3. *$\mathcal{S}(\mathbb{R}^d)$ with the countable family of semi-norms $\|\cdot\|_{\alpha,\beta}$ is a Fréchet space.*

Definitions 5.2.4. The topological dual space of $\mathcal{S}(\mathbb{R}^d)$, i.e. the space of continuous linear maps $\mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$, is denoted by $\mathcal{S}'(\mathbb{R}^d)$ and called the space of *tempered distributions.* Given $\phi \in \mathcal{S}(\mathbb{R}^d)$ define the associated tempered distribution $T_\phi \in \mathcal{S}'(\mathbb{R}^d)$ by $T_\phi \psi := \int_{\mathbb{R}^d} \phi(x)\psi(x)\,dx$.

Lemma 5.2.5. *Let $T \in \mathcal{S}(\mathbb{R}^d)^*$ be a linear functional. If there is one semi-norm $\|\cdot\|_{\alpha,\beta}$ and a constant $C \geq 0$ such that $|T(\phi)| \leq C\|\phi\|_{\alpha,\beta}$ for all $\phi \in \mathcal{S}(\mathbb{R}^d)$, then $T \in \mathcal{S}'(\mathbb{R}^d)$.*

Example 5.2.6. The δ-distribution. There is no function δ_b such that

$$\int_{\mathbb{R}^d} \delta_b(x)\phi(x)\,dx = \phi(b).$$

However, δ_b is a tempered distribution.

Remark 5.2.7. We can add distributions, multiply by scalars, and act on them with linear operators. We cannot in general multiply two distributions, or take square roots.

Definition 5.2.8. For $T \in \mathcal{S}'(\mathbb{R}^d)$ define the *distributional derivative* $\partial_x^\alpha T \in \mathcal{S}'(\mathbb{R}^d)$ by

$$(\partial_x^\alpha T)(\phi) = T\left((-1)^{|\alpha|}\partial_x^\alpha \phi\right)$$

for all $\phi \in \mathcal{S}(\mathbb{R}^d)$.

Remarks 5.2.9. (i) $\partial_x^\alpha : \mathcal{S}' \to \mathcal{S}'$ is continuous since it is the adjoint of the continuous map $(-1)^{|\alpha|}\partial_x^\alpha : \mathcal{S} \to \mathcal{S}$.

(ii) $\partial_x^\alpha : \mathcal{S}' \to \mathcal{S}'$ is an extension of the usual derivative $\partial_x^\alpha : \mathcal{S} \to \mathcal{S}$ in the sense that for $\phi \in \mathcal{S}$, $\partial_x^\alpha T_\phi = T_{\partial_x^\alpha \phi}$.

Examples 5.2.10. (i) The derivative of of a δ-distribution:

$$(\partial_x^\alpha \delta_b)(\phi) = \delta_b\left((-1)^{|\alpha|}\partial_x^\alpha \phi\right) = (-1)^{|\alpha|}\partial_x^{|\alpha|}\phi(b).$$

(ii) Let $\theta(x) := 0$ for $x \leq 0$, 1 for $x > 0$. Then $\partial_x^1 T_\theta = \frac{d}{dx}T_\theta = \delta_0$.

5.3 The Fourier Transform

Example 5.3.1. The free Schrödinger equation:

$$i\partial_t \psi(t,x) = -\frac{1}{2}\Delta_x \psi(t,x)$$

with initial conditions $\psi(0,\cdot) \in L^2(\mathbb{R}^d)$. This equation describes the free dynamics of a quantim particle in $\mathbb{R}^d$. The probability of finding the particle in a region $\Omega \subseteq \mathbb{R}^d$ at time $t \in \mathbb{R}$ is $\mathbb{P}_t(X \in \Omega) := \int_\Omega |\psi(t,x)|^2 \, dx$.

Definitions 5.3.2. Let $\psi \in L^1(\mathbb{R}^d)$, then

$$\hat{\psi}(k) = (\mathcal{F}\psi)(k) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ik\cdot x}\psi(x) \, dx$$

is the *Fourier transform* of ψ, and

$$\check{\psi}(k) = (\mathcal{F}^{-1}\psi)(k) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ik\cdot x}\psi(x) \, dx$$

is the *inverse Fourier transform* of ψ.

Theorem 5.3.3. *(i) $\mathcal{F}$ and $\mathcal{F}^{-1}$ are continuous linear mappings $\mathcal{S} \to \mathcal{S}$.*

(ii) *For all multi-indices* $\alpha, \beta \in \mathbb{N}_0^d$ *we have*

$$\left((ik)^\alpha \partial_k^\beta \mathcal{F}\psi\right)(k) = \left(\mathcal{F}\partial_x^\alpha (-ix)^\beta \psi\right)(k)$$

and, in particular,

$$\widehat{x\psi}(k) = i\nabla_k \hat{\psi}(k),$$
$$\widehat{\nabla_x \psi}(k) = ik\hat{\psi}(k).$$

(iii) $\mathcal{F}^{-1}\mathcal{F} = \mathcal{F}\mathcal{F}^{-1} = \mathrm{id}_{\mathcal{S}}$.

(iv) *For* $\phi, \psi \in \mathcal{S}(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} \hat{\psi}(x)\phi(x)\, dx = \int_{\mathbb{R}^d} \psi(k)\hat{\phi}(k)\, dk$.

(v) *For* $\phi \in \mathcal{S}(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} |\phi(x)|^2\, dx = \int_{\mathbb{R}^d} |\hat{\phi}(k)|^2\, dk$, *i.e.* $\|\phi\|_{L^2} = \|\hat{\phi}\|_{L^2}$.

Corollary 5.3.4. $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ *and* $\mathcal{F}^{-1} : \mathcal{S} \to \mathcal{S}$ *can be extended uniquely to continuous linear operators* $\mathcal{F} : L^2 \to L^2$ *and* $\mathcal{F}^{-1} : L^2 \to L^2$ *satisfying* $\mathcal{F}^{-1}\mathcal{F} = \mathcal{F}\mathcal{F}^{-1} = \mathrm{id}_{L^2}$.

Corollary 5.3.5. *For* $T \in \mathcal{S}'$ *define* $\hat{T}(\phi) = (\mathcal{F}T)(\phi) := T(\hat{\phi})$ *for* $\phi \in \mathcal{S}$, *i.e.* $\mathcal{F}_{\mathcal{S}'} = \mathcal{F}'_{\mathcal{S}}$. *Then* $\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$ *is a continuous linear map that extends* $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ *in the sense that for* $\phi \in \mathcal{S}$, $\widehat{T_\phi} = T_{\hat{\phi}}$.

Definition 5.3.6. Let $g : \mathbb{R}^d \to \mathbb{C}$ be a function for which the map $M_g : \mathcal{S} \to \mathcal{S} : \phi \mapsto g\phi$ is continuous. Then the *pseudo-differential operator* $g(-i\nabla x)$ is defined by $g(-i\nabla x)T := \mathcal{F}^{-1}g(k)\mathcal{F}T$, $T \in \mathcal{S}'$. This extends the usual derivative, since for $g(k) = k^\alpha$, $f(-i\nabla x) = \partial_x^\alpha$.

Example 5.3.7. Solution for free Schrödinger equation:

$$\psi(t,x) = e^{\frac{i}{2}\Delta_x t}\psi(0,x).$$