UNIVERSITY OF WARWICK

MA453 LIE ALGEBRAS

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1. DEFINITIONS AND BASIC PRINCIPLES

An *algebra* over $\mathbb{C}$ is a vector space A over $\mathbb{C}$ with a bilinear multiplication $A \times A \to A$, $(x, y) \mapsto xy$. Bilinearity means that for all $x, y \in A$ and $\lambda \in \mathbb{C}$

$$\begin{aligned}(x_1 + x_2)y &= x_1 y + x_2 y\\ x(y_1 + y_2) &= xy_1 + xy_2\\ (\lambda x)y &= x(\lambda y) = \lambda(xy)\end{aligned}$$

All the algebras in this course will be over $\mathbb{C}$. The main advantage of $\mathbb{C}$ is that it is algebraically closed.

An *associative algebra* is an algebra A such that for all $x, y, z \in A$, $x(yz) = (xy)z$.

A *Lie algebra*[†] is an algebra L with multiplication $L \times L \to L$, $(x, y) \mapsto [xy]$ such that

$$[xx] = 0 \text{ for all } x \in L,$$
$$[[xy]z] + [[yz]x] + [[zx]y] = 0 \text{ for all } x, y, z \in L \text{ – the \textit{Jacobi identity}.}$$

Unless otherwise specified, L shall be an arbitrary Lie algebra. Where dictated by requirements of clarity, we shall write $[x, y]$ for $[xy]$.

Lemma 1.1. *For all* $x, y \in L$, $[xy] = -[yx]$.

Proof.

$$0 = [x + y, x + y] = [xx] + [xy] + [yx] + [yy] = [xy] + [yx]$$

■

We say that Lie multiplication is *anticommutative*.

Lemma 1.2. *Suppose A is an associative algebra. Then A can be made into a Lie algebra by defining* $[xy] = xy - yx$.

Proof. $[xx] = 0$ is clear.

$$\begin{aligned}[[xy]z] &= [xy - yx, z] = xyz - yxz - zxy + zyx\\ [[yz]x] &= yzx - zyx - xyz + xzy\\ [[zx]y] &= zxy - xzy - yxz + yxz\end{aligned}$$

[†] After Marius Sophus Lie (1842-99).

All terms cancel, so we have the Jacobi identity.

∎

The Lie algebra obtained in this way is denoted $[A]$.

We can also multiply subspaces in the following way: let L be a Lie algebra and H,K subspaces of L. We define $[HK]$ to be the smallest subspace containing all the Lie products $[hk]$ for $h \in H, k \in K$. So

$$[HK] = \{[h_1k_1] + \ldots + [h_rk_r] \mid h_i \in H, k_i \in K\}.$$

Lemma 1.3. *If H,K are subspaces of L then $[HK] = [KH]$.*

Proof.

$$[HK] \ni [h_1k_1] + \ldots + [h_rk_r] = -[k_1h_1] - \ldots - [k_rh_r] \in [KH]$$

∎

Multiplication of subspaces is commutative.

$H \subseteq L$ is called a *subalgebra* of L if H is a subspace of L and $[HH] \subseteq H$. That is, a subalgebra of L is a subset of L that is itself a Lie algebra under the same operations as L.

A subset $I \subseteq L$ is called an *ideal* of L if I is a subspace of L and $[IL] \subseteq I$. We will write $I \triangleleft L$.

Note. Since $[IL] = [LI]$, $[IL] \subseteq I \Leftrightarrow [LI] \subseteq I$.

Every ideal of L is also a subalgebra of L, but the converse is not true.

Example. Consider $M_2 = \{2 \times 2 \text{ matrices over } \mathbb{C}\}$. $[M_2]$ is a Lie algebra. The subset T of elements of $[M_2]$ of trace zero form an ideal of $[M_2]$. The subset U of elements of $[M_2]$ with upper-right element zero form a subalgebra of $[M_2]$, but not an ideal.

Proposition 1.4. *(i) If H,K are subalgebras then $H \cap K$ is a subalgebra.*
(ii) If $H,K \triangleleft L$ then $H \cap K \triangleleft L$.
(iii) If $H \triangleleft L$ and K is a subalgebra then $H + K$ is a subalgebra.
(iv) If $H,K \triangleleft L$ then $H + K \triangleleft L$.

Proof. (i) $H \cap K$ is certainly a subspace.

$$[H \cap K, H \cap K] \subseteq [HH] \subseteq H$$
$$[H \cap K, H \cap K] \subseteq [KK] \subseteq K$$

(ii)

$$[H \cap K, L] \subseteq [HL] \subseteq H$$
$$[H \cap K, L] \subseteq [KL] \subseteq K$$

(iii)

$$[H + K, H + K] \subseteq [HH] + [HK] + [KH] + [KK] \subseteq H + H + H + K \subseteq H + K$$

(iv)

$$[H + K, L] \subseteq [HL] + [KL] \subseteq H + K$$

■

Note. The sum of two subalgebras need not be a subalgebra.

We can form *factor algebras*: let $I \lhd L$. In particular, I is an additive subgroup so we can form the factor group L/I; the elements of L/I are the cosets $I + x$ for $x \in L$.

$$(I + x) + (I + y) = I + (x + y)$$
$$\lambda(I + x) = I + \lambda x$$

We define $[I + x, I + y] = I + [xy]$. We do need to check that this is well-defined, i.e. that if $I + x = I + x'$ and $I + y = I + y'$ then $I + [xy] = I + [x'y']$. We can find $i_1, i_2 \in I$ such that $x' = i_1 + x$ and $y' = i_2 + y$. So

$$\begin{aligned}[x'y'] &= [i_1 + x, i_2 + y] \\ &= [i_1 i_2] + [i_1 y] + [x i_2] + [xy] \\ &\in I + [xy]\end{aligned}$$

So the coset containing $[xy]$ is the same as that containing $[x'y']$.

It is easy to verify that L/I is a Lie algebra.

A *homomorphism* of Lie algebras is a linear map $\theta : L_1 \to L_2$ such that for all $x, y \in L_1$, $\theta([xy]) = [\theta(x), \theta(y)]$. If θ is bijective it is called an *isomorphism* and we write $L_1 \cong L_2$.

Proposition 1.5. *Let $\theta : L_1 \to L_2$ be a homomorphism with kernel K. Then $K \lhd L_1$, $\mathrm{im}(\theta)$ is a subalgebra of L_2 and $L_1/K \cong \mathrm{im}(\theta)$.*

Proof. Let $x,y\in L_1$. $[\theta(x),\theta(y)]=\theta[xy]\in\theta(L_1)$, so $\mathrm{im}(\theta)$ is a subalgebra of L_2.

Now let $x\in K$ and $y\in L_1$. Then

$$\theta[xy]=[\theta(x),\theta(y)]=[0,\theta(y)]=0$$

so $y\in K$. Hence, $K\lhd L_1$.

Now let $x,y\in L_1$.

$$\theta(x)=\theta(y)\Leftrightarrow\theta(x-y)=0\Leftrightarrow x-y\in K\Leftrightarrow K+x=K+y$$

So $\theta(x)\mapsto K+x$ is a bijection between $\mathrm{im}(\theta)$ and L_1/K. We now check that this bijection is an isomorphism of Lie algebras: let $x,y,z\in L_1$.

$$\begin{aligned}[\theta(x),\theta(y)]=\theta(z)&\Leftrightarrow\theta[xy]=\theta(z)\\&\Leftrightarrow K+[xy]=K+z\\&\Leftrightarrow[K+x,K+y]=K+z\end{aligned}$$

So $\mathrm{im}(\theta)\cong L_1/K$.

■

Proposition 1.6. *Let $I\lhd L$ and H a subalgebra of L. Then $I+H$ and $I\cap H$ are subalgebras (by 1.4) and*
(i) $I\lhd I+H$,
(ii) $I\cap H\lhd H$,
(iii) $(I+H)/I\cong H/(I\cap H)$.

Proof. (i) $[I,I+H]\subseteq[I,L]\subseteq I$.

(ii)

$$[I\cap H,H]\subseteq[IH]\subseteq I$$
$$[I\cap H,H]\subseteq[HH]\subseteq H$$

(iii) We can form $(I+H)/I$ and $H/(I\cap H)$. Elements of $(I+H)/I$ have the form $I+i+h=I+h$ for $h\in H$. Define a map $\theta:H\to(I+H)/I$ by $\theta(h)=I+h$. This map is a homomorphism since $[I+h,I+h']=I+[hh']$. It is surjective: $\mathrm{im}(\theta)=(I+H)/I$. Consider $\ker(\theta)$: $h\in\ker(\theta)\Leftrightarrow I+h=I\Leftrightarrow h\in I$. So $\ker(\theta)=H\cap I$. By 1.5 $H/\ker(\theta)\cong\mathrm{im}(\theta)$, so $(I+H)/I\cong H/(I\cap H)$.

■

Note. In this course we shall consider only finite-dimensional Lie algebras over $\mathbb{C}$. In this case,

$$\dim(L/I) = \dim(L) - \dim(I)$$

To prove this, select a basis $e_1,\ldots,e_r$ of I and extend to a basis $e_1,\ldots,e_n$ of L. Each element of L has the form $\lambda_1 e_1 + \ldots + \lambda_n e_n$; each element of L/I has the form $I + \lambda_{r+1}e_{r+1} + \ldots + \lambda_n e_n = (I + \lambda_{r+1}e_{r+1}) + \ldots + (I + \lambda_n e_n)$. $I + e_{r+1},\ldots, I + e_n$ form a basis for L/I. So $\dim(L/I) = n - r = \dim(L) - \dim(I)$.

Examples. If $\dim(L) = 1$, L has basis x. $[xx] = 0$, so $[LL] = 0$.

If $\dim(L) = 2$ let x, y be a basis for L. $[xx] = [yy] = 0$, but $[xy] = -[yx] = ?$ Possibly $[LL] = 0$. If $[LL] \neq 0$ then $\dim([LL]) = 1$. Let x' be a basis for $[LL]$ and x', y' a basis for L itself. Then we have $[x'y'] = \lambda x'$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. Re-choose $y'' = \lambda^{-1} y'$: we then have that $[xy] = x$. Hence, we have two Lie algebras of dimension 2.

2. REPRESENTATIONS AND MODULES OF LIE ALGEBRAS

Recall that $M_n = \{n \times n \text{ matrices over } \mathbb{C}\}$ and that $[M_n]$ is the Lie algebra of such matrices with $[AB] = AB - BA$.

A *representation* of a Lie algebra L is a homomorphism $\rho : L \to [M_n]$ for some $n \in \mathbb{N}$. I.e.,

$$\rho[xy] = [\rho(x), \rho(y)] = \rho(x)\rho(y) - \rho(y)\rho(x).$$

If ρ is a representation of L then so is ρ' given by $\rho'(x) = T^{-1}\rho(x)T$ where T is a non-singular $n \times n$ matrix independent of x. We say that two representations are *equivalent* if there is a non-singular T such that $\rho'(x) = T^{-1}\rho(x)T$ holds for all $x \in L$.

An L-*module* is a vector space V over $\mathbb{C}$ with a map $V \times L \to V$ such that
(i) $(v, x) \mapsto vx$ is linear in both v and x;
(ii) $v[xy] = (vx)y - (vy)x$ for all $v \in V$ and $x, y \in L$.

We shall only deal with finite-dimensional L-modules in this course.

A *submodule* W of V is a subspace of V such that $wx \in W$ for all $w \in W, x \in L$ i.e. a subspace closed under the right action of the element of L.

Proposition 2.1. *Let V be an L-module with basis $e_1, \ldots, e_n$. Let $x \in L$ and let $e_i x = \sum_{i=1}^{n} \rho_{ij}(x) e_j$. Let $\rho(x)$ be the matrix with ij th entry $\rho_{ij}(x)$. Then $x \mapsto \rho(x)$ is a representation of L and a different choice of basis gives an equivalent representation.*

Proof. The linear transformation $v \mapsto vx$ has matrix $\rho(x)$; $v \mapsto (vx)y$ has matrix $\rho(x)\rho(y)$; $v \mapsto (vy)x$ has matrix $\rho(y)\rho(x)$; $v \mapsto (vx)y - (vy)x$ has matrix $\rho(x)\rho(y) - \rho(y)\rho(x)$. That is, the linear transformation $v \mapsto v[xy]$ has matrices $\rho[xy]$ and $\rho(x)\rho(y) - \rho(y)\rho(x)$. So $\rho[xy] = \rho(x)\rho(y) - \rho(y)\rho(x)$, so ρ is a representation of L.

Now take a new basis $f_1, \ldots, f_n$ of V. The linear transformation $v \mapsto vx$ is represented by a matrix $T^{-1}\rho(x)T$, where $e_i = \sum_{i=1}^{n} T_{ij} f_j$. So we get a representation $x \mapsto T^{-1}\rho(x)T$ that is equivalent to ρ.

■

An L-module is called *irreducible* if it has no submodules except itself and 0; otherwise it is said to be *reducible*.

An L-module V is called *decomposable* if there are submodules $V_1, V_2 \neq 0$ of V such that $V = V_1 \oplus V_2$; otherwise it is said to be *indecomposable*.

Proposition 2.2. *L is itself an L-module under the map $L \times L \to L : (x, y) \mapsto [xy]$.*

This is the *adjoint* L-module; we define $\operatorname{ad} y : L \to L$ by $(\operatorname{ad} y)x = [xy]$.

Proof. It is sufficient to show that for all $x, y, z \in L$, $[z[xy]] = [[zx]y] - [[zy]x]$. This follows immediately from the Jacobi identity and the anticommutativity of Lie multiplication.

■

A *derivation* of a Lie algebra L is a linear map $D : L \to L$ such that for all $x, y \in L$,

$$D[xy] = [Dx, y] + [x, Dy].$$

Proposition 2.3. *Let $x \in L$. Then $\operatorname{ad} x$ is a derivation of L.*

Proof. Linearity is clear. We need to check that $(\operatorname{ad} x)[yz] = [(\operatorname{ad} x)y, z] + [y, (\operatorname{ad} x)z]$. This is true if and only if the Jacobi identity is true.

■

Let V be an L-module, W a subspace of V and H a subspace of L. We define WH to be the subspace spanned by wh for all $w \in W, h \in H$.

If W is a submodule of V, V/W is itself an L-module under the action $(W + v)x = W + vx$ for $v \in V, x \in L$. This action is well-defined because

$$(W + v)x = (W + v')x \Rightarrow v - v' \in W \Rightarrow (v - v')x \in W \Rightarrow W + vx = W + v'x.$$

3. ABELIAN, NILPOTENT AND SOLUBLE LIE ALGEBRAS

A Lie algebra L is *abelian* if $[LL]=0$, i.e. $[xy]\equiv 0$.

Define $L^1 = L$ and inductively define $L^{n+1}=[L^n L]$.

Proposition 3.1. *For each* $n\in\mathbb{N}$, $L^n \triangleleft L$.

Proof. It is sufficient to show that if $H,K\triangleleft L$ then $[HK]\triangleleft L$. Let $x\in H, y\in K, z\in L$. Is $[[xy]z]\in[HK]$?

$$[[xy]z] = -[[yz]x]-[[zx]y]$$

But $y,[yz]\in K$ and $x,[zx]\in H$ so $[[HK]L]\subseteq[HK]$.

Clearly $L^1 = L\triangleleft L$. If we assume inductively that $L^n\triangleleft L$ then the above workings show that $L^{n+1}=[L^n L]\triangleleft L$, and the result follows.

■

Proposition 3.2. $L = L^1 \supseteq L^2 \supseteq L^3 \supseteq \dots$

Proof. For each n, $L^{n+1}=[L^n L]\subseteq L^n$ since $L^n\triangleleft L$.

■

L is *nilpotent* if there is an $n\in\mathbb{N}$ such that $L^n = 0$.

Clearly every abelian Lie algebra is nilpotent as $L^2 = 0$.

Example. Let L be the Lie algebra of upper-triangular $n\times n$ matrices with zeroes on the principal diagonal; $\dim(L)=\frac{1}{2}n(n-1)$; L is a subalgebra of $[M_n]$. Define subspaces H_i by requiring that elements of H_i have zeroes on and below the $(i-1)$th diagonal above the principal diagonal.

Lie multiplication shows that $[H_i,L]\subseteq H_{i-1}$. We show that $L^i\subseteq H_i$ by induction on i. If $i=1$ then $L=L^1=H_1$. Assume $L^i\subseteq H_i$ for $i=r$. Then

$$\begin{aligned} L^{r+1} &= [L^r L] \\ &\subseteq [H^r L] \\ &\subseteq H_{r+1} \end{aligned}$$

In particular, $L^n\subseteq H_n = 0$, so L is nilpotent.

Proposition 3.3. *For all* $m,n \geq 1$, $\left[L^m L^n\right] \subseteq L^{m+n}$.

Proof. Use induction on n. If $n=1$ then $\left[L^m L^1\right]=\left[L^m L\right]=L^{m+1}$. Assume for $n=r$ and consider $n=r+1$:

$$\begin{aligned}\left[L^m L^{r+1}\right] &= \left[L^m\left[L^r L\right]\right] \\ &= \left[\left[L^r L\right]L^m\right] \\ &\subseteq \left[\left[L L^m\right]L^r\right]+\left[\left[L^m L^r\right]L\right] \text{by the Jacobi identity}\end{aligned}$$

So

$$\begin{aligned}\left[L^m L^{r+1}\right] &\subseteq \left[L^{m+1} L^r\right]+\left[\left[L^m L^r\right]L\right] \\ &\subseteq L^{m+1+r}+\left[L^{m+r} L\right] \\ &\subseteq L^{m+r+1}\end{aligned}$$

■

We now inductively define another sequence of subspaces of L:

$$\begin{aligned}L^{(0)} &= L \\ L^{(i+1)} &= \left[L^{(i)} L^{(i)}\right]\end{aligned}$$

The $L^{(i)}$ are all ideals of L, so $L^{(i+1)} \subseteq L^{(i)}$, so $L = L^{(0)} \supseteq L^{(1)} \supseteq L^{(2)} \supseteq \ldots$.

We say that L is *soluble* if there is an $n \in \mathbb{N}$ such that $L^{(n)} = 0$.

Proposition 3.4. *(i)* $L^{(n)} \subseteq L^{2^n}$.
(ii) Every nilpotent Lie algebra is soluble.

Proof. (i) Induction on n: if $n=0$ $L^{(0)} = L = L^1$. Assume for $n=r$:

$$\begin{aligned}L^{(r+1)} &= \left[L^{(r)} L^{(r)}\right] \\ &\subseteq \left[L^{2^r} L^{2^r}\right] \\ &\subseteq L^{2^{r+1}}\end{aligned}$$

(ii) Suppose L is nilpotent. Then there is an n such that $L^{2^n} = 0$. By (i) $L^{(n)} = 0$ also, so L is soluble.

■

Example. Let L be the set of all $n \times n$ matrices with zeroes below the principal diagonal. $[LL] \subseteq L$; L is a subalgebra of $[M_n]$. Define subspaces H_i by requiring that elements of H_i have zeroes on and below the ith diagonal above the principal diagonal.

We have that $[H_i H_i] \subseteq H_{i+1}$. We show that $L^{(i)} \subseteq H_i$ by induction on i. If $i = 0$ then $L^{(0)} = L = H_0$. Assume for $i = r$. Then $L^{(r+1)} = \left[L^{(r)} L^{(r)}\right] \subseteq [H_r H_r] \subseteq H_{r+1}$. In particular, $L^{(n)} \subseteq H_n$ so L is soluble.

However, L is not nilpotent. To see this, consider

$$A = \begin{pmatrix} 0 & \mu_1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & \mu_{n-1} \\ 0 & & & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$AB = \begin{pmatrix} 0 & \mu_1\lambda_2 & & 0 \\ & 0 & \ddots & \\ & & \ddots & \mu_{n-1}\lambda_n \\ 0 & & & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} 0 & \mu_1\lambda_1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & \mu_{n-1}\lambda_{n-1} \\ 0 & & & 0 \end{pmatrix}$$

$$[AB] = AB - BA = \begin{pmatrix} 0 & \mu_1(\lambda_2 - \lambda_1) & & 0 \\ & 0 & \ddots & \\ & & \ddots & \mu_{n-1}(\lambda_n - \lambda_{n-1}) \\ 0 & & & 0 \end{pmatrix}$$

By choosing the λ_i all unequal and the μ_j suitably we can get any desired matrix

$$[AB] = \begin{pmatrix} 0 & v_1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & v_{n-1} \\ 0 & & & 0 \end{pmatrix}$$

Let K be the subspace consisting of all matrices of this form.

$$K \subseteq [KL] \subseteq [[KL]L] \subseteq \ldots \subseteq [L \ldots L] = L^i$$

So $K \subseteq L^i$ for all i and $K \neq 0$, so $L^i \neq 0$ for all i. Hence, L is not nilpotent.

Proposition 3.5. *(i) Every subalgebra of a soluble Lie algebra is soluble.*
(ii) Every factor algebra of a soluble Lie algebra is soluble.
(iii) If $I \lhd L$ and $I, L/I$ are soluble then L is soluble.

Proof. (i) Let H be a subalgebra of L. L is soluble so $L^{(n)} = 0$ for some n. We show that $H^{(i)} \subseteq L^{(i)}$ for all i. This is true for $i = 0$. Assume it is true for $i = r$.

$$H^{(r+1)} = \left[H^{(r)}H^{(r)}\right] \subseteq \left[L^{(r)}L^{(r)}\right] = L^{(r+1)}$$

So $H^{(n)} = 0$ and H is soluble.

(ii) $I \lhd L$. We need to show that $(L/I)^{(i)} = \left(I + L^{(i)}\right)/I$. This is true for $i = 0$; assume it for $i = r$:

$$\begin{aligned}(L/I)^{(r+1)} &= [(L/I)^{(r)}, (L/I)^{(r)}] \\ &= [\tfrac{I+L^{(r)}}{I}, \tfrac{I+L^{(r)}}{I}] \\ &= \tfrac{\left[I+L^{(r)}, I+L^{(r)}\right]}{I} \\ &= \tfrac{I+\left[L^{(r)}L^{(r)}\right]}{I} \\ &= \tfrac{I+L^{(r+1)}}{I}\end{aligned}$$

If $L^{(n)} = 0$ then $(L/I)^{(n)} = I/I$, the zero subspace of L/I.

(iii) Suppose I and L/I are soluble. L/I soluble $\Leftrightarrow (L/I)^{(m)} = I/I$ for some m. $\frac{I+L^{(m)}}{I} = \frac{I}{I}$, so $L^{(m)} \subseteq I$. I is soluble so $I^{(n)} = 0$ for some n.

$$L^{(m+n)} = \left(L^{(m)}\right)^{(n)} \subseteq L^{(n)} = 0$$

So $L^{(m+n)} = 0$; L is soluble.

■

Proposition 3.6. *Let H, K be soluble ideals of L. Then $H + K$ is a soluble ideal of L.*

Proof. We know that $H + K \lhd L$. By 1.6, $\frac{H+K}{H} \cong \frac{K}{H \cap K}$. K is soluble, so $K/(H \cap K)$ is soluble. Hence, $(H + K)/H$ is soluble. By the previous proposition, since H is soluble, $H + K$ is soluble.

■

Corollary 3.7. Any Lie algebra L has a unique maximal soluble ideal.

Proof. Since $\dim(L) < \infty$ L certainly has a maximal soluble ideal. Let H, K be two maximal soluble ideals of L. Then $H + K$ is a soluble ideal. $H \subseteq H + K$ and H is maximal, so $H = H + K$. Similarly $K = H + K$, so $H = K$.

■

This maximal soluble ideal of L is called the *soluble radical* L, usually denoted R. If $R = 0$ we say that L is *semisimple.*

L/R is semisimple. For if R'/R is the soluble radical of L/R then since R'/R and R are both soluble, so is R'. Hence $R' \subseteq R$. So $R' = R$ and $R'/R = R/R$ is the zero subspace of L/R.

A Lie algebra L is called *simple* if it has no ideals other than 0 and L.[†]

If $\dim(L) = 1$ then L is certainly simple. There are other simple Lie algebras. If L is abelian and simple then $\dim(L) = 1$, since $[LL] = 0$. If L is soluble and simple and $\dim(L) = 1$ then $L \neq [LL]$, so $[LL] = 0$.

If $I \lhd L$ then the ideals of L/I have the form J/I for $J \lhd L$, $J \supseteq I$. So L/I is simple if and only if I is maximal.

A *composition series* of L is a sequence of subalgebras

$$L = K_0 \supset K_1 \supset \ldots \supset K_r = 0$$

where $K_{i+1} \lhd K_i$ is maximal. The factor algebras K_{i-1}/K_i are all simple Lie algebras and are known as the *composition factors* of L.

Proposition 3.8. *L is soluble if and only if all composition factors in a composition series of L are 1-dimensional.*

Proof. Let $L = K_0 \supset K_1 \supset \ldots \supset K_r = 0$ be a composition series. L is soluble, so K_i is soluble, so K_i/K_{i+1} is soluble. So $\dim(K_i/K_{i+1}) = 1$. Conversely, suppose that $\dim(K_i/K_{i+1}) = 1$. Then certainly K_i/K_{i+1} is soluble (even abelian). K_{r-1} is soluble and K_{r-2}/K_{r-1} is soluble, so K_{r-2} is soluble. K_{r-3}/K_{r-2} is soluble, so K_{r-3} is soluble. Eventually we see that $K_0 = L$ is soluble.

■

[†] The classification of the simple Lie algebras was completed in the 1890's by Élie Cartan and Wilhelm Killing, working independently.

4. REPRESENTATIONS OF NILPOTENT LIE ALGEBRAS

We shall first discuss representations of abelian Lie algebras.

Proposition 4.1. *Let L be abelian. Then every irreducible L-module has dimension 1. Every linear map $L \to \mathbb{C}$ is a 1-dimensional representation of L.*

Proof. A 1-dimensional representation of L is by definition a linear map $\lambda : L \to \mathbb{C}$ such that $\lambda([xy]) = \lambda(x)\lambda(y) - \lambda(y)\lambda(x)$. But the RHS of this equation is zero; since L is abelian the LHS is always zero, too. So every linear map $L \to \mathbb{C}$ is a representation of L.

Let V be an irreducible L-module and let $x \in L$; consider the linear map $V \to V$, $v \mapsto vx$. Let w be an eigenvector of this map; i.e. $w \neq 0$ and $wx = \lambda x$ for some $\lambda \in \mathbb{C}$, where λ is the eigenvalue. Let $W = \{ v \in V \mid vx = \lambda v \}$, the eigenspace. W is a subspace of V. Since $w \neq 0$, $W \neq 0$. We shall show that W is a submodule of V.

Let $v \in W, y \in L$.

$$(vy)x = (vx)y + \underbrace{v[yx]}_{\substack{=0 \\ \because L \text{ abelian}}} = (vx)y = (\lambda v)y = \lambda(vy)$$

So $vy \in W$, which shows that W is a submodule of V. But V is irreducible so $V = W$. So $vx = \lambda v$ for all $v \in V$. Hence each $x \in L$ acts on V by scalar multiplication. So every subspace of V is a submodule. Hence $\dim(V) = 1$.

■

We now recall some linear algebra.

Let $A \in M_n$. Then the *characteristic polynomial* of A is $\chi(t) = \det(tI_n - A)$. For non-singular $T \in M_n$, A and $T^{-1}AT$ have the same characteristic polynomial:

$$\begin{aligned} \det(tI_n - T^{-1}AT) &= \det(T^{-1}(tI_n - A)T) \\ &= \det(T^{-1})\det(tI_n - A)\det(T) \\ &= \det(tI_n - A) \end{aligned}$$

If V is an n-dimensional vector space over $\mathbb{C}$ and $\theta : V \to V$ is a linear map we define the characteristic polynomial of θ to be the characteristic polynomial of any matrix representing θ.

$\chi(t) \in \mathbb{C}[t]$ factorizes into linear factors:

$$\chi(t) = (t-\lambda_1)^{m_1} \ldots (t-\lambda_r)^{m_r} \text{ with } \lambda_i \neq \lambda_j \text{ for } i \neq j .$$

The λ_i are the *eigenvalues*, each with multiplicity m_i.

Question 1: Is there a decomposition of V into a direct sum of subspaces, one for each λ_i?

Answer 1: Yes.

There is an *eigenvector* $v_i \in V$ with eigenvalue λ_i, i.e. $\theta(v_i) = \lambda_i v_i$. The *eigenspace* for θ with respect to the eigenvalue λ_i is

$$\begin{aligned} \mathrm{ES}(\theta, \lambda_i) &= \{ v \in V \mid \theta(v) = \lambda_i v \} \\ &= \{ v \in V \mid (\theta - \lambda_i I)v = 0 \} \end{aligned}$$

Question 2: Is $\dim(\mathrm{ES}(\theta, \lambda_i)) = m_i$?

Question 3: Is V the direct sum of the eigenspaces of the λ_i?

Example. Let $\dim(V) = 2$. Let $\{e_1, e_2\}$ be a basis for V and take θ such that $\theta : e_1 \mapsto e_2 \mapsto 0$. The matrix of θ is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\chi(t) = \begin{vmatrix} t & -1 \\ 0 & t \end{vmatrix} = t^2$$

θ has eigenvalues $0, 0$. The eigenspace of θ with eigenvalue 0 is $\mathbb{C}e_2$, so

$$\dim(\mathrm{ES}(\theta, 0)) = 1 \neq 2 = \text{multiplicity of } 0 .$$

So

Answer 2: No.

Answer 3: No.

The *generalized eigenspace* of θ with respect to the eigenvalue λ_i is

$$\begin{aligned}\mathrm{GES}(\theta,\lambda_i) &= \{v \in V \mid v \text{ is annihilated by some power of } (\theta - \lambda_i I)\} \\ &= \{v \in V \mid \exists N \in \mathbb{N} \text{ s.t. } (\theta - \lambda_i I)^N v = 0\}\end{aligned}$$

So, in the above example, $\mathrm{GES}(\theta,0) = V$.

Proposition 4.2. (The Decomposition Theorem) *Let V be a vector space of dimension n over $\mathbb{C}$ and let $\theta : V \to V$ be a linear map with characteristic polynomial*

$$\chi(t) = \prod_{i=1}^{r} (t - \lambda_i)^{m_i}$$

with λ_i distinct and $\sum_{i=1}^{r} m_i = n$. Let $V_i = \mathrm{GES}(\theta,\lambda_i)$. Then
(i) $V = V_1 \oplus \ldots \oplus V_r$;
(ii) $\dim(V_i) = m_i$;
(iii) $\theta(V_i) \subseteq V_i$;
(iv) The characteristic polynomial of $\theta|_{V_i}$ is $(t - \lambda_i)^{m_i}$;
(v) $V_i = \{v \in V \mid (\theta - \lambda_i I)^{m_i} v = 0\}$.

Proof. The proof (omitted) uses the Cayley-Hamilton Theorem, i.e. that $\chi(\theta) : V \to V$ satisfies $\chi(\theta) = 0$.

■

Theorem 4.3. *Let L be a nilpotent Lie algebra and V an L-module. Let $y \in L$ and $\rho(y) : V \to V : v \mapsto vy$. Then the generalized eigenspaces V_i of V with respect to $\rho(y)$ are all submodules of V.*

Note. This does not hold for arbitrary Lie algebras: we need the nilpotency condition.

Recall. Leibnitz's formula for differentiation:

$$D^n(fg) = \sum_{i=0}^{n} \binom{n}{i} (D^{n-i} f)(D^i g)$$

where D denotes the action of differentiation (once).

We first prove

Proposition 4.4. *Let L be a Lie algebra and V an L-module. Let $v \in V$, $x, y \in L$ and $\alpha, \beta \in \mathbb{C}$. Then*

$$(vx)(\rho(y)-(\alpha+\beta)I)^n=\sum_{i=0}^{n}\binom{n}{i}v(\rho(y)-\alpha I)^{n-i}\left(x(\operatorname{ad}y-\beta I)^i\right)$$

Note. If $\alpha=\beta=0$ then

$$(vx)(\rho(y))^n=\sum_{i=0}^{n}\binom{n}{i}v\left(\rho(y)^{n-i}\right)\left(x(\operatorname{ad}y)^i\right)$$

Proof. We use induction on n.

If $n=1$ then $\mathrm{LHS}=(vx)y-(\alpha+\beta)vx$ and

$$\begin{aligned}\mathrm{RHS}&=v(\rho(y)-\alpha I)+v(x(\operatorname{ad}y-\beta I))\\&=(vy)x-avx+v[xy]-\beta vx\end{aligned}$$

By the module axioms, $\mathrm{LHS}=\mathrm{RHS}$.

Now assume the result for $n=r$.

$$\begin{aligned}(vx)(\rho(y)-(\alpha+\beta)I)^{r+1}&=\left(\sum_{i=0}^{r}\binom{r}{i}v(\rho(y)-\alpha I)^{r-i}\left(x(\operatorname{ad}y-\beta I)^i\right)\right)(\rho(y)-(\alpha+\beta)I)\\&=\sum_{i=0}^{r}\binom{r}{i}v(\rho(y)-\alpha I)^{r-i}\rho(x_i)(\rho(y)-(\alpha+\beta)I)\end{aligned}$$

where $x_i=x(\operatorname{ad}y-\beta I)^i$. Now

$$\begin{aligned}\rho(x_i)\rho(y)&=\rho(y)\rho(x_i)+\rho[x_iy]\\\rho(x_i)(\rho(y)-(\alpha+\beta)I)&=(\rho(y)-\alpha I)\rho(x_i)+\rho(x_i(\operatorname{ad}y-\beta I))\\&=(\rho(y)-\alpha I)\rho(x_i)\rho(x_{i+1})\end{aligned}$$

So

$$\begin{aligned}(vx)(\rho(y)-(\alpha+\beta)I)^{r+1}&=\sum_{i=0}^{r}\binom{r}{i}v(\rho(y)-\alpha I)^{r-i+1}\rho(x_i)+\sum_{i=0}^{r}\binom{r}{i}v(\rho(y)-\alpha I)^{r-i}\rho(x_{i+1})\\&=\sum_{i=0}^{r}\binom{r}{i}v(\rho(y)-\alpha I)^{r-i+1}\rho(x_i)+\sum_{i=1}^{r+1}\binom{r}{i-1}v(\rho(y)-\alpha I)^{r-i+1}\rho(x_i)\\&=\sum_{i=0}^{r+1}\left(\binom{r}{i}+\binom{r}{i-1}\right)v(\rho(y)-\alpha I)^{r-i+1}\rho(x_i)\end{aligned}$$

which is

$$\sum_{i=0}^{r+1}\binom{r+1}{i}v(\rho(y)-\alpha I)^{r-i+1}\left(x(\operatorname{ad} y-\beta I)^{i}\right)$$

■

We shall call the formula of Proposition 4.4 the *Leibnitz formula for Lie algebras*.

Proof. (of 4.3.) Consider the map $\rho(y): V \to V$. Let V_i be the generalized eigenspace of this map with eigenvalue λ_i. Let $v \in V_i$ and $x \in L$. To show that $vx \in V_i$ we require that $(vx)(\rho(y)-\lambda_i I)^N = 0$ for suitably large N. Apply Leibnitz with $\alpha = \lambda_i$, $\beta = 0$:

$$(vx)(\rho(y)-\lambda_i I)^N = \sum_{i=0}^{N}\binom{N}{i}v(\rho(y)-\lambda_i I)^{N-i}\left(x(\operatorname{ad} y)^{i}\right)$$

$v \in V_i$ so $v(\rho(y)-\lambda_i I)^{N-i} = 0$ if $N-i$ is sufficiently large. Since L is nilpotent $x(\operatorname{ad} y)^i = 0$ if i is suitably large. Thus, if N is suitably large, $(vx)(\rho(y)-\lambda_i I)^N = 0$, and so $vx \in V_i$. Thus, each generalized eigenspace is a submodule of V.

■

Corollary 4.5. *If L is a nilpotent Lie algebra and V is an indecomposable L-module then for all $y \in L$ the linear map $v \mapsto vy$ has only one eigenvalue.*

Proof. We know that $V = V_1 \oplus \ldots \oplus V_r$ for generalized eigenspaces V_i of $\rho(y)$. These are all submodules. Since V is indecomposable, $r = 1$.

■

Proposition 4.6. *Let L be a nilpotent Lie algebra and V an indecomposable L-module. Let $y \in L$ have a single eigenvalue $\lambda(y)$ on V. Then the map $y \mapsto \lambda(y)$ is a 1-dimensional representation of L.*

Proof. Let $\dim(V) = n$. It is clear that $y \mapsto \lambda(y)$ is linear. We must also show that

$$\lambda[xy] = \lambda(x)\lambda(y) - \lambda(y)\lambda(x).$$

The RHS is clearly zero, so we need to show that the LHS is zero as well. Consider the trace function:

$$\begin{aligned}\operatorname{tr}(A) &= \sum_i \alpha_{ii} \\ &= \sum \text{eigenvalues of } A \\ &= -\left(\text{coefficient of } t^{n-1} \text{ in } \chi(t)\right) \\ &\quad \operatorname{tr}(AB) = \operatorname{tr}(BA)\end{aligned}$$

Consider $\rho[xy]:V\to V$; this has only one eigenvalue, $\lambda[xy]$.

$$\operatorname{tr}(\rho[xy]) = n\lambda[xy]$$
$$\begin{aligned}\operatorname{tr}(\rho[xy]) &= \operatorname{tr}(\rho(x)\rho(y)-\rho(y)\rho(x)) \\ &= \operatorname{tr}(\rho(x)\rho(y))-\operatorname{tr}(\rho(y)\rho(x)) \\ &= 0\end{aligned}$$

So $n\lambda[xy]=0$ and $\lambda[xy]=0$, as required.

■

Proposition 4.7. *Let L be a nilpotent Lie algebra and V an indecomposable L-module. Let $y\in L$ and let $\lambda(y)$ be the unique eigenvalue of $\rho(y)$. Define $\sigma(y):V\to V$ by*

$$\sigma(y)=\rho(y)-\lambda(y)I\,.$$

Then
(i) σ is a representation of L;
(ii) $\sigma(y)$ is a nilpotent linear map for all $y\in L$.

Proof. (i) We must show that

$$\sigma[xy]=\sigma(x)\sigma(y)-\sigma(y)\sigma(x)$$

$$\begin{aligned}\text{RHS} &= (\rho(x)-\lambda(x)I)(\rho(y)-\lambda(y)I)-(\rho(y)-\lambda(y)I)(\rho(x)-\lambda(x)I) \\ &= \rho(x)\rho(y)-\rho(y)\rho(x) \\ &= \rho[xy] \\ &= \sigma[xy]+\lambda[xy]I \\ &= \sigma[xy]\end{aligned}$$

(ii) $\rho(y)$ has characteristic polynomial $(t-\lambda(y))^n$, so $\sigma(y)=\rho(y)-\lambda(y)I$ has characteristic polynomial

$$\begin{aligned}\det(tI-\sigma(y)) &= \det((t+\lambda(y))I-\rho(y)) \\ &= (t+\lambda(y)-\lambda(y))^n \\ &= t^n\end{aligned}$$

So, by the Cayley-Hamilton Theorem, $\sigma(y)$ satisfies $\sigma(y)^n=0$.

■

A representation $\sigma:L\to[M_n]$ is called a *nil representation* if each matrix $\sigma(y)$ for $y\in L$ is nilpotent.

Proposition 4.8. *Let L be a nilpotent Lie algebra and σ a nil representation of L. Then σ is equivalent to a representation under which each $x \in L$ is represented by a matrix with zeroes on and below the principal diagonal.*

Proof. Let V be an L-module giving representation σ. Suppose V is irreducible. L is nilpotent, so $L^m = 0$ for some m. So $VL^m = 0$. We show that $VL = 0$ by descending induction, i.e. that $VL^i = 0 \Rightarrow VL^{i-1} = 0$.

Let $x \in L^{i-1}$. $\sigma(x)$ is nilpotent so $\sigma(x)^k = 0$ for some k, i.e. $((vx)x)\ldots x = 0$ (with k x's). So there is a $v \in V$ such that $v \neq 0$ and $vx = 0$. Let U be the set of all such v; we claim U is a submodule of V. Let $u \in U$, $y \in L$.

$$(uy)x = \underbrace{(ux)}_{=0}y + u\underbrace{[yx]}_{\in L^i} = 0$$

So $uy \in U$; hence U is a submodule of V. $U \neq 0$ and V is irreducible, so $U = V$. Hence, $Vx = 0$ for all $x \in L^{i-1}$, i.e. $VL^{i-1} = 0$. $VL^m = 0$ and $VL^i = 0 \Rightarrow VL^{i-1} = 0$, so $VL = 0$. But in this situation every subspace of V is a submodule. Since V is irreducible we have that $\dim(V) = 1$. So $x \mapsto (0) \in [M_1]$.

If the module V is not irreducible then

$$V = V_0 \supset V_1 \supset \ldots \supset V_n = 0$$

where each V_{i+1} is a maximal proper submodule of V_i. V gives a nil representation, so V_i gives a nil representation; V_i/V_{i+1} gives a nil representation of L. But V_i/V_{i+1} is irreducible, so $\dim(V_i/V_{i+1}) = 1$ and $(V_i/V_{i+1})L = 0$, i.e. $V_iL \subset V_{i+1}$.

Choose a basis $e_1, \ldots, e_n$ of V adapted to the chain of subspaces, i.e.

$$\begin{gathered} V_0 \text{ has basis } e_1, \ldots, e_n, \\ V_1 \text{ has basis } e_2, \ldots, e_n \\ \vdots \end{gathered}$$

$V_ix \subset V_{i+1}$ so the matrix representing x with respect to this basis has the required upper triangular form.

■

Corollary 4.9. *Let L be a nilpotent Lie algebra and V an indecomposable L-module. Then we can choose a basis for V such that the matrix representation of x has the form*

$$\begin{pmatrix} \lambda(x) & & * \\ & \ddots & \\ 0 & & \lambda(x) \end{pmatrix}$$

i.e. zeroes below the principal diagonal, and all elements on the principal diagonal equal.

Proof. Follows from 4.5, 4.6, 4.7 and 4.8.

■

Corollary 4.10. *Let L be a nilpotent Lie algebra and V an irreducible L-module. Then* $\dim(V)=1$.

We now consider arbitrary L-modules.

Let L be a nilpotent Lie algebra and V any L-module. A *weight* of V is a 1-dimensional representation $\lambda : L \to \mathbb{C}$ such that there is a $v \in V \setminus \{0\}$ annihilated by some power of $\rho(x) - \lambda(x)I$ for all $x \in L$, where $\rho(x) : V \to V : v \mapsto vx$.

If λ is a weight of V the corresponding *weight space* V_λ of V is

$$V_\lambda = \{ v \in V \mid v \text{ annihilated by some power of } \rho(x) - \lambda(x)I \ \forall x \in L \}$$

V_λ is a subspace of V.

Theorem 4.11. (The Weight Space Decomposition Theorem) *Let L be a nilpotent Lie algebra and V an L-module. Then*
(i) V has only finitely many weights;
(ii) V is the direct sum of its weight spaces;
(iii) each weight space is a submodule of V;
(iv) a basis can be chosen for each λ-weight space V_λ such that the matrix representation on V_λ has the form

$$x \mapsto \begin{pmatrix} \lambda(x) & & * \\ & \ddots & \\ 0 & & \lambda(x) \end{pmatrix}$$

Proof. V may be expressed as a direct sum of indecomposable submodules. Each indecomposable submodule determines a weight λ by 4.6. Let W_λ be the direct sum of all indecomposable components with weight λ. Then $V = \oplus_\lambda W_\lambda$. We need to show that $V_\lambda = W_\lambda$.

Certainly $W_\lambda \subseteq V_\lambda$. Take $v \in V_\lambda$; since $V = \oplus_\mu W_\mu$ we can write $v = \sum_\mu v_\mu$ for $v_\mu \in W_\mu$. For some N, $v(\rho(x) - \lambda(x)I)^N = 0$. So $\sum_\mu v_\mu(\rho(x) - \lambda(x)I)^N = 0$ and each $v_\mu(\rho(x) - \lambda(x)I)^N \in W_\mu$. So each $v_\mu(\rho(x) - \lambda(x)I)^N = 0$. Suppose $\lambda \neq \mu$, so there is an $x \in L$ such that $\lambda(x) \neq \mu(x)$. By 4.9, $\rho(x)$ is represented on W_μ by a matrix of the form

$$\begin{pmatrix} \mu(x) & & * \\ & \ddots & \\ 0 & & \mu(x) \end{pmatrix}$$

so $\rho(x) - \lambda(x)I$ is represented on W_μ by

$$\begin{pmatrix} \mu(x) - \lambda(x) & & * \\ & \ddots & \\ 0 & & \mu(x) - \lambda(x) \end{pmatrix}$$

Choose $x \in L$ such that $\lambda(x) \neq \mu(x)$. Then the matrix of $\rho(x) - \lambda(x)I$ is non-singular on W_μ. So the matrix on $(\rho(x) - \lambda(x)I)^N$ on W_μ is non-singular. So

$$v_\mu(\rho(x) - \lambda(x)I)^N = 0 \Rightarrow v_\mu = 0.$$

So $v_\mu = 0$ for all $\mu \neq \lambda$. So $v = \sum_\mu v_\mu = v_\lambda$. Hence $v \in W_\lambda$, so $V_\lambda \subseteq W_\lambda$, so $V_\lambda = W_\lambda$.

(i) $V_\lambda \neq 0 \Rightarrow W_\lambda \neq 0$ so λ is one of the finite number of weights in our decomposition of V. So there are only finitely many weights.

(ii) $V = \oplus_\lambda W_\lambda$ and $V_\lambda = W_\lambda$ so $V = \oplus_\lambda V_\lambda$.

(iii) $V_\lambda = W_\lambda$ is a submodule of V.

(iv) Follows from 4.9.

■

The decomposition $V = \oplus_\lambda V_\lambda$ is called the *weight space decomposition* of V.

5. CARTAN SUBALGEBRAS

Proposition 5.1. *Let L be a nilpotent Lie algebra. Then the adjoint representation of L is a nil representation.*

Proof. The adjoint representation comes from the L-module L itself, $\operatorname{ad} x : y \mapsto [yx]$. If $y \in L$ then we have $[yx] \in L^2$, $[[yx]x] \in L^3$ and so on. But L is nilpotent, so $L^m = 0$ for some $m \in \mathbb{N}$. I.e., $[[yx]x \ldots x]$ ($m-1$ x's) is zero. So $(\operatorname{ad} x)^{m-1} = 0$; $\operatorname{ad} x$ is nil.

■

The converse is also true.

Theorem 5.2. (Engel's Theorem) *If L is a Lie algebra for which the adjoint representation is a nil representation then L is nilpotent.*

Proof. Suppose not. Choose a maximal nilpotent subalgebra N of L. $[LN] \subseteq L$, so we can regard L as an N-module. $[NN] \subseteq N$, so N is an N-submodule of L. Let M be an N-submodule of L containing N such that M/N is an irreducible N-module. Since N is nilpotent, $\dim(M/N) = 1$ by 4.10. So $\dim(M) = \dim(N) + 1$.

L gives a nil representation of L, and so L gives a nil representation of N. So M gives a nil representation of N. So M/N gives a nil representation of N, $n \mapsto (\alpha)$, nil if and only if $\alpha = 0$. So $(M/N)x \subseteq N/N$ for all $x \in N$. So $[MN] \subseteq N$. Since $\dim(M) = \dim(N) + 1$, $M = N + \mathbb{C}m$. Hence,

$$\begin{aligned}
[MM] &= [N + \mathbb{C}m, N + \mathbb{C}m] \\
&\subseteq [NN] + [N\mathbb{C}m] \\
&\subseteq N + N \\
&= N \\
&\subseteq M
\end{aligned}$$

Thus, M is a subalgebra of L. Since $[NM] \subseteq N$, we also have that $N \lhd M$.

We know $M^2 \subseteq N$. We shall show that for each $i > 0$ there exists an integer n_i such that $M^{n_i} \subseteq N^i$. We use induction on i. If $i = 1$ take $n_i = 2$, since $M^2 \subseteq N$. Assume the statement is true for $i = r$. Then we have an n_r such that $M^{n_r} \subseteq N^r$.

$$\begin{aligned}M^{n_r+1} &= \left[M^{n_r}M\right]\\ &= \left[M^{n_r}, N+\mathbb{C}m\right]\\ &\subseteq \left[M^{n_r}N\right]+M^{n_r}\operatorname{ad}m\\ &\subseteq \left[N^rN\right]+M^{n_r}\operatorname{ad}m\\ &\subseteq N^{r+1}+M^{n_r}\operatorname{ad}m\end{aligned}$$

We now show by induction on j that $M^{n_r+j} \subseteq N^{r+1}M^{n_r}(\operatorname{ad}m)^j$. This is true for $j=1$; assume it for $j=k$:

$$\begin{gathered}M^{n_r+k} \subseteq N^{r+1}+M^{n_r}(\operatorname{ad}m)^k\\ M^{n_r+k+1} \subseteq \left[N^{r+1},M\right]+\left[M^{n_r}(\operatorname{ad}m)^k, M\right]\\ N \triangleleft M \Rightarrow N^{r+1} \triangleleft M \Rightarrow \left[N^{r+1},M\right] \subseteq N^{r+1}\end{gathered}$$

So $M^{n_r+j} \subseteq N^{r+1}M^{n_r}(\operatorname{ad}m)^j$.

The adjoint representation of L is nil, so $(\operatorname{ad}m)^j=0$ for large j. So, for such j, $M^{n_{r+1}} \subseteq N^{r+1}$, so there exists an n_r such that $M^{n_r} \subseteq N^r$.

N is nilpotent, so $N^r=0$ for some r, hence $M^{n_r}=0$. So M is nilpotent, which is a contradiction. So L is nilpotent.

■

We now consider arbitrary Lie algebras. Consider elements $x \in L$ "as far as possible" from 0, in that $\operatorname{ad}0$ has all eigenvalues 0.

We say that $x \in L$ is *regular* if $\operatorname{ad}x : L \to L$ has as few eigenvalues zero as possible.

Example. Let $L=\{A \in [M_2] \mid \operatorname{tr}(A)=0\}$. $\dim(L)=3$. Basis of L:

$$e=\begin{pmatrix}0&1\\0&0\end{pmatrix},\ h=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\ f=\begin{pmatrix}0&0\\1&0\end{pmatrix}$$
$$[he]=2e,\ [hf]=-2f,\ [ef]=h$$

$x \in L$ has the form $ae+bh+cf$ for $a,b,c \in \mathbb{C}$.

$$\begin{aligned}[ex]&=-2be+ch\\ [hx]&=2ae-2cf\\ [fx]&=-ah+2bf\end{aligned}$$

The matrix of $\operatorname{ad}x$ with respect to the basis (e,f,h) is

$$\begin{pmatrix} -2b & c & 0 \\ 2a & 0 & -2c \\ 0 & -a & 2b \end{pmatrix}$$

This has characteristic polynomial

$$\begin{vmatrix} t+2b & -c & 0 \\ -2a & t & 2c \\ 0 & a & t-2b \end{vmatrix} = (t+2b)(t^2-2bt-2ac)+2a(-ct+2bc)$$
$$= t^3-2bt^2-2act+2bt^2-4b^2t-4abc-2acb+4abc$$
$$= t^3-4t(b^2+ac)$$

So the multiplicity of zero as an eigenvalue is

$$1 \text{ if } b^2+ac \neq 0$$
$$3 \text{ is } b^2+ac = 0$$

So $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ is regular if and only if $\begin{vmatrix} a & b \\ c & -a \end{vmatrix} \neq 0$.

Lemma 5.3. *Let M be a subalgebra of L. Then the set of all $x \in L$ such that $[Mx] \subseteq M$ is a subalgebra $\mathcal{N}(M)$ containing M, and $M \lhd \mathcal{N}(M)$. Moreover, $\mathcal{N}(M)$ is the largest subalgebra in which M is an ideal.*

Proof. Easy – see Exercise Sheet 1.

■

We call $\mathcal{N}(M)$ the *idealizer* (or *normalizer*) of M.

Theorem 5.4. *Let x be a regular element of L. Let $H = \mathrm{GES}(\mathrm{ad}\,x, 0)$. Then*
(i) H is a subalgebra of L;
(ii) H is nilpotent;
(iii) $H = \mathcal{N}(H)$.

Proof. (i) Let $y, z \in H$; we need to show that $[yz] \in H$. By Leibnitz,

$$[yx](\mathrm{ad}\,x)^n = \sum_{i=0}^{n} \binom{n}{i} \left[y(\mathrm{ad}\,x)^{n-i}, z(\mathrm{ad}\,x)^i\right]$$

$y \in H$, so $y(\operatorname{ad} x)^{n-i} = 0$ if $n-i$ is large. $z \in H$, so $z(\operatorname{ad} x)^{i} = 0$ if i is large. Hence, $[yz](\operatorname{ad} x)^{n} = 0$ for large n. So $[yz] \in H$, and H is a subalgebra.

(iii) We show that $H = \mathcal{N}(H)$. If $z \in \mathcal{N}(H)$ then $[Hz] \subseteq H$. Now $x \in H$ since $[xx] = 0$. So $[xz], [zx] \in H$. So $[zx]$ is annihilated by some power of $\operatorname{ad} x$. So $z \in H$, so $H \supseteq \mathcal{N}(H)$, hence $H = \mathcal{N}(H)$.

(ii) We show that H is nilpotent by Engel's Theorem, i.e. we show that the adjoint representation of H is nil. Let $\dim(L) = n$, $\dim(H) = l$. Choose a basis $e_1, \ldots, e_l$ of H and extend to a basis $e_1, \ldots, e_n$ of L. Let $y \in H$, $y = \lambda_1 e_1 + \ldots + \lambda_l e_l$, $\lambda_i \in \mathbb{C}$. Consider $\operatorname{ad} y : L \to L$. H is invariant under $\operatorname{ad} y$ since H is a subalgebra. Hence we also have $\operatorname{ad} y : H \to H$ and $\operatorname{ad} y : L/H \to L/H$.

Let $\chi_L(t)$ be the characteristic polynomial of $\operatorname{ad} y$ on L; let $\chi_H(t)$ be the characteristic polynomial of $\operatorname{ad} y$ on H; let $\chi_{L/H}(t)$ be the characteristic polynomial of $\operatorname{ad} y$ on L/H. We claim that $\chi_L(t) = \chi_H(t)\chi_{L/H}(t)$. $\operatorname{ad} y : L \to L$ has a matrix of the block form

$$A = \begin{pmatrix} B_{l \times l} & 0_{l \times (n-l)} \\ D_{(n-l) \times l} & C_{(n-l) \times (n-l)} \end{pmatrix}$$

$$\begin{aligned} \chi_L(t) &= \det(tI_n - A) \\ &= \det\begin{pmatrix} tI_l - B & 0 \\ B & tI_{n-l} - C \end{pmatrix} \\ &= \det(tI_l - B)\det(tI_{n-l} - C) \\ &= \chi_H(t)\chi_{L/H}(t) \end{aligned}$$

Given that $y = \lambda_1 e_1 + \ldots + \lambda_l e_l$, how do the coefficients of $\chi_L(t)$, $\chi_H(t)$ and $\chi_{L/H}(t)$ depend on the λ_i? The entries in A are linear functions of the λ_i. The coefficients in $\chi_L(t)$ etc. are polynomial functions of the λ_i.

Let $\chi_{L/H}(t) = b_0 + b_1 t + b_2 t^2 + \ldots$. We claim that b_0 is not the zero polynomial, for in the special case $y = x$ b_0 is non-zero. Let $\chi_H(t) = t^m(a_0 + a_1 t + a_2 t^2 + \ldots)$, where a_0 is not the zero polynomial. We know that $m \le l$ since $\chi_H(t)$ has degree l. So $\chi_L(t) = t^m(a_0 b_0 + \ldots)$ and $a_0 b_0$ is not the zero polynomial. Choose $\lambda_1, \ldots, \lambda_l$ such that $a_0 b_0 \neq 0$. For this y we have that $\operatorname{ad} y$ has eigenvalue 0 with multiplicity m. So, by the regularity of x, $m \ge l$; hence $m = l$.

So $\chi_H(t) = t^l(a_0 + \ldots)$ has degree l, and so is a multiple of t^l. Hence $\chi_H(t) = t^l$ since characteristic polynomials are monic. By the Cayley-Hamilton Theorem, $\operatorname{ad} y : H \to H$ satisfies $(\operatorname{ad} y)^l = 0$, so the adjoint representation of H is nil.

Hence, by Engel's Theorem, H is nilpotent.

■

The generalized eigenspace of $\operatorname{ad} x$ with eigenvalue zero where $x \in L$ is regular is called a *Cartan subalgebra* of L.

Any two Cartan subalgebras of L have equal dimension; this is called the *rank* of L.

Any Cartan subalgebra is nilpotent and is its own idealizer.

Example. Let $L = \{ A \in [M_2] \mid \operatorname{tr}(A) = 0 \}$.

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is a regular element of L. H will be the Cartan subalgebra given by $\operatorname{ad} h$. $\dim(H) = 1$. So

$$H = \mathbb{C}h = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \middle| \; a \in \mathbb{C} \right\}$$

In general, let H be a Cartan subalgebra of L. Then $[LH] \subseteq L$, so we can regard L as an H-module and decompose L as $L = \oplus_\lambda L_\lambda$, where the L_λ are the weight spaces of L as an H-module.

Consider the special case $\lambda = 0$, i.e. $0 : H \to \mathbb{C}$. L_0 is the 0-weight space.

Proposition 5.5. $L_0 = H$. *Thus, 0 is a weight of* H *on* L.

Proof. By definition,

$$L_0 = \left\{ y \in L \;\middle|\; (\operatorname{ad} x)^k y = 0 \text{ for some } k \text{ and all } x \in H \right\}.$$

But H is nilpotent, so $H^r = 0$, so $[[yx]x \ldots x]$ (with $r-1$ x's) is zero. So $H \subseteq L_0$.

Now suppose if possible that $H \neq L_0$. Then L_0/H is an H-module. By 4.11 the representation of H on L_0/H can be given by matrices

$$z \mapsto \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$$

So there is a non-zero element of L_0/H that is annihilated by all $z \in H$. So there exists a $y \in L_0 \setminus H$ such that $[yz] \in H$ for all $z \in H$. Hence $y \in \mathcal{N}(H)$. But $H = \mathcal{N}(H)$, so $y \in H$, a contradiction. Hence $H = L_0$.

■

Hence we have the *Cartan decomposition* of L as

$$L = H \oplus \left(\oplus_{\lambda \neq 0} L_\lambda\right)$$

The non-zero weights are called *roots*. Let Φ be the set of all roots – a finite set. Then

$$L = H \oplus \left(\oplus_{\alpha \in \Phi} L_\alpha\right)$$

By Lie multiplication we know that $[HH] \subseteq H$ and $[L_\alpha H] \subseteq L_\alpha$.

Proposition 5.6. *Let $\alpha, \beta \in \Phi$. Then*
(i) $[L_\alpha L_\beta] \subseteq L_{\alpha+\beta}$ if $\alpha + \beta \in \Phi$;
(ii) $[L_\alpha L_\beta] \subseteq H = L_0$ if $a + \beta = 0$;
(iii) $[L_\alpha L_\beta] = 0$ if $\alpha + \beta \notin \Phi$ and $\alpha + \beta \neq 0$.

Proof. (i) Let $y \in L_\alpha$, $z \in L_\beta$, $x \in H$. By Leibnitz,

$$[yz]\left(\operatorname{ad} x - \left(\alpha(x) + \beta(x)\right)I\right)^N = \sum_{i=0}^{N} \binom{N}{i} \left[y\left(\operatorname{ad} x - \alpha(x)I\right)^{N-i}, z\left(\operatorname{ad} x - \beta(x)I\right)^i\right]$$

$y \in L_\alpha$, so y is annihilated by large powers of $\left(\operatorname{ad} x - \alpha(x)I\right)$; similarly z is annihilated by large powers of $\left(\operatorname{ad} x - \beta(x)I\right)$. So $[yz]\left(\operatorname{ad} x - \left(\alpha(x) + \beta(x)\right)I\right)^N = 0$ for large N. Hence, $[yz] \in L_{\alpha+\beta}$.

(ii) If $\alpha + \beta = 0$, $[yz] \in L_0 = H$ by 5.5, so $[L_\alpha L_{-\alpha}] \subseteq H$.

(iii) If $\alpha + \beta \notin \Phi \cup \{0\}$ then we deduce that $[yz] = 0$, otherwise there would be a non-zero element in the $(\alpha + \beta)$-weight space.

■

Consider $[L_\alpha L_{-\alpha}] \subseteq H$ for $\alpha \in \Phi$, and $\beta : H \to \mathbb{C}$. Consider the restriction $\beta : [L_\alpha L_{-\alpha}] \to \mathbb{C}$.

Proposition 5.7. *Let $\alpha \in \Phi$. Consider the subspace $[L_\alpha L_{-\alpha}] \subseteq H$. Let $\beta \in \Phi$. Then β restricted to $[L_\alpha L_{-\alpha}]$ is a rational multiple of α.*

Proof. If $-\alpha \notin \Phi$ then $L_{-\alpha} = 0$ and there is nothing to prove, so assume $-\alpha \in \Phi$. Let $\beta \in \Phi$ and consider the functions

$$\ldots, -2\alpha + \beta, -\alpha + \beta, \beta, \alpha + \beta, 2\alpha + \beta, \ldots,$$

all linear functions on H. Since Φ is finite there exist integers p, q such that

$$-p\alpha + \beta, \ldots, -\alpha + \beta, \beta, \alpha + \beta, \ldots, q\alpha + \beta$$

are roots but $-(p+1)\alpha + \beta$ and $(q+1)\alpha + \beta$ are not. If $-(p+1)\alpha + \beta = 0$ the result is clear; similarly if $(q+1)\alpha + \beta = 0$. So we can assume that $-(p+1)\alpha + \beta \neq 0$ and $(q+1)\alpha + \beta \neq 0$.

Let $M = L_{-p\alpha+\beta} \oplus \ldots \oplus L_{q\alpha+\beta}$. M is a subspace of L. Take $y \in L_\alpha$, $z \in L_{-\alpha}$. Then $[yz] \in L_0 = H$.

$M \operatorname{ad} y \subseteq M$ since $\operatorname{ad} y$ takes $L_{i\alpha+\beta}$ to $L_{(i+1)\alpha+\beta}$ and $L_{q\alpha+\beta}$ to 0.
$M \operatorname{ad} z \subseteq M$ since $\operatorname{ad} y$ takes $L_{i\alpha+\beta}$ to $L_{(i-1)\alpha+\beta}$ and $L_{-p\alpha+\beta}$ to 0.

Let $x = [yz] \in H$; $M \operatorname{ad} x \subseteq M$ by the above. We now calculate $\operatorname{tr}_M(\operatorname{ad} x)$ in two different ways:

$$\begin{aligned}
\operatorname{tr}_M(\operatorname{ad} x) &= \operatorname{tr}_M \operatorname{ad}[yz] \\
&= \operatorname{tr}_M(\operatorname{ad} y \operatorname{ad} z - \operatorname{ad} z \operatorname{ad} y) \\
&= \operatorname{tr}_M(\operatorname{ad} y \operatorname{ad} z) - \operatorname{tr}_M(\operatorname{ad} z \operatorname{ad} y) \\
&= 0
\end{aligned}$$

$\operatorname{ad} x$ acts on $L_{i\alpha+\beta}$ as

$$\begin{pmatrix} (i\alpha+\beta)(x) & & * \\ & \ddots & \\ 0 & & (i\alpha+\beta)(x) \end{pmatrix}$$

by 4.11. Hence,

$$\mathrm{tr}_{L_{i\alpha+\beta}}(\mathrm{ad}\,x) = (ia+\beta)(x)\dim(L_{i\alpha+\beta})$$

So

$$\begin{aligned}\mathrm{tr}_M(\mathrm{ad}\,x) &= \sum_{i=-p}^{q} \mathrm{tr}_{L_{i\alpha+\beta}}(\mathrm{ad}\,x) \\ &= \sum_{i=-p}^{q} (i\alpha+\beta)(x)\dim(L_{i\alpha+\beta}) \\ &= \alpha(x)\sum_{i=-p}^{q} i\dim(L_{i\alpha+\beta}) + \beta(x)\sum_{i=-p}^{q}\dim(L_{i\alpha+\beta})\end{aligned}$$

Equating the two traces gives

$$\beta(x)\underbrace{\sum_{i=-p}^{q}\dim(L_{i\alpha+\beta})}_{>0} = -\alpha(x)\sum_{i=-p}^{q} i\dim L_{i\alpha+\beta}$$

And so

$$\beta(x) = -\alpha(x)\frac{\sum_{i=-p}^{q} i\dim(L_{i\alpha+\beta})}{\sum_{i=-p}^{q}\dim(L_{i\alpha+\beta})}$$

So there exists an $r_{\beta,\alpha} \in \mathbb{Q}$ such that $\beta = r_{\beta,\alpha}\alpha$ on $[L_\alpha L_{-\alpha}]$.

■

6. THE KILLING FORM

We define a map $L \times L \to \mathbb{C}$ by $(x, y) \mapsto \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y)$. Define $\langle x, y \rangle = \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y)$. The map $(x, y) \mapsto \langle x, y \rangle$ is called the *Killing form*.

Proposition 6.1. *(i) $\langle \cdot, \cdot \rangle$ is bilinear;*
(ii) $\langle \cdot, \cdot \rangle$ is symmetric;
(iii) $\langle \cdot, \cdot \rangle$ is invariant, i.e. $\langle [xy], z \rangle = \langle x, [yz] \rangle$ for all $x, y, z \in L$.

Proof. (i) Easy.

(ii) Follows from the identity $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

(iii)

$$\begin{aligned}
\langle [xy], z \rangle &= \operatorname{tr}(\operatorname{ad}[xy] \operatorname{ad} z) \\
&= \operatorname{tr}((\operatorname{ad} x \operatorname{ad} y - \operatorname{ad} y \operatorname{ad} x) \operatorname{ad} z) \\
&= \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y \operatorname{ad} z) - \operatorname{tr}(\operatorname{ad} y \operatorname{ad} x \operatorname{ad} z) \\
&= \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y \operatorname{ad} z) - \operatorname{tr}(\operatorname{ad} x \operatorname{ad} z \operatorname{ad} y) \\
&= \operatorname{tr}(\operatorname{ad} x (\operatorname{ad} y \operatorname{ad} z - \operatorname{ad} z \operatorname{ad} y)) \\
&= \operatorname{tr}(\operatorname{ad} x \operatorname{ad}[yz]) \\
&= \langle x, [yz] \rangle
\end{aligned}$$

■

The Killing form is called *non-degenerate* if $\langle x, y \rangle = 0 \;\; \forall y \in L \;\; \Rightarrow x = 0$.

The Killing form is *identically zero* if $\langle x, y \rangle = 0 \;\; \forall x, y \in L$.

Proposition 6.2. *Let $I \lhd L$ and $x, y \in I$. Then $\langle x, y \rangle_I = \langle x, y \rangle_L$. Thus, the Killing form of L restricted to I is the Killing form of I.*

Proof. Choose a basis of I and extend to a basis of L. With respect to this basis, since I is an ideal, $\operatorname{ad} x$ is represented by a matrix of the block form

$$\begin{pmatrix} A_1 & 0 \\ A_2 & 0 \end{pmatrix}$$

Similarly, $\operatorname{ad} y$ is represented by

$$\begin{pmatrix} B_1 & 0 \\ B_2 & 0 \end{pmatrix}$$

So $\operatorname{ad} x \operatorname{ad} y$ is represented by

$$\begin{pmatrix} A_1 B_1 & 0 \\ A_2 B_1 & 0 \end{pmatrix}$$

Thus, $\operatorname{tr}(\operatorname{ad} x \operatorname{ad} y) = \operatorname{tr}(A_1 B_1)$. But A_1 is the matrix of $\operatorname{ad} x$ on I and B_1 is the matrix of $\operatorname{ad} y$ on I. Thus,

$$\langle x, y \rangle_I = \operatorname{tr}(A_1 B_1) = \langle x, y \rangle_L .$$

■

For any subspace M of L define the *perpendicular space* $M^\perp$ by

$$M^\perp = \{ x \in L \mid \langle x, y \rangle = 0 \ \forall y \in M \}.$$

$M^\perp$ is also a subspace of L.

Lemma 6.3. $I \lhd L \Rightarrow I^\perp \lhd L$.

Proof. Let $x \in I^\perp$ and $y \in L$; we show that $[xy] \in I^\perp$. Let $z \in I$.

$$\langle [xy], z \rangle = \langle x, [yz] \rangle = 0$$

$[yz] \in I$, $x \in I^\perp$. Hence $[xy] \in I^\perp$.

■

In particular, $L^\perp \lhd L$:

$$L^\perp = \{ x \in L \mid \langle x, y \rangle = 0 \ \forall y \in L \}$$

So $L^\perp = 0$ iff $\langle \cdot, \cdot \rangle$ is non-degenerate; $L^\perp = L$ iff $\langle \cdot, \cdot \rangle$ is identically zero.

Proposition 6.4. *Let L be a Lie algebra with $L \neq 0$, $L^2 = L$. Let H be a Cartan subalgebra of L. Then there is an $x \in H$ such that $\langle x, x \rangle \neq 0$.*

Proof. Consider the Cartan decomposition of L as an H-module:

$$L = \oplus_\lambda L_\lambda$$
$$L^2 = [LL] = [\oplus_\lambda L_\lambda, \oplus_\mu L_\mu] = \sum_{\lambda,\mu} [L_\lambda L_\mu]$$

Now $[L_\lambda L_\mu] \subseteq L_{\lambda+\mu}$ by 5.6, where $L_{\lambda+\mu} = 0$ if $\lambda + \mu$ is not a weight. Now $L_0 = H$, so, since $L^2 = L$, we have

$$H = \sum_\lambda [L_\lambda L_{-\lambda}] = [HH] + \sum_{\alpha\in\Phi} [L_\alpha L_{-\alpha}]$$

Now L is not nilpotent since $L^2 = L$, but H is nilpotent, so $H \neq L$. So there exists a root $\beta \in \Phi$. β is a 1-dimensional representation of H, $\beta \neq 0$. β vanishes on $[HH]$ since if $x, y \in H$ then $\beta[xy] = \beta(x)\beta(y) - \beta(y)\beta(x) = 0$. So $H = [HH]$.

Hence, there exists an $\alpha \in \Phi$ with $[L_\alpha L_{-a}] \neq 0$. In particular, $L_{-\alpha} \neq 0$. Also, β does not vanish on $[L_\alpha L_{-\alpha}]$. Choose $x \in [L_\alpha L_{-\alpha}]$ such that $\beta(x) \neq 0$. By definition, $\langle x, x \rangle = \mathrm{tr}(\mathrm{ad}\, x \,\mathrm{ad}\, x)$. $\mathrm{ad}\, x$ acts on L_λ by

$$\begin{pmatrix} \lambda(x) & & 0 \\ & \ddots & \\ 0 & & \lambda(x) \end{pmatrix}$$

by 4.11. So $(\mathrm{ad}\, x)^2$ acts on L_λ by

$$\begin{pmatrix} \lambda(x)^2 & & 0 \\ & \ddots & \\ 0 & & \lambda(x)^2 \end{pmatrix}$$

So $\langle x, x \rangle = \sum_\lambda \dim(L_\lambda)\lambda(x)^2$. However, by 5.7 there exists an $r_{\lambda,\alpha} \in \mathbb{Q}$ such that $\lambda(x) = r_{\lambda,\alpha}\alpha(x)$ since $x \in [L_\alpha L_{-\alpha}]$. So $\langle x, x \rangle = \alpha(x)^2 \sum_\lambda \dim(L_\lambda) r_{\lambda,\alpha}^2$. In particular, $\beta(x) = r_{\beta,\alpha}\alpha(x)$. Now $\beta(x) \neq 0$, so $\alpha(x) = 0$ and $r_{\beta,\alpha} \neq 0$. So

$$\langle x, x \rangle = \underbrace{\alpha(x)^2}_{\neq 0} \underbrace{\sum_\lambda \dim(L_\lambda) r_{\lambda,\alpha}^2}_{>0}$$

So $\langle x, x \rangle \neq 0$.

■

Theorem 6.5. *If the Killing form of L is identically zero then L is soluble.*

Proof. Use induction on $\dim(L)$. If $\dim(L)=1$ then L is certainly soluble. If $\dim(L)>1$, $L^2 \neq L$ by 6.4, and $L^2 \lhd L$. The Killing form on L^2 is the restriction of that on L, and so is identically zero. $\dim(L^2)<\dim(L)$; by induction L^2 is soluble. L/L^2 is abelian, and so soluble. Hence L is soluble.

■

Note. The converse is not true. Consider a Lie algebra of dimension 2, basis $\{x,y\}$, with $[xy]=x$.

$$\operatorname{ad} y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ (\operatorname{ad} y)^2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

So $\langle y,y\rangle = \operatorname{tr}\left((\operatorname{ad} y)^2\right)=1$.

For which Lie algebras is the Killing form non-degenerate?

Let R be the soluble radical of L. Then L is semisimple if and only if $R=0$.

Theorem 6.6. *The Killing form on L is non-degenerate if and only if L is semisimple.*

Proof. Suppose the Killing form on L is degenerate. Then $L^\perp \neq 0$. $L^\perp \lhd L$ by 6.3. So the Killing form of L restricted to $L^\perp$ is the Killing form of $L^\perp$. Hence, the Killing form of $L^\perp$ is identically zero, since $x,y \in L^\perp \Rightarrow \langle x,y\rangle = 0$. Hence, by 6.5, $L^\perp$ is soluble. $L^\perp$ is a non-zero soluble ideal of L, and so L is not semisimple.

Conversely, suppose L is not semisimple. Then $R \neq 0$, so

$$R = R^{(0)} \supseteq R^{(1)} \supseteq \ldots R^{(k-1)} \supseteq R^{(k)} = 0$$

Let $I = R^{(k-1)}$; $I \neq 0$ but $I^2 = 0$, and $I \lhd L$. Choose a basis of I and extend to a basis of L. Let $x \in I$ and $y \in L$. With respect to this basis, $\operatorname{ad} x : L \to L$ has matrix

$$\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}$$

$\operatorname{ad} y : L \to L$ has matrix

$$\begin{pmatrix} B_1 & 0 \\ B_2 & B_3 \end{pmatrix}$$

So $\operatorname{ad} x \operatorname{ad} y : L \to L$ has matrix

$$\begin{pmatrix} 0 & 0 \\ AB_1 & 0 \end{pmatrix}$$

Hence $\langle x, y \rangle = \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y) = 0$. Thus $I \subseteq L^{\perp}$. Thus $L^{\perp} \neq 0$, which implies that the Killing form is degenerate.

■

Suppose L_1, L_2 are Lie algebras. We can define the *direct sum* $L_1 \oplus L_2$ to be the set of pairs $(x_1, x_2) \in L_1 \times L_2$ with $[(x_1, x_2)(y_1, y_2)] = ([x_1 y_1], [x_2 y_2])$. Similarly, we can define $L_1 \oplus \ldots \oplus L_k$.

The 1-dimensional Lie algebra is simple, and is called the *trivial* simple Lie algebra.

Theorem 6.7. *A Lie algebra is semisimple if and only if it is the direct sum of simple nontrivial Lie algebras.*

Proof. Let L be a semisimple Lie algebra. If L is simple L is nontrivial and there is nothing to prove. So assume L is not simple. Choose a minimal non-zero ideal $I \lhd L$, $I \neq L$. Consider $I^{\perp}$: $I^{\perp} \lhd L$ as well.

$$x \in I^{\perp} \Leftrightarrow \langle x, y \rangle = 0 \text{ for all } y \in I.$$

The Killing form is non-degenerate on L. This gives $\dim(I)$ linearly independent conditions on x, since the form is non-degenerate. So $\dim(I^{\perp}) = \dim(L) - \dim(I)$ by the Rank-Nullity Formula.

Now consider $I \cap I^{\perp}$. $I \cap I^{\perp}$ is an ideal, so the Killing form on $I \cap I^{\perp}$ is the restriction of that on L. But $x, y \in I \cap I^{\perp} \Rightarrow \langle x, y \rangle = 0$. So $I \cap I^{\perp}$ is soluble by 6.5. Since L is semisimple, $I \cap I^{\perp} = 0$.

$$\begin{aligned} \dim(I + I^{\perp}) &= \dim(I) + \dim(I^{\perp}) - \dim(I \cap I^{\perp}) \\ &= \dim(I) + \dim(I^{\perp}) \\ &= \dim(L) \end{aligned}$$

So $L = I + I^{\perp}$, and $I \cap I^{\perp} = 0$, so $L = I \oplus I^{\perp}$ as a direct sum of subspaces. Let $x \in I$ and $y \in I^{\perp}$. Then $[xy] \in [II^{\perp}] \subseteq I \cap I^{\perp} = 0$. So $L = I \oplus I^{\perp}$ as a direct sum of Lie algebras:

$$[a + b, a' + b'] = [aa'] + [bb']$$

We now show that I is simple. Let $J \triangleleft I$.

$$[JL] = [JI] + [JI^{\perp}] = [JI] \subseteq J$$

So J is an ideal of L contained in I. But I is minimal, so either $J = 0$ or $J = I$. Hence, I is simple.

We now show that $I^{\perp}$ is semisimple. Let J be a soluble ideal of $I^{\perp}$.

$$[JL] = [JI] + [JI^{\perp}] = [JI^{\perp}] \subseteq J$$

Thus, J is a soluble ideal of L. Since L is semisimple, $J = 0$, and so $I^{\perp}$ is semisimple.

So $L = I \oplus I^{\perp}$, I simple and $I^{\perp}$ semisimple. By induction, $I^{\perp}$ is the direct sum of simple nontrivial Lie algebras. Hence, L has this property.

Conversely, let $L = L_1 \oplus \ldots \oplus L_r$, where each L_i is simple and nontrivial. L_i is semisimple, and so its Killing form is non-degenerate by 6.6. For each i, $L_i \triangleleft L$. If $i \neq j$ and $x_i \in L_i$, $x_j \in L_j$, then $\langle x_i, x_j \rangle = 0$. For if $y \in L$ then $y \operatorname{ad} x_i = [yx_i] \in L_i$; $y \operatorname{ad} x_j = [yx_j] \in L_j$. So $y \operatorname{ad} x_i \operatorname{ad} x_j = [[yx_i]x_j] \in L_i \cap L_j = 0$. Thus $\operatorname{ad} x_i \operatorname{ad} x_j = 0$, so $\operatorname{tr}(\operatorname{ad} x_i \operatorname{ad} x_j) = 0$, so $\langle x_i, x_j \rangle = 0$.

We show $L^{\perp} = 0$. Let $x \in L^{\perp}$, $x = x_1 + \ldots + x_r$, $x_i \in L_i$. Let $y_i \in L_i$. Then $\langle x, y_i \rangle = \langle x_i, y_i \rangle = 0$, since $x \in L^{\perp}$. So $\langle x_i, y_i \rangle = 0$ for all $y_i \in L_i$. This implies $x_i = 0$ since the Killing form on L_i is non-degenerate. So $x = 0$, and so $L^{\perp} = 0$. Hence the Killing form on L is non-degenerate, and so L is semisimple.

■

7. THE LIE ALGEBRA $\mathfrak{sl}_n(\mathbb{C})$

$$\mathfrak{sl}_n(\mathbb{C}) = \{ A \in [M_n] \mid tr(A) = 0 \}$$

$\mathfrak{sl}_n(\mathbb{C})$ is an ideal of the Lie algebra $[M_n] = \mathfrak{gl}_n(\mathbb{C})$.

Theorem 7.1. $\mathfrak{sl}_n(\mathbb{C})$ *is a simple Lie algebra.*

Proof. Every ideal of $\mathfrak{sl}_n(\mathbb{C})$ is an ideal of $[M_n]$. We shall show that if $I \triangleleft [M_n]$ and $I \subseteq \mathfrak{sl}_n(\mathbb{C})$ then $I = \mathfrak{sl}_n(\mathbb{C})$ or $I = 0$.

Let I be a non-zero ideal of $[M_n]$ contained in $\mathfrak{sl}_n(\mathbb{C})$. Let $x \in I \setminus \{0\}$. Then

$$x \in \sum_{p,q} x_{pq} E_{pq} \text{ , } x_{pq} \in \mathbb{C} \text{ not all zero,}$$

where E_{pq} is an $n \times n$ matrix with 1 in the (p,q)th position and zeroes elsewhere.

Case 1: Suppose $\exists i \neq j$ such that $x_{ij} \neq 0$. Then

$$[E_{ii}x] = \sum_q x_{iq} E_{iq} - \sum_p x_{pi} E_{pi} \in I$$
$$[[E_{ii}x]E_{jj}] = x_{ij} E_{ij} + x_{ji} E_{ji} \in I$$
$$[E_{ii} - E_{jj}, x_{ij}E_{ij} - x_{ji}E_{ji}] = 2x_{ij}E_{ij} - 2x_{ji}E_{ji} \in I$$

So $4x_{ij}E_{ij} \in I$, $x_{ij} \neq 0$, so $E_{ij} \in I$.

Case 2: Suppose $x_{ij} = 0 \ \forall i \neq j$. Then $x = \sum_p x_{pp} E_{pp}$; $\sum_p x_{pp} = 0$, so not all the x_{pp} are equal. Suppose $x_{ii} \neq x_{jj}$.

$$[xE_{ij}] = (x_{ii} - x_{jj}) E_{ij} \in I$$

So $E_{ij} \in I$.

So in either case $E_{ij} \in I$ for some $i \neq j$. Let $q \neq i, j$. Then $[E_{ij}, E_{iq}] = E_{iq} \in I$. So I contains all E_{iq} with $q \neq i$. Let $p \neq i, q$. Then $[E_{pi}, E_{iq}] = E_{pq} \in I$. So $E_{pq} \in I$ for all $p \neq q$. For $p \neq q$, $[E_{pq}, E_{qp}] = E_{pp} - E_{qq} \in I$. So $I = \mathfrak{sl}_n$. Hence, $\mathfrak{sl}_n(\mathbb{C})$ is simple.

■

It is easy to see that $\dim(\mathfrak{sl}_n(\mathbb{C})) = n^2 - 1$; assume $n \geq 2$. We now find a regular element of $\mathfrak{sl}_n(\mathbb{C})$.

Proposition 7.2. *Let*

$$x = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \in \mathfrak{sl}_n(\mathbb{C})$$

with $\sum_i \lambda_i = 0$ *and* $i \neq j \Rightarrow \lambda_i \neq \lambda_j$. *Then* x *is regular.*

Proof. Take a basis of $\mathfrak{sl}_n(\mathbb{C})$:

$$\{E_{ij} \mid i \neq j\} \cup \{E_{11} - E_{22}, E_{22} - E_{33}, \ldots, E_{n-1,n-1} - E_{nn}\}$$

Consider the matrix of $\operatorname{ad} x$ with respect to this basis:

$$[E_{ij} x] = (\lambda_j - \lambda_i) E_{ij} \text{ for } i \neq j$$
$$[E_{ii} - E_{i+1,i+1}, x] = 0$$

So the $(n^2 - 1) \times (n^2 - 1)$ matrix of $\operatorname{ad} x$ with respect to this basis is

$$\begin{pmatrix} \ddots & & & & 0 \\ & \lambda_j - \lambda_i & & & \\ & & \ddots & & \\ & & & 0 & \\ 0 & & & & \ddots \end{pmatrix}$$

The characteristic polynomial is $t^{n-1} \prod_{i \neq j} (t - \lambda_i + \lambda_j)$; the multiplicity of zero as an eigenvalue of $\operatorname{ad} x$ is $n - 1$.

Now let $y \in \mathfrak{sl}_n(\mathbb{C})$. $\operatorname{ad} y$ is similar to a matrix in Jordan canonical form, say

$$\begin{pmatrix} J_{m_1}(\mu_1) & & 0 \\ & \ddots & \\ 0 & & J_{m_r}(\mu_r) \end{pmatrix} \tag{$*$}$$

where $J_m(\mu)$ is the $m \times m$ matrix

$$\begin{pmatrix} \mu & 1 & & 0 \\ & \mu & \ddots & \\ & & \ddots & 1 \\ 0 & & & \mu \end{pmatrix}$$

$J_m(\mu) = \mu I_m + J_m$, where $J_m = J_m(0)$. J_m^k has 1's on the k th diagonal above the principal diagonal and zeroes elsewhere. We claim that $J_m(\mu)$ commutes with any matrix of the form $\alpha I_m + \beta J_m + \beta J_m^2 + \ldots$, so $(*)$ commutes with all matrices of the block form

$$\begin{pmatrix} \alpha_1 & \beta_1 & & 0 & & & & & \\ & \alpha_1 & \ddots & & \cdots & & 0 & & \\ & & \ddots & \beta_1 & & & & & \\ 0 & & & \alpha_1 & & & & & \\ & \vdots & & & \ddots & & \vdots & & \\ & & & & & \alpha_r & \beta_r & & 0 \\ & & & & & & \alpha_r & \ddots & \\ & 0 & & & \cdots & & & \ddots & \beta_r \\ & & & & & 0 & & & \alpha_r \end{pmatrix}$$

These matrices form a vector space of dimension $m_1 + m_2 + \ldots + m_r = n$.

Assume $m_1\alpha_1 + \ldots + m_r\alpha_r = 0$, i.e. the matrix is in $\mathfrak{sl}_n(\mathbb{C})$. We have a vector space of dimension $n-1$. All these matrices lie in the zero eigenspace of $\operatorname{ad} y$, so the multiplicity of zero as an eigenvalue of $\operatorname{ad} y$ is at least $n-1$.

Thus, x is regular.

■

Proposition 7.3. *The subalgebra of diagonal matrices in* $\mathfrak{sl}_n(\mathbb{C})$ *is a Cartan subalgebra.*

Proof. Let

$$x = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$\lambda_i \neq \lambda_j$ for $i \neq j$, $\sum_i \lambda_i = 0$. x is regular. Let

$$H = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \in \mathfrak{sl}_n(\mathbb{C}) \right\}$$

$\dim(H) = n-1$. If $y \in H$ then $[yx] = 0$, so $H \subseteq \mathrm{ES}(\operatorname{ad} x, 0) \subseteq \mathrm{GES}(\operatorname{ad} x, 0)$. But

$$\dim(H) = n - 1 = \dim(\mathrm{GES}(\operatorname{ad} x, 0)).$$

Thus, H is a Cartan subalgebra of $\mathfrak{sl}_n(\mathbb{C})$.

■

Proposition 7.4. *Let $L = \mathfrak{sl}_n(\mathbb{C})$ and let H be the diagonal subalgebra. Then*

$$L = H \oplus \left(\oplus_{i \neq j} \mathbb{C}E_{ij}\right)$$

is the Cartan decomposition of L with respect to H.

Proof. Clearly $L = H \oplus \left(\oplus_{i \neq j} \mathbb{C}E_{ij}\right)$ as a direct sum of vector spaces. Let

$$h = \lambda_1 E_{11} + \ldots + \lambda_n E_{nn} \in H.$$

Then $[E_{ij} h] = (\lambda_j - \lambda_i) E_{ij}$. So $\mathbb{C}E_{ij}$ is an H-module giving a 1-dimensional representation

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \mapsto \lambda_j - \lambda_i \neq 0 \text{ since } i \neq j.$$

So this 1-dimensional representation of H is a weight and $\mathbb{C}E_{ij}$ lies in the weight space.

$$\dim(H) + \sum \dim(\text{weight space}) = \dim(L) = \dim(H) + \sum \dim(\mathbb{C}E_{ij})$$

So $\mathbb{C}E_{ij}$ is the full weight space, giving $L = H \oplus \left(\oplus_{i \neq j} \mathbb{C}E_{ij}\right)$ as a direct sum of Lie algebras.

■

Summary. $\mathfrak{sl}_n(\mathbb{C}) = [n \times n \text{ matrices of trace } 0]$. $\mathfrak{sl}_n(\mathbb{C})$ is simple.

$$H = \left\{ \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \;\middle|\; \sum_{i=1}^{n} \lambda_i = 0 \right\}$$

is a Cartan subalgebra. $L = H \oplus \left(\oplus_{i \neq j} \mathbb{C} E_{ij} \right)$. The roots are

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \mapsto \lambda_j - \lambda_i \text{ for } i \neq j .$$

8. THE CARTAN DECOMPOSITION

Throughout this chapter, let L be a semisimple Lie algebra and H a Cartan subalgebra of L.

$$L = \oplus_\lambda L_\lambda,\ \lambda \text{ weights – the Cartan decomposition}$$

$$H = L_0$$
$$L = H \oplus \left(\oplus_{\alpha\in\Phi} L_\alpha\right)$$

Proposition 8.1. *If $x \in L_\lambda$, $y \in L_\mu$, $\lambda \neq -\mu$ then $\langle x, y\rangle = 0$.*

Proof.

$$\langle x, y\rangle = \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y)$$
$$L_V \operatorname{ad} x \subseteq L_{V+\lambda}$$
$$L_V \operatorname{ad} x \operatorname{ad} y \subseteq L_{V+\lambda+\mu}$$

So if $\lambda + \mu \neq 0$ then $V + \lambda + \mu \neq V$. Choose a basis of L adapted to a Cartan decomposition. $\operatorname{ad} x \operatorname{ad} y$ is then represented by a block matrix with zero blocks on the diagonal:

$$\begin{pmatrix} 0 & & & * \\ & 0 & & \\ & & \ddots & \\ * & & & 0 \end{pmatrix}$$

So $\langle x, y\rangle = \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y) = 0$.

■

Proposition 8.2. *If $\alpha \in \Phi$ then $-\alpha \in \Phi$.*

Proof. Suppose if possible that $-\alpha \notin \Phi$. Then $L_{-\alpha} = 0$ and $L_\alpha \neq 0$. Let $x \in L_\alpha$. $\langle x, y\rangle = 0$ for all $y \in L_\lambda$, for all λ. Hence $\langle x, y\rangle = 0$ for all $y \in L$. But $\langle \cdot,\cdot\rangle$ is non-degenerate, so $x = 0$. Thus $L_\alpha = 0$, a contradiction.

■

Proposition 8.3. *The Killing form of L remains non-degenerate on restriction to H; i.e. if $x \in H$ and $\langle x, y\rangle = 0$ for all $y \in H$ then $x = 0$.*

Note. We are saying that the Killing form of L restricted to H is non-degenerate, *not* that the Killing form of H is non-degenerate. In fact, the Killing form of H is degenerate since H is nilpotent.

Proof. Let $x \in H$ satisfy $\langle x, y \rangle = 0$ for all $y \in H$. We also have $\langle x, y \rangle = 0$ for all $y \in L_\lambda$, $\lambda \neq 0$, by 8.1. So $\langle x, y \rangle = 0$ for all $y \in L$. So $x = 0$.

■

Proposition 8.4. $[HH] = 0$*; i.e.* H *is abelian.*

Proof. Suppose $x \in [HH]$ and let $y \in H$.

$$\langle x, y \rangle = \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y)$$
$$L = \oplus_\lambda L_\lambda \text{, each } L_\lambda \text{ an } H\text{-module.}$$

On L_λ we have $\operatorname{ad} x$ represented by

$$\begin{pmatrix} \lambda(x) & & * \\ & \ddots & \\ 0 & & \lambda(x) \end{pmatrix}$$

and $\operatorname{ad} y$ represented by

$$\begin{pmatrix} \lambda(y) & & * \\ & \ddots & \\ 0 & & \lambda(y) \end{pmatrix}$$

Hence, $\operatorname{ad} x \operatorname{ad} y$ is represented by

$$\begin{pmatrix} \lambda(x)\lambda(y) & & * \\ & \ddots & \\ 0 & & \lambda(x)\lambda(y) \end{pmatrix}$$

So $\langle x, y \rangle = \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y) = \sum_\lambda \dim(L_\lambda)\lambda(x)\lambda(y)$. Let $h_1, h_2 \in H$. Then

$$\lambda[h_1 h_2] = \lambda(h_1)\lambda(h_2) - \lambda(h_2)\lambda(h_1) = 0$$

Since λ is a 1-dimensional representation. Hence $\lambda(x)=0$ for all $x \in [HH]$, so $\langle x,y \rangle = 0$ for all $y \in H$. Since $\langle \cdot,\cdot \rangle$ is non-degenerate, $x=0$. So $[HH]=0$; i.e. H is abelian.

■

Let H^* denote the *dual space* of H, $\mathrm{Hom}(H,\mathbb{C})$. Then $\dim(H^*)=\dim(H)$. We can define a map $H \to H^*$, $h \mapsto h^*$, by $h^*(x)=\langle h,x \rangle$ for $x \in H$.

Lemma 8.5. *The map $h \mapsto h^*$ is an isomorphism of vector spaces.*

Proof. The map is clearly linear. Suppose h is in the kernel of this map, i.e. $h^*=0$. Then $\langle h,x \rangle = 0$ for all $x \in H$. Since the Killing form is non-degenerate, $h=0$. So the kernel is trivial.

$$\dim(\text{image}) = \dim(H) - \dim(\text{kernel}) = \dim(H) = \dim(H^*)$$

■

Let $\alpha \in \Phi$. Then $\alpha \in H^*$, so there is a unique $h_\alpha \in H$ such that $\alpha(x) = \langle h_\alpha, x \rangle$ for all $x \in H$.

Proposition 8.6. *The h_α, as defined above, and taken over all $\alpha \in \Phi$, span H.*

Proof. Suppose not. Then there is an $x \in H$, $x \neq 0$, such that $\langle h_\alpha, x \rangle = 0$ for each $\alpha \in \Phi$. Hence $\alpha(x)=0$ for all $\alpha \in \Phi$. Let $y \in H$.

$$\langle x,y \rangle = \mathrm{tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y) = \sum\nolimits_\lambda \dim(L_\lambda)\lambda(x)\lambda(y)$$

Since $\lambda(x)=0$ for all weights λ, $\langle x,y \rangle = 0$ for all $y \in H$. So $x=0$, a contradiction.

■

Proposition 8.7. *For $\alpha \in \Phi$, $h_\alpha \in [L_\alpha L_{-\alpha}]$, a subspace of H.*

Proof. Consider L_α as an H-module: it contains a 1-dimensional submodule $\mathbb{C}e_\alpha$, $e_\alpha \neq 0$. $[e_\alpha x] = \alpha(x)e_\alpha$ for $x \in H$. Let $y \in L_{-\alpha}$. We show that $[ye_\alpha] = \langle y, e_\alpha \rangle h_\alpha$. To see this let $z = [ye_\alpha] - \langle y, e_\alpha \rangle h_\alpha$ and let $x \in H$.

$$\begin{aligned}\langle z,x\rangle &= \langle [ye_\alpha],x\rangle - \langle y,e_\alpha\rangle\langle h_\alpha,x\rangle \\ &= \langle y,[e_\alpha x]\rangle - \langle y,e_\alpha\rangle\alpha(x) \\ &= \alpha(x)\langle y,e_\alpha\rangle - \langle y,e_\alpha\rangle\alpha(x) \\ &= 0\end{aligned}$$

So $\langle z,x\rangle = 0$ for all $x \in H$, so $z = 0$. So $[ye_\alpha] = \langle y,e_\alpha\rangle h_\alpha$. We can find $y \in L_{-\alpha}$ such that $\langle y,e_\alpha\rangle \neq 0$. If not, e_α is orthogonal to $L_{-\alpha}$ and to each L_λ with $\lambda \neq -\alpha$, so $e_\alpha \in L^\perp$, so $e_\alpha = 0$, a contradiction. Choose such a y, then

$$h_\alpha = \left[\frac{y}{\langle y,e_\alpha\rangle}e_\alpha\right] \in [L_{-\alpha}L_\alpha]$$

■

Proposition 8.8. *Let* $\alpha \in \Phi$. *Then* $\langle h_\alpha,h_\alpha\rangle \neq 0$.

Proof. Let $\beta \in \Phi$. We know that $h_\alpha \in [L_{-\alpha}L_\alpha]$. There exists $r_{\beta,\alpha} \in \mathbb{Q}$ such that $\beta = r_{\beta,\alpha}\alpha$ on $[L_{-\alpha}L_\alpha]$.

$$\langle h_\beta,h_\alpha\rangle = \beta(h_\alpha) = r_{\beta,\alpha}\alpha(h_\alpha) = r_{\beta,\alpha}\langle h_\alpha,h_\alpha\rangle$$

If $\langle h_\alpha,h_\alpha\rangle = 0$ then $\langle h_\beta,h_\alpha\rangle = 0$ for all $\beta \in \Phi$. But the set $\{h_\beta \mid \beta \in \Phi\}$ spans H. Hence $\langle x,h_\alpha\rangle = 0$ for all $x \in H$. This implies $h_\alpha = 0$, so $\alpha = 0$, a contradiction.

■

Theorem 8.9. *If* $\alpha \in \Phi$ *then* $\dim(L_\alpha) = 1$.

Note. Of course, $\dim(H) = \dim(L_0)$ is not generally 1.

Proof. Let M be the subspace of L given by

$$M = \mathbb{C}e_\alpha \oplus \mathbb{C}h_a \oplus L_{-\alpha} \oplus L_{-2\alpha} \oplus \ldots$$

where $\mathbb{C}e_\alpha$ is a 1-dimensional H-submodule of L_α. There are only finitely many summands since Φ is finite. Recall from the proof of 8.7 that there is an $e_{-\alpha} \in L_{-\alpha}$ such that $[e_{-\alpha}e_\alpha] = h_\alpha$. Also, for any $y \in L_{-\alpha}$, $[ye_\alpha] = \langle y,e_\alpha\rangle h_\alpha$.

We first show that $M \,\mathrm{ad}\, e_\alpha \subseteq M$:

$$[e_\alpha e_\alpha] = 0$$
$$[h_\alpha e_\alpha] = -[e_\alpha h_\alpha] = -\alpha(h_\alpha)e_\alpha$$

Let $y \in L_{-\alpha}$. Then

$$[y e_\alpha] = \langle y, e_\alpha \rangle h_\alpha \in \mathbb{C}h_\alpha$$
$$[L_{-i\alpha} e_\alpha] \subseteq L_{-(i-1)\alpha} \text{ for } i \geq 2$$

Secondly, we show that $M \operatorname{ad} e_{-\alpha} \subseteq M$.

$$[e_\alpha e_{-\alpha}] = -h_\alpha$$
$$[h_\alpha e_{-\alpha}] = -[e_{-\alpha} h_\alpha] = \alpha(h_\alpha)e_{-\alpha} \in L_{-\alpha}$$
$$[L_{-i\alpha} e_{-\alpha}] \subseteq L_{-(i+1)\alpha} \text{ for } i \geq 1$$

$[e_{-\alpha} e_\alpha] = h_\alpha$, so $M \operatorname{ad} h_a \subseteq M$. We now calculate $\operatorname{tr}_M(\operatorname{ad} h_\alpha)$ in two different ways.

$$\operatorname{tr}_M(\operatorname{ad} h_\alpha) = \operatorname{tr}_M(\operatorname{ad} e_{-a} \operatorname{ad} e_\alpha - \operatorname{ad} e_\alpha \operatorname{ad} e_{-\alpha}) = 0$$

$$\begin{aligned} \operatorname{tr}_M(\operatorname{ad} h_\alpha) &= \alpha(h_\alpha) + 0 - \dim(L_{-\alpha})a(h_\alpha) - 2\dim(L_{-2\alpha})\alpha(h_\alpha) - 3\dim(L_{-3\alpha})\alpha(h_\alpha) - \ldots \\ &= \alpha(h_\alpha)(1 - \dim(L_{-\alpha}) - 2\dim(L_{-2\alpha}) - \ldots) \\ &= \langle h_\alpha, h_\alpha \rangle (1 - \dim(L_{-\alpha}) - 2\dim(L_{-2\alpha}) - \ldots) \end{aligned}$$

Since $\langle h_\alpha, h_\alpha \rangle \neq 0$, $1 - \dim(L_{-\alpha}) - 2\dim(L_{-2\alpha}) - \ldots = 0$. Since $-\alpha \in \Phi$, $\dim(L_{-\alpha}) \geq 1$. Hence $\dim(L_{-\alpha}) = 1$ and $\dim(L_{-i\alpha}) = 0$ for $i \geq 2$.

So, interchanging $\alpha \leftrightarrow -\alpha$, we have that for each $\alpha \in \Phi$, $\dim(L_\alpha) = 1$ and $\dim(L_{i\alpha}) = 0$ for $i \geq 2$, $i \in \mathbb{N}$.

∎

We have the following easy corollary:

Corollary 8.10. *If $\alpha \in \Phi$, $m\alpha \in \Phi$ and $m \in \mathbb{Z}$ then $m = \pm 1$.*

Now let $\alpha, \beta \in \Phi$, $\beta \neq \pm\alpha$. Consider

$$-p\alpha + \beta, \ldots - \alpha + \beta, \beta, \alpha + \beta, 2\alpha + \beta, \ldots, q\alpha + \beta$$

There are integers p, q such that all of the above are roots but $-(p+1)\alpha + \beta$ and $(q+1)\alpha + \beta$ are not roots. This collection is called the α*-chain of roots through* β.

Note that $\beta+(q+1)\alpha, \beta-(p+1)\alpha \neq 0$, so $L_{\beta+(q+1)\alpha}=0$, $L_{\beta-(p+1)\alpha}=0$. Let

$$M = L_{-p\alpha+\beta} \oplus \ldots \oplus L_{q\alpha+\beta}.$$

Choose $e_\alpha \in L_\alpha$, $e_{-\alpha} \in L_{-\alpha}$ such that $[e_{-\alpha} e_\alpha] = h_\alpha$. We claim that $M \operatorname{ad} e_\alpha \subseteq M$, $M \operatorname{ad} e_{-\alpha} \subseteq M$ and $M \operatorname{ad} h_\alpha \subseteq M$.

$$L_{i\alpha+\beta} \operatorname{ad} e_\alpha \subseteq L_{(i+1)\alpha+\beta}$$
$$L_{i\alpha+\beta} \operatorname{ad} e_{-\alpha} \subseteq L_{(i-1)\alpha+\beta}$$

So $M \operatorname{ad} h_\alpha \subseteq M$. We now calculate $\operatorname{tr}_M(\operatorname{ad} h_\alpha)$ in two different ways:

$$\operatorname{tr}_M(\operatorname{ad} h_\alpha) = \operatorname{tr}_M(\operatorname{ad} e_{-a} \operatorname{ad} e_\alpha - \operatorname{ad} e_\alpha \operatorname{ad} e_{-\alpha}) = 0$$

$$\operatorname{tr}_M(\operatorname{ad} h_\alpha) = \sum\nolimits_{i=-p}^{q} (i\alpha+\beta)(h_\alpha) \text{ since } \dim(L_{i\alpha+\beta}) = 1.$$

So $\sum\nolimits_{i=-p}^{q} (i\alpha+\beta)(h_\alpha) = 0$; $\left(\sum\nolimits_{i=-p}^{q} i\right)\alpha(h_\alpha) + \sum\nolimits_{i=-p}^{q} \beta(h_\alpha) = 0$.

$\Rightarrow$ $$\left(\tfrac{p(p+1)}{2} - \tfrac{q(q+1)}{2}\right)\langle h_\alpha, h_\alpha\rangle + (p+q+1)\langle h_\beta, h_\alpha\rangle = 0$$
$\Rightarrow$ $$\left(\tfrac{q-p}{2}\right)(p+q+1)\langle h_\alpha, h_\alpha\rangle + (p+q+1)\langle h_\beta, h_\alpha\rangle = 0$$
$\Rightarrow$ $$2\langle h_\beta, h_\alpha\rangle / \langle h_\alpha, h_\alpha\rangle = p-q$$

Thus we have proved:

Proposition 8.11. *Let $\alpha, \beta \in \Phi$, $\beta \neq \pm\alpha$. Take the α-chain of roots through β,*

$$-p\alpha+\beta, \ldots -\alpha+\beta, \beta, \alpha+\beta, \ldots, q\alpha+\beta.$$

Then

$$\frac{2\langle h_\alpha, h_\beta\rangle}{\langle h_\alpha, h_\alpha\rangle} = p-q.$$

Corollary 8.12. *If $\alpha \in \Phi$, $\xi\alpha \in \Phi$ and $\xi \in \mathbb{C}$ then $\xi = \pm 1$.*

Proof. If $\xi \neq \pm 1$ set $\beta = \xi\alpha$. Then

$$\frac{2\langle h_\alpha, h_\beta\rangle}{\langle h_\alpha, h_\alpha\rangle} = 2\xi = p - q$$

So $2\xi \in \mathbb{Z}$. If $\xi \in \mathbb{Z}$ then $\xi = \pm 1$ by 8.10. If $\xi \notin \mathbb{Z}$ then $p \not\equiv q$ modulo 2. The α-chain of roots through β is

$$-p\alpha + \beta, \ldots -\alpha + \beta, \beta, \alpha + \beta, \ldots, q\alpha + \beta$$

But $\beta = \frac{p-q}{2}\alpha$, with p and q not both zero. So $\frac{\alpha}{2}$ appears in the α-chain. This implies $\alpha, \frac{\alpha}{2} \in \Phi$, which contradicts 8.10.

■

Proposition 8.13. *For all $\alpha, \beta \in \Phi$, $\langle h_\alpha, h_\beta\rangle \in \mathbb{Q}$.*

Proof. If $\beta \neq \pm\alpha$ then by 8.11 $\langle h_\alpha, h_\beta\rangle / \langle h_\alpha, h_\alpha\rangle \in \mathbb{Q}$. We show that $\langle h_\alpha, h_\alpha\rangle \in \mathbb{Q}$.

$$\begin{aligned}\langle h_\alpha, h_\alpha\rangle &= \operatorname{tr}(\operatorname{ad} h_\alpha \operatorname{ad} h_\alpha) \\ &= \sum_{\beta\in\Phi} \beta(h_\alpha)^2 \\ &= \sum_{\beta\in\Phi} \langle h_\alpha, h_\beta\rangle^2 \\ \frac{1}{\langle h_\alpha, h_\alpha\rangle} &= \sum_{\beta\in\Phi} \left(\frac{\langle h_\alpha, h_\beta\rangle}{\langle h_\alpha, h_\alpha\rangle}\right)^2 \in \mathbb{Q}\end{aligned}$$

So $\langle h_\alpha, h_\alpha\rangle \in \mathbb{Q}$; so $\langle h_\alpha, h_\beta\rangle \in \mathbb{Q}$.

■

9. THE ROOT SYSTEM AND THE WEYL GROUP

$\{h_\alpha \mid \alpha \in \Phi\}$ spans H, so we can find a subset $\{h_{\alpha_1}, \ldots, h_{\alpha_l}\}$ that forms a basis for H; $\dim(H) = l$.

Proposition 9.1. *Let $\alpha \in \Phi$. Then $h_\alpha = \sum_{i=1}^{l} \mu_i h_{\alpha_i}$ for some $\mu_i \in \mathbb{Q}$.*

Proof. We know this for $\mu_i \in \mathbb{C}$. Let $\langle h_{\alpha_i}, h_{\alpha_j} \rangle = \xi_{ij} \in \mathbb{Q}$. The matrix $\Xi = (\xi_{ij})$ is non-singular, since if it were singular we would have $\eta_1, \ldots, \eta_l$ not all zero such that $\sum_{i=1}^{l} \eta_i \xi_{ij} = 0$. Then

$$\left\langle \sum_{i=1}^{l} \eta_i h_{\alpha_i}, h_{\alpha_i} \right\rangle = \sum_{i=1}^{l} \eta_i \xi_{ij} = 0$$

So $\left\langle \sum_{i=1}^{l} \eta_i h_{\alpha_i}, x \right\rangle = 0$ for all $x \in H$. $\langle \cdot, \cdot \rangle$ is non-degenerate, so all $\eta_i = 0$, which is a contradiction.

$$\begin{gathered} \langle h_\alpha, h_{\alpha_1} \rangle = \mu_1 \xi_{11} + \ldots + \mu_l \xi_{l1} \\ \vdots \\ \langle h_\alpha, h_{\alpha_l} \rangle = \mu_1 \xi_{1l} + \ldots + \mu_l \xi_{ll} \end{gathered}$$

We have l linear equations in l unknowns with a non-singular coefficient matrix, all the entries of which are rational. Hence, by Cramer's Rule, there is a unique solution $(\mu_i) \in \mathbb{Q}^l$

■

Let $H_{\mathbb{Q}}$ be the set of all $\sum_{i=1}^{l} \mu_i h_{\alpha_i}$, $\mu_i \in \mathbb{Q}$. $\dim_{\mathbb{Q}}(H_{\mathbb{Q}}) = l$. $H_{\mathbb{Q}}$ is independent of the choice of basis; all $h_\alpha \in H_{\mathbb{Q}}$.

Let $H_{\mathbb{R}}$ be the set of all $\sum_{i=1}^{l} \mu_i h_{\alpha_i}$, $\mu_i \in \mathbb{R}$. $\dim_{\mathbb{R}}(H_{\mathbb{R}}) = l$.

Proposition 9.2. *Let $x \in H_{\mathbb{R}}$. Then $\langle x, x \rangle \in \mathbb{R}_{\geq 0}$ and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$.*

Proof. Let $x \in H_{\mathbb{R}}$, $x = \sum_{i=1}^{l} \mu_i h_{\alpha_i}$.

$$\begin{aligned}\langle x,x\rangle &= \sum_i \sum_j \mu_i\mu_j \langle h_{\alpha_i}, h_{\alpha_j}\rangle \\ &= \sum_i \sum_j \mu_i\mu_j \operatorname{tr}\left(\operatorname{ad} h_{\alpha_i} \operatorname{ad} h_{\alpha_j}\right) \\ &= \sum_i \sum_j \mu_i\mu_j \sum_{\alpha\in\Phi} \alpha\left(h_{\alpha_i}\right)\alpha\left(h_{\alpha_j}\right) \\ &= \sum_\alpha \sum_i \sum_j \mu_i\mu_j \alpha\left(h_{\alpha_i}\right)\alpha\left(h_{\alpha_j}\right) \\ &= \sum_\alpha \left(\sum_i \mu_i \alpha\left(h_{\alpha_i}\right)\right)^2\end{aligned}$$

So $\langle x,x\rangle \in \mathbb{R}$ and $\langle x,x\rangle \geq 0$. Suppose $\langle x,x\rangle = 0$. Then for all $\alpha\in\Phi$, $\sum_i \mu_i\alpha\left(h_{\alpha_i}\right) = 0$. In particular, $\sum_i \mu_i\alpha_j\left(h_{\alpha_i}\right) = 0$ for $j = 1,\ldots,l$; $\sum_i \mu_i \langle h_{\alpha_i}, h_{\alpha_j}\rangle = \sum_i \mu_i \xi_{ij} = 0$ for all j. Ξ is non-singular, so $\mu_i = 0$ for all i, so $x = 0$.

■

So all $h_\alpha \in H_{\mathbb{R}}$; $\dim_{\mathbb{R}}(H_{\mathbb{R}}) = l$. We introduce a total order on $H_{\mathbb{R}}$: let $x\in H_{\mathbb{R}}$, $x = \sum_i \mu_i h_{\alpha_i}$. If $x \neq 0$ we say $x \succ 0$ if the first non-zero μ_i is positive; if $x\neq 0$ we say $x \prec 0$ if the first non-zero μ_i is negative. We have trichotomy: for each $x \in H_{\mathbb{R}}$ precisely one of $x=0$, $x\prec 0$, $x \succ 0$ is true.

So, for $\alpha\in\Phi$, $h_\alpha \prec 0$ or $h_\alpha \succ 0$. Define $\alpha \prec 0$ if $h_\alpha \prec 0$ and $\alpha\succ 0$ if $h_\alpha \succ 0$. Define

$$\Phi^+ = \{\alpha\in\Phi \mid \alpha\succ 0\}, \text{ the positive roots, and}$$
$$\Phi^- = \{\alpha\in\Phi \mid \alpha\prec 0\}, \text{ the negative roots.}$$

Clearly, $\Phi = \Phi^+ \cup \Phi^-$.

A *fundamental root* is a positive root that is not the sum of two positive roots. Let Π be the set of fundamental roots.

Proposition 9.3. *(i) Every positive root is a sum of fundamental roots.*
(ii) $\{h_\alpha \mid \alpha\in\Pi\}$ *is a basis of* $H_{\mathbb{R}}$.
(iii) If $\alpha,\beta\in\Pi$ *and* $\alpha\neq\beta$ *then* $\langle h_\alpha, h_\beta\rangle \leq 0$.

Proof. (i) Let $\alpha\in\Phi^+$. If $\alpha\in\Pi$ we are done. If $\alpha\notin\Pi$ the there exist $\beta,\gamma\in\Phi^+$ such that $\alpha=\beta+\gamma$ with $\beta,\gamma\prec\alpha$. Repeat to get the result.

(iii) Let $\alpha,\beta\in\Pi$ with $\alpha\neq\beta$. Then $\alpha-\beta\notin\Phi$ since if not

$$\alpha = (\alpha-\beta)+\beta \text{ or } \beta = (\beta-\alpha)+\alpha$$

so either α or β would be a sum of positive roots. Consider the α-chain of roots through β:

$$\beta, \alpha+\beta, \ldots, q\alpha+\beta$$

$$\Rightarrow \qquad 2\frac{\langle h_\alpha, h_\beta\rangle}{\langle h_\alpha, h_\alpha\rangle} = p - q = -q \qquad \text{by 8.11.}$$

$$\Rightarrow \qquad \langle h_\alpha, h_\beta\rangle \le 0 \qquad \text{since } \langle h_\alpha, h_\alpha\rangle \ge 0 \text{ by 9.2.}$$

(ii) By (i), the h_α for $\alpha \in \Phi$ span H. We show the h_α are linearly independent. Suppose not: then there exist $\mu_i \in \mathbb{R}$ not all zero such that

$$\sum_{\alpha_i \in \Pi} \mu_i h_{\alpha_i} = 0$$

Rearrange this sum, taking all the positive μ_i to one side. Then

$$x = \mu_{i_1} h_{\alpha_{i_1}} + \ldots + \mu_{i_r} h_{\alpha_{i_r}} = \mu_{j_1} h_{\alpha_{j_1}} + \ldots + \mu_{j_s} h_{\alpha_{j_s}}$$

$$\mu_{i_u}, \mu_{j_v} > 0,\ i_u, j_v \text{ distinct for } 1 \le u \le r,\ 1 \le v \le s.$$

Then

$$\langle x, x\rangle = \left\langle \mu_{i_1} h_{\alpha_{i_1}} + \ldots + \mu_{i_r} h_{\alpha_{i_r}}, \mu_{j_1} h_{\alpha_{j_1}} + \ldots + \mu_{j_s} h_{\alpha_{j_s}} \right\rangle \le 0$$

by (iii). So $x = 0$, a contradiction.

■

Note. Φ^+ can be chosen in many different ways. However, Π is determined by Φ^+ and Φ^+ is determined by Π.

Example. Let $L = \mathfrak{sl}_n(\mathbb{C})$. The roots are

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \mapsto \lambda_j - \lambda_i \text{ for } j \ne i$$

Define Φ^+ to be the roots with $j > i$. Then the fundamental roots are

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \mapsto \lambda_{i+1} - \lambda_i \text{ for } 1 \le i \le n-1$$

$$\lambda_j - \lambda_i = (\lambda_{i+1} - \lambda_i) + (\lambda_{i+2} - \lambda_{i+1}) + \ldots + (\lambda_j - \lambda_{j-1})$$

$$\dim(H) = n - 1 = l \text{, the rank of } L.$$

For each $\alpha \in \Phi$ we define $s_\alpha : H_{\mathbb{R}} \to H_{\mathbb{R}}$ by

$$s_\alpha(x) = x - 2\frac{\langle h_\alpha, x\rangle}{\langle h_\alpha, h_\alpha\rangle} h_\alpha$$

s_α is linear and $s_\alpha(h_\alpha) = -h_\alpha$. The set of x such that $\langle x, x\rangle = 0$ forms a hyperplane i.e. a subspace of codimension 1. s_α is the reflection of $H_{\mathbb{R}}$ in the hyperplane orthogonal to h_α.

$$s_\alpha^2 = \mathrm{id}$$

$$s_\alpha = s_{-\alpha}$$

Let W be the group of all non-singular linear maps $H_{\mathbb{R}} \to H_{\mathbb{R}}$ generated by $\{ s_\alpha \mid \alpha \in \Phi \}$. W is called the *Weyl group*.†

Proposition 9.4. *(i) W is a finite group.*
(ii) W is a group of isometries, i.e. for all $x, y \in H_{\mathbb{R}}$, $w \in W$, $\langle w(x), w(y)\rangle = \langle x, y\rangle$.
(iii) For each $\alpha \in \Phi$ and $w \in W$ there is a $\beta \in \Phi$ such that $w(h_\alpha) = h_\beta$.

Proof. (ii) Let $x, y \in H_{\mathbb{R}}$. Then

$$\begin{aligned}\langle s_\alpha(x), s_\alpha(y)\rangle &= \left\langle x - 2\frac{\langle h_\alpha, x\rangle}{\langle h_\alpha, h_\alpha\rangle} h_\alpha, y - 2\frac{\langle h_\alpha, y\rangle}{\langle h_\alpha, h_\alpha\rangle} h_\alpha \right\rangle \\ &= \langle x, y\rangle - 4\frac{\langle h_\alpha, x\rangle\langle h_\alpha, y\rangle}{\langle h_\alpha, h_\alpha\rangle} - 4\frac{\langle h_\alpha, x\rangle\langle h_\alpha, y\rangle}{\langle h_\alpha, h_\alpha\rangle^2}\langle h_\alpha, h_\alpha\rangle \\ &= \langle x, y\rangle\end{aligned}$$

So s_α is an isometry; so w is an isometry for all $w \in W$.

(iii) Now consider $s_\alpha(h_\beta)$:

† After Hermann Weyl.

$$s_\alpha(h_\alpha) = h_{-\alpha}$$
$$s_\alpha(h_{-\alpha}) = h_\alpha$$

So suppose $\beta \neq \pm\alpha$. Consider the α-chain of roots through β,

$$-p\alpha + \beta, \ldots, \beta, \ldots, q\alpha + \beta$$

$$\begin{aligned} s_\alpha(h_\beta) &= h_\beta - 2\frac{\langle h_\alpha, h_\beta \rangle}{\langle h_\alpha, h_\alpha \rangle} h_\alpha \\ &= h_\beta - (p-q)h_\alpha \\ &= h_{\beta + (q-p)\alpha} \end{aligned}$$

Now $\beta + (q-p)\alpha \in \Phi$ since $-p \leq p - q \leq q$. So s_α permutes the h_β for $\beta \in \Phi$. Hence $w \in W$ permutes the h_β for $\beta \in \Phi$. Note that

$$\beta + ((q-p)\alpha + \beta) = (-p\alpha + \beta) + (q\alpha + \beta)$$

so s_α inverts the h_β in a given α-chain.

(i) We have a homomorphism from W to the group of permutations of the h_α for $\alpha \in \Phi$. Φ is finite, so the image of this homomorphism is finite. If $w \in W$ is in the kernel then $w(h_\alpha) = h_\alpha$ for all $\alpha \in \Phi$. Since the h_α span $H_{\mathbb{R}}$, $w = \mathrm{id}$. Hence, W is finite. ■

Proposition 9.5. *Given any root $\alpha \in \Phi$ there exists a fundamental root $\alpha_i \in \Pi$ and a $w \in W$ such that $h_\alpha = w(h_{\alpha_i})$.*

Proof. Each $\alpha \in \Phi$ has the form $\alpha = n_1\alpha_1 + \ldots + n_l\alpha_l$, $n_i \in \mathbb{Z}$. If $\alpha \in \Phi^+$ then all $n_i \geq 0$; if $\alpha \in \Phi^-$ then all $n_i \leq 0$. We may assume $\alpha \in \Phi^+$ since if $\alpha \in \Phi^-$ then use $h_\alpha = s_\alpha(h_{-\alpha})$. The quantity $n_1 + \ldots + n_l$ is called the *height* of α, $\mathrm{ht}(\alpha)$. We use induction on $\mathrm{ht}(\alpha)$. If $\mathrm{ht}(\alpha) = 1$ we are done, so assume $\mathrm{ht}(\alpha) > 1$. By 8.12, at least two $n_i > 0$.

$$0 < \langle h_\alpha, h_\alpha \rangle = \sum_i n_i \langle h_{\alpha_i}, h_\alpha \rangle$$

All $n_i \geq 0$, so there exists i such that $\langle h_{\alpha_i}, h_\alpha \rangle > 0$. Let $s_{\alpha_i}(h_\alpha) = h_\beta$.

$$h_\beta = h_\alpha - 2\frac{\langle h_{\alpha_i}, h_\alpha \rangle}{\langle h_{\alpha_i}, h_{\alpha_i} \rangle} h_{\alpha_i}$$
$$\beta = \alpha - 2\frac{\langle h_{\alpha_i}, h_\alpha \rangle}{\langle h_{\alpha_i}, h_{\alpha_i} \rangle} \alpha_i$$

So $\mathrm{ht}(\beta) < \mathrm{ht}(\alpha)$. Passing from α to β changes only one n_i, hence β has at least one $n_j > 0$, so $\beta \in \Phi^+$. By induction, $\beta = w'(h_{\alpha_j})$ for some $w' \in W$ and some $\alpha_j \in \Pi$. Thus, taking $w = s_{\alpha_i} w' \in W$,

$$h_\alpha = s_{\alpha_i}(h_\beta) = s_{\alpha_i} w'(h_{\alpha_j}) = w(h_{\alpha_j}).$$

■

Proposition 9.6. *The Weyl group W is generated by $s_{\alpha_1}, \ldots, s_{\alpha_l}$ for $\Pi = \{\alpha_1, \ldots, \alpha_l\}$.*

Proof. Suppose W_0 is the subgroup generated by $\{s_{\alpha_i} \mid \alpha_i \in \Pi\}$. To show $W = W_0$ we show $s_\alpha \in W_0$ for all $\alpha \in \Phi$. The proof of 9.5 shows that $h_\alpha = w(h_{\alpha_i})$ for some $\alpha_i \in \Pi$ and some $w \in W$. Consider $w s_{\alpha_i} w^{-1} \in W_0$.

$$w s_{\alpha_i} w^{-1}(h_\alpha) = w s_{\alpha_i}(h_{\alpha_i}) = w(-h_{\alpha_i}) = -h_\alpha$$

Let $x \in H_{\mathbb{R}}$ be such that $\langle h_\alpha, x \rangle = 0$. Then

$$\Rightarrow \qquad \langle w^{-1}(h_\alpha), w^{-1}(x) \rangle = 0$$
$$\Rightarrow \qquad \langle h_{\alpha_i}, w^{-1}(x) \rangle = 0$$
$$\Rightarrow \qquad w s_{\alpha_i} w^{-1}(x) = w w^{-1}(x) = x$$

Hence, $w s_{\alpha_i} w^{-1} = s_\alpha$. Then $s_\alpha \in W_0$, so $W = W_0$.

■

Example. $L = \mathfrak{sl}_3(\mathbb{C})$; $\dim(L) = 8$.

$$H = \left\{ \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \middle| \ \lambda_1 + \lambda_2 + \lambda_3 = 0 \right\}$$

$\dim(H) = 2$.

$$L = H \oplus \mathbb{C}E_{12} \oplus \mathbb{C}E_{23} \oplus \mathbb{C}E_{13} \oplus \mathbb{C}E_{21} \oplus \mathbb{C}E_{32} \oplus \mathbb{C}E_{31}$$

Let

$$h = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$
$$[E_{ij}h] = (\lambda_j - \lambda_i)E_{ij}$$

The roots are

$$\begin{array}{ccc} \alpha_1 : h \mapsto \lambda_2 - \lambda_1 & \alpha_2 : h \mapsto \lambda_3 - \lambda_2 & \alpha_1 + \alpha_2 : h \mapsto \lambda_3 - \lambda_1 \\ -\alpha_1 : h \mapsto \lambda_1 - \lambda_2 & -\alpha_2 : h \mapsto \lambda_2 - \lambda_3 & -\alpha_1 - \alpha_2 : h \mapsto \lambda_1 - \lambda_3 \end{array}$$
$$\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2)\}$$
$$\Pi = \{\alpha_1, \alpha_2\}$$

Consider the corresponding vectors $h_\alpha \in H$. Let

$$h = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \in H, \; h' = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix} \in H$$

$$\begin{aligned} \langle h, h' \rangle &= \operatorname{tr}(\operatorname{ad} h \operatorname{ad} h') \\ &= 2(\lambda_2 - \lambda_1)(\mu_2 - \mu_1) + 2(\lambda_3 - \lambda_2)(\mu_3 - \mu_2) + 2(\lambda_3 - \lambda_1)(\mu_3 - \mu_1) \\ &= 2(2\lambda_1\mu_1 + 2\lambda_2\mu_2 + 2\lambda_3\mu_3 - (\lambda_1\mu_2 + \lambda_1\mu_3 + \lambda_2\mu_1 + \lambda_2\mu_3 + \lambda_3\mu_1 + \lambda_3\mu_2)) \\ &= 4(\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3) - 2(\lambda_1 + \lambda_2 + \lambda_3)(\mu_1 + \mu_2 + \mu_3) + 2(\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3) \\ &= 6(\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3) \\ &= 6\operatorname{tr}(hh') \end{aligned}$$

h_{α_1} satisfies $\langle h_{\alpha_1}, h \rangle = \alpha_1(h) = \lambda_2 - \lambda_1$, so

$$h_{\alpha_1} = \frac{1}{6}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Similarly,

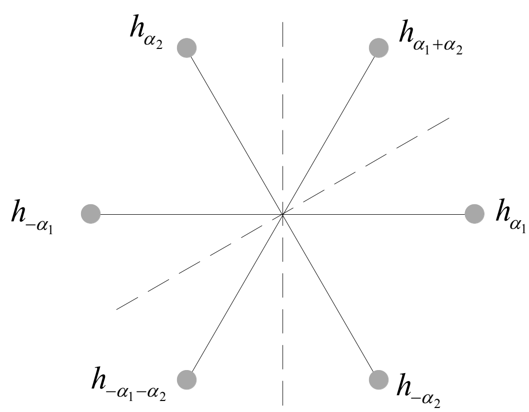
$$h_{\alpha_2} = \frac{1}{6}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For $x \in H_{\mathbb{R}}$ define $|x| = \sqrt{\langle x, x \rangle}$. With this notation, $|h_{\alpha_1}| = |h_{\alpha_2}| = 1/\sqrt{3}$ and

$$\langle h_{\alpha_1}, h_{\alpha_2} \rangle = 6\tfrac{1}{6}\tfrac{1}{6}(-1) = -\tfrac{1}{6}$$

The angle between h_{α_1} and h_{α_2} is given by the cosine formula:

$$\langle h_{\alpha_1}, h_{\alpha_2} \rangle = |h_{\alpha_1}||h_{\alpha_2}|\cos\theta$$
$$\theta = 2\pi/3$$



$$W = \{\mathrm{id}, s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_1}s_{\alpha_2}, s_{\alpha_2}s_{\alpha_1}, s_{\alpha_1}s_{\alpha_2}s_{\alpha_1} = s_{\alpha_2}s_{\alpha_1}s_{\alpha_2} = s_{\alpha_1+\alpha_2}\}$$

10. THE DYNKIN DIAGRAM

We shall consider the geometrical properties of the h_α for $\alpha \in \Phi$.

Proposition 10.1. *Let $\alpha, \beta \in \Phi$, $\beta \neq \pm\alpha$. Then*
(i) the angle between α and β is one of

$$\pi/6,\ \pi/4,\ \pi/3,\ \pi/2,\ 2\pi/3,\ 3\pi/4,\ 5\pi/6;$$

(ii) if the angle is $\pi/3$ or $2\pi/3$, h_α and h_β have the same length;
(iii) if the angle is $\pi/4$ or $3\pi/4$, the ratio of the lengths of h_α and h_β is $\sqrt{2}$;
(iv) if the angle is $\pi/6$ or $5\pi/6$, the ratio of the lengths of h_α and h_β is $\sqrt{3}$.

Proof. Let $\theta_{\alpha\beta}$ be the angle between h_α and h_β. We have

$$\langle h_\alpha, h_\beta \rangle = |h_\alpha||h_\beta|\cos\theta_{\alpha\beta}$$

$$\Rightarrow \quad \langle h_\alpha, h_\beta \rangle^2 = \langle h_\alpha, h_\alpha \rangle \langle h_\beta, h_\beta \rangle \cos^2\theta_{\alpha\beta}$$

$$\Rightarrow \quad 4\cos^2\theta_{\alpha\beta} = \frac{2\langle h_\alpha, h_\beta \rangle}{\langle h_\alpha, h_\alpha \rangle} \frac{2\langle h_\beta, h_\alpha \rangle}{\langle h_\beta, h_\beta \rangle}$$

By 8.11, both factors on the RHS are integers, so $4\cos^2\theta_{\alpha\beta} \in \mathbb{Z}$. $0 \le \cos^2\theta_{\alpha\beta} < 1$, so $0 \le 4\cos^2\theta_{\alpha\beta} < 4$, so $4\cos^2\theta_{\alpha\beta} \in \{0,1,2,3\}$.

$$\Rightarrow \quad \cos^2\theta_{\alpha\beta} \in \left\{0, \pm\tfrac{1}{2}, \pm\tfrac{1}{\sqrt{2}}, \pm\tfrac{\sqrt{3}}{2}\right\}$$

$$\Rightarrow \quad \theta_{\alpha\beta} \in \{\pi/2, \pi/3, 2\pi/3, \pi/4, 3\pi/4, \pi/6, 5\pi/6\}$$

$$4\cos^2\theta_{\alpha\beta} = \left(2\tfrac{\langle h_\alpha, h_\beta \rangle}{\langle h_\alpha, h_\alpha \rangle}\right)\left(2\tfrac{\langle h_\beta, h_\alpha \rangle}{\langle h_\beta, h_\beta \rangle}\right)$$

Suppose $\theta_{\alpha\beta}$ is $\pi/3$ or $2\pi/3$, so $4\cos^2\theta_{\alpha\beta} = 1$. $1 = 1.1 = (-1).(-1)$, so $|h_\alpha| = |h_\beta|$.

Suppose $\theta_{\alpha\beta}$ is $\pi/4$ or $3\pi/4$, so $4\cos^2\theta_{\alpha\beta} = 2$. $2 = 1.2 = 2.1 = (-1).(-2) = (-2).(-1)$. So one of $\langle h_\alpha, h_\alpha \rangle$ and $\langle h_\beta, h_\beta \rangle$ is twice the other, so one of $|h_\alpha|$, $|h_\beta|$ is $\sqrt{2}$ times the other.

Suppose $\theta_{\alpha\beta}$ is $\pi/6$ or $5\pi/6$, so $4\cos^2\theta_{\alpha\beta} = 3$. $3 = 1.3 = 3.1 = (-1).(-3) = (-3).(-1)$. So, as above, one of $|h_\alpha|$, $|h_\beta|$ is $\sqrt{3}$ times the other.

■

Proposition 10.2. *Let $\alpha \in \Phi$. Then every α - chain of roots has at most four roots in it.*

Proof. Consider the α - chain of roots through β with β as the first root:

$$\beta, \alpha + \beta, \ldots, q\alpha + \beta$$

By 8.11, $2\langle h_\alpha, h_\beta \rangle / \langle h_\alpha, h_\alpha \rangle = -q$. The LHS is $0, -1, -2$ or -3 by 10.1. So $q \leq 3$. So the length of the α -chain is at most 4.

■

Let

$$a_{ij} = 2\frac{\langle h_{\alpha_i}, h_{\alpha_j} \rangle}{\langle h_{\alpha_i}, h_{\alpha_i} \rangle}$$

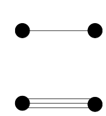
and $A = (a_{ij})$. A is called the *Cartan matrix*; the a_{ij} are the *Cartan integers*.

Proposition 10.3. *The Cartan matrix has the following properties:*
(i) for each i, $a_{ii} = 2$;
(ii) for $i \neq j$, $a_{ij} \in \{0, -1, -2, -3\}$;
(iii) $a_{ij} = -2 \Rightarrow a_{ji} = -1$; $a_{ij} = -3 \Rightarrow a_{ji} = -1$;
(iv) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$.

Proof. If $i \neq j$ then $4\cos^2\theta_{\alpha\beta} = a_{ij}a_{ji}$.
(i) Clear.
(ii) Follows from 10.1, 9.3.
(iii) Follows from 10.1.
(iv) Clear.

■

We incorporate this information into a graph. The *Dynkin*[†] *diagram* is a graph Δ with l vertices, one for each fundamental root. If $i \neq j$ then vertices i, j are joined by $n_{ij} = a_{ij}a_{ji}$ edges, $0 \leq n_{ij} \leq 3$. The Dynkin diagram may be disconnected, as in



[†] After E.B. Dynkin. Also due to H.S.M. Coxeter.

It splits into connected components and the Cartan matrix splits into corresponding blocks; off-diagonal blocks are zero:

$$A = \left(\begin{array}{c|c|c} * & & 0 \\ \hline & \ddots & \\ \hline 0 & & * \end{array}\right)$$

We define a corresponding quadratic form Q:

$$Q(x_1,\ldots,x_l) = \sum_{i=1}^{l} 2x_i^2 - \sum_{1\le i\ne j\le l} \sqrt{n_{ij}}\, x_i x_j$$

Recall the correspondence between quadratic forms on $\mathbb{R}$ and real symmetric matrices:

$$\begin{aligned} M &= (m_{ij}) \text{ symmetric} \\ xMx^{\mathrm{T}} &= \sum_{i,j} m_{ij} x_i x_j \end{aligned}$$

The matrix of $Q(x_1,\ldots,x_l)$ is

$$\begin{pmatrix} 2 & -\sqrt{n_{12}} & -\sqrt{n_{13}} & & \\ -\sqrt{n_{12}} & 2 & -\sqrt{n_{23}} & \ddots & \\ -\sqrt{n_{13}} & -\sqrt{n_{23}} & 2 & \ddots & \\ & \ddots & \ddots & \ddots & \\ & & & & 2 \end{pmatrix}$$

Proposition 10.4. *The quadratic form $Q(x_1,\ldots,x_l)$ is positive definite, i.e. $Q(x_1,\ldots,x_l)\ge 0$ and $Q(x_1,\ldots,x_l)=0 \Leftrightarrow x_1=\ldots=x_l=0$.*

Proof.

$$4\cos^2\theta_{ij} = a_{ij}a_{ji} = n_{ij}$$

$$2\cos\theta_{ij} = -\sqrt{n_{ij}}$$

$$\langle h_{\alpha_i}, h_{\alpha_j}\rangle = |h_{\alpha_i}||h_{\alpha_j}|\cos\theta_{ij}$$

$$Q(x_1,\ldots,x_l) = \sum_{i,j=1}^{l} 2\frac{\langle h_{\alpha_i}, h_{\alpha_j}\rangle}{|h_{\alpha_i}||h_{\alpha_j}|} x_i x_j = 2\left\langle \sum_i \frac{x_i}{|h_{\alpha_i}|} h_{\alpha_i}, \sum_j \frac{x_j}{|h_{\alpha_j}|} h_{\alpha_j} \right\rangle = 2\langle y, y\rangle$$

where $y = \sum_i x_i h_{\alpha_i} / |h_{\alpha_i}|$. So $Q(x_1,\ldots,x_l)\ge 0$. If $Q(x_1,\ldots,x_l)=0$ then $\langle y,y\rangle = 0$, so $y=0$, so all $x_i=0$. The converse is clear.

■

Recall. Any quadratic form can be diagonalized; there exists a non-singular real $l \times l$ matrix P such that $PMP^{\mathrm{T}} = D$, a diagonal matrix. Let $y = xP^{-1}$; then $xMx^{\mathrm{T}} = yDy^{\mathrm{T}}$.

Proposition 10.5. *Let $M = (m_{ij})$ be an $l \times l$ real symmetric matrix. Then the associated quadratic form $\sum_{i,j} m_{ij} x_i x_j$ is positive definite if and only if all leading minors of M have positive determinant. (The leading minors are*

$$(m_{11}), \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \ldots, M \text{ .)}$$

Proof. We use induction on l. Assume the quadratic form is positive definite. If $l = 1$, $M = (m_{11})$. $m_{11} x^2 > 0 \Leftrightarrow m_{11} > 0$. Suppose $l > 1$. $\sum_{i,j=1}^{l} m_{ij} x_i x_j$ is still positive definite as it is the original with $x_l = 0$. By induction, the first $l-1$ leading minors of M have positive determinant; we require that $\det(M) > 0$. $xMx^{\mathrm{T}} = yDy^{\mathrm{T}}$, D diagonal with entries $d_1, \ldots, d_l > 0$. Now if $PMP^{\mathrm{T}} = D$,

$$\det(P)^2 \det(M) = \det(D) > 0 .$$

Conversely, suppose that all leading minors of M have positive determinant. The same is true of the smaller $(l-1) \times (l-1)$ leading minor. By induction, $\sum_{i,j=1}^{l-1} m_{ij} x_i x_j$ is positive definite. So we have a diagonal form in new coordinates $y_1, \ldots, y_l$:

$$\sum_{i,j=1}^{l-1} m_{ij} x_i x_j = \sum_{k=1}^{l-1} d_k x_k^2 \text{ with } d_k > 0 .$$
$$\sum_{i,j=1}^{l} m_{ij} x_i x_j = \sum_{k=1}^{l-1} d_k x_k^2 + 2e_1 y_1 x_l + \ldots + 2e_{l-1} y_{l-1} x_l + e x_l^2$$

This may be diagonalized by a further transformation of coordinates:

$$z_i = y_i + \tfrac{e_i}{d_i} x_i$$

We get $d_1 z_1^2 + \ldots d_{l-1} z_{l-1}^2 + f x_l^2$. So there is a non-singular P such that

$$PMP^{\mathrm{T}} = \begin{pmatrix} d_1 & & & 0 \\ & \ddots & & \\ & & d_{l-1} & \\ 0 & & & f \end{pmatrix}$$
$$\det(P)^2 \det(M) = f \prod_{i=1}^{l-1} d_i$$

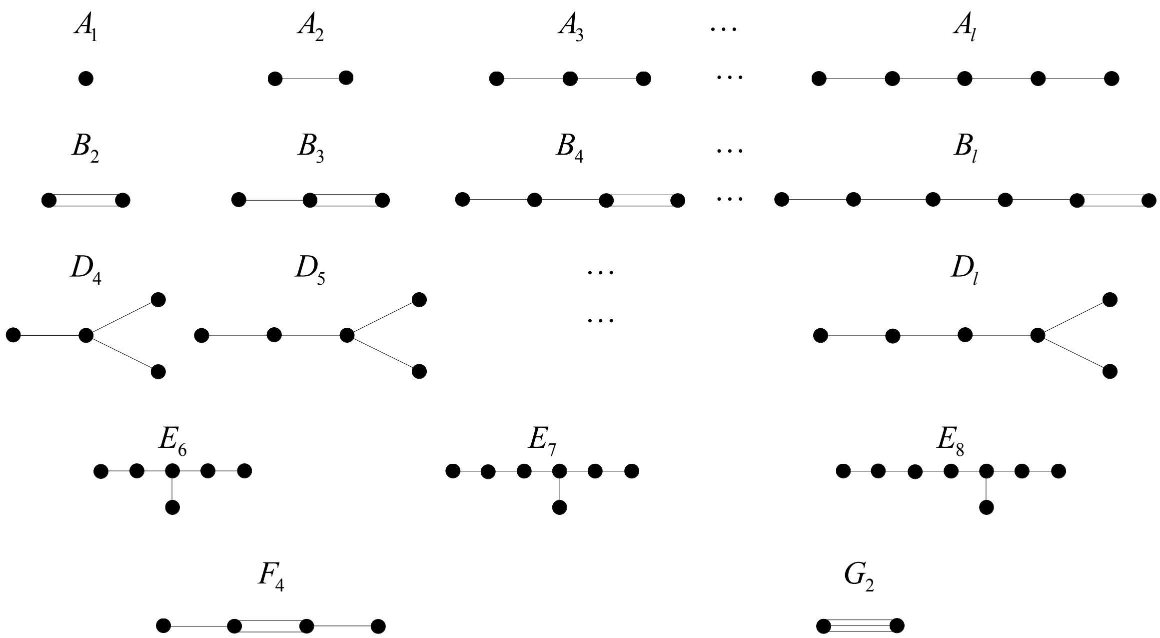
We assume $\det(M) > 0$, so $f \prod_{i=1}^{l-1} d_i > 0$, so $f > 0$. Thus, the form is positive definite.

■

We consider graphs with the following properties:
(i) the graph is connected;
(ii) any two distinct vertices are joined by 0, 1, 2 or 3 edges;
(iii) the associated quadratic form is positive definite.

The Dynkin diagram of a semisimple Lie algebra has connected components satisfying (i)-(iii). It is possible to determine all graphs satisfying (i)-(iii).

Theorem 10.6. *The only graphs satisfying (i)-(iii) are*



Proof. The given graphs clearly satisfy (i) and (ii). We show that they satisfy (iii). We show $Q(x_1,\ldots,x_l)$ is positive definite by induction on l. If $l=1$ we have $Q(x_1) = 2x_1^2$, which is positive definite. Suppose $l > 1$. There is a vertex l such that when it is removed we have another graph on the list. By induction, $Q(x_1,\ldots,x_{l-1})$ is positive definite, so all leading minors of the matrix of $Q(x_1,\ldots,x_{l-1})$ have positive determinant. To complete the induction we show that the matrix of $Q(x_1,\ldots,x_l)$ has positive determinant.

Let Y_l be a graph of l vertices and y_l the determinant of the matrix of the associated quadratic form. In the case $l=1$, $a_1 = |2| = 2$. In the case $l=2$ we have

$$a_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

$$b_2 = \begin{vmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{vmatrix} = 2$$

$$c_2 = \begin{vmatrix} 2 & -\sqrt{3} \\ -\sqrt{3} & 2 \end{vmatrix} = 1$$

Suppose $l \geq 3$. Remove a vertex l joined to just one other vertex $l-1$ by a single edge. If Y_l is the given graph, let Y_{l-1} be the graph with vertex l removed in this way, and let Y_{l-2} be the graph with vertices l and $l-1$ removed in this way.

$$y_l = \det(Y_l) = \begin{vmatrix} & & & & 0 \\ & & & & \vdots \\ & & & & 0 \\ & & & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{vmatrix} = 2y_{l-1} - (-1)(-1)y_{l-2} = 2y_{l-1} - y_{l-2}$$

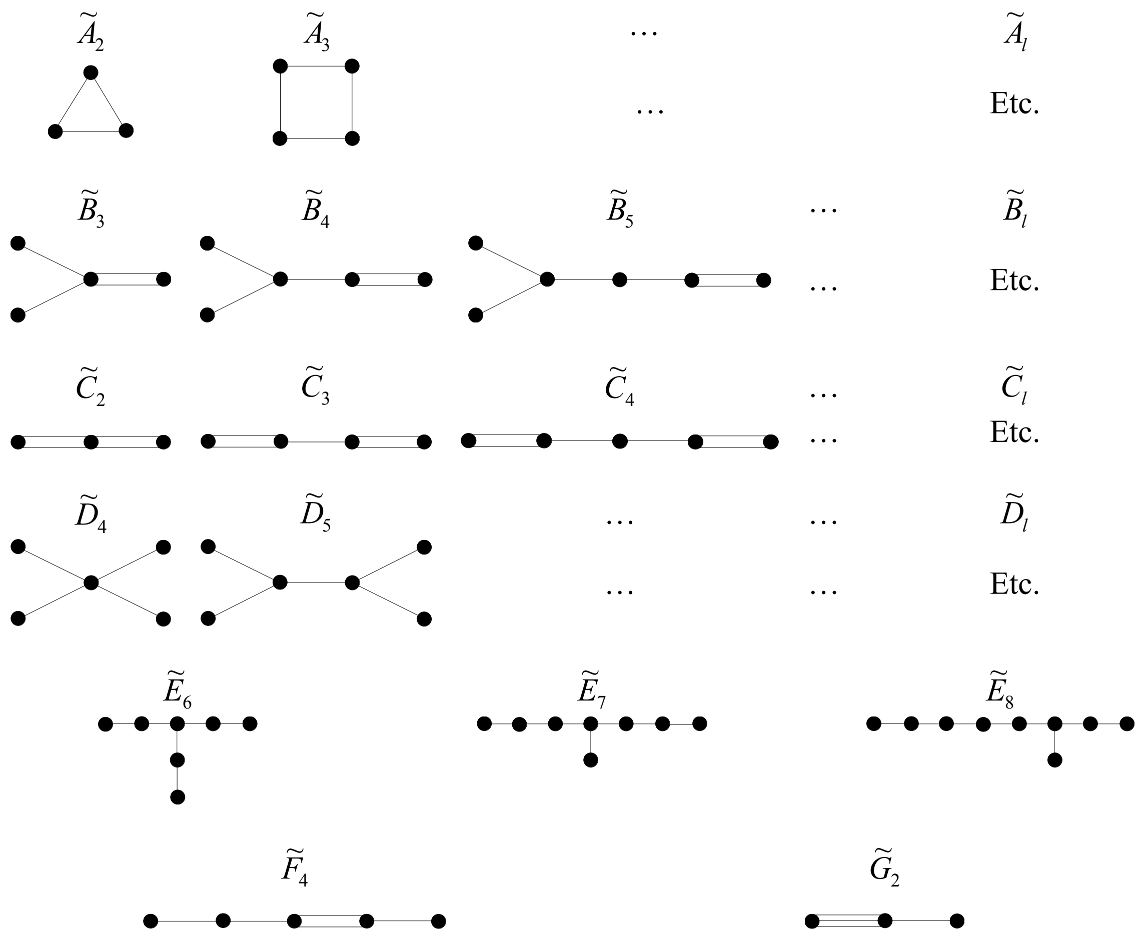
Hence:

Type A_l	$a_l = 2a_{l-1} - a_{l-2} \Rightarrow a_l = l+1$
Type B_l	$b_l = 2b_{l-1} - b_{l-2} \Rightarrow b_l = 2$
Type D_l	$d_4 = 2a_3 - a_1^2 = 4$
	$d_5 = 2d_4 - a_3 = 4$
	$\Rightarrow d_l = 4$ by induction
Type E_6	$e_6 = 2d_5 - a_4 = 3$
Type E_7	$e_7 = 2e_6 - d_5 = 2$
Type E_8	$e_8 = 2e_7 - e_6 = 1$
Type F_4	$f_4 = 2b_3 - a_2 = 1$
Type G_2	$g_2 = 1$

Hence, $Q(x_1, \ldots, x_l)$ is positive definite in each case.

In order to show the converse, i.e. that the graphs on our list are the only possible ones, we shall first require some additional results.

Proposition 10.7. *For each of the following graphs the corresponding quadratic form* $Q(x_1, \ldots, x_l)$ *has determinant zero.*



Proof. In most cases we can calculate the determinant as before, but not in types $\widetilde{A}_l$, $\widetilde{C}_l$.

Type $\widetilde{A}_l$

$$\begin{vmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & 0 & \\ & -1 & \ddots & & \\ & 0 & & \ddots & -1 \\ -1 & & & -1 & 2 \end{vmatrix} = 0$$

since the row sum is $(0,\ldots,0)$

Type $\widetilde{C}_l$

$$\begin{vmatrix} & & & 0 & \\ & & & \vdots & \\ & & & 0 & \\ & & & 2 & -\sqrt{2} \\ 0 & \cdots & 0 & -\sqrt{2} & 2 \end{vmatrix} = 2b_l - \left(-\sqrt{2}\right)\left(-\sqrt{2}\right)b_{l-1} = 0$$

Type $\widetilde{B}_l$ $\qquad \widetilde{b}_3 = 2b_3 - a_1^2 = 2.2 - 2^2 = 0$

Type $\widetilde{D}_l$ $\qquad \widetilde{d}_4 = 2d_4 - a_1^3 = 2.4 - 2^3 = 0$

Type $\widetilde{E}_6$ $\qquad \widetilde{e}_6 = 2e_6 - a_5 = 2.3 - 6 = 0$

Type $\widetilde{E}_7$ $\qquad$ $\widetilde{e}_7 = 2e_7 - d_6 = 2.2 - 4 = 0$

Type $\widetilde{E}_8$ $\qquad$ $\widetilde{e}_8 = 2e_8 - e_7 = 2.1 - 2 = 0$

Type $\widetilde{F}_4$ $\qquad$ $\widetilde{f}_4 = 2f_4 - b_3 = 2.1 - 2 = 0$

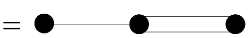
Type $\widetilde{G}_2$ $\qquad$ $\widetilde{g}_2 = 2g_2 - a_1 = 2.1 - 2 = 0$

■

Lemma 10.8. *Let Y be a graph in which any two vertices are joined by at most three edges. Suppose the corresponding quadratic form is positive definite. Suppose Y' is a graph obtained from Y by omitting some of the vertices, or by reducing the number of edges, or both. Then the quadratic form for Y' is also positive definite.*

We call Y' a *subgraph* of Y.

Example.

$Y' =$  is a subgraph of $Y =$ 

Proof. The quadratic form for Y is

$$Q(x_1,\ldots,x_l) = \sum_{i=1}^{l} 2x_i^2 - \sum_{1\le i\ne j\le l} \sqrt{n_{ij}}\, x_i x_j \,.$$

The quadratic form for Y' is

$$Q'(x_1,\ldots,x_m) = \sum_{i=1}^{m} 2x_i^2 - \sum_{1\le i\ne j\le m} \sqrt{n'_{ij}}\, x_i x_j \,,$$

with $m \le l$ and $n'_{ij} \le n_{ij}$. Suppose Q' is not positive definite. Then there exists $(y_1,\ldots,y_m) \ne 0$ with $Q'(y_1,\ldots,y_m) \le 0$. Consider $Q(|y_1|,\ldots,|y_m|,0,\ldots,0)$. This is

$$\begin{aligned}\sum_{i=1}^{m} 2y_i^2 - \sum_{1\le i\ne j\le m} \sqrt{n_{ij}}\,|y_i||y_j| &\le \sum_{i=1}^{m} 2y_i^2 - \sum_{1\le i\ne j\le m} \sqrt{n'_{ij}}\,|y_i||y_j| \\ &\le \sum_{i=1}^{m} 2y_i^2 - \sum_{1\le i\ne j\le m} \sqrt{n_{ij}}\, y_i y_j \\ &\le Q'(y_1,\ldots,y_m)\end{aligned}$$

So $Q(|y_1|,\ldots,|y_m|,0,\ldots,0) \le Q'(y_1,\ldots,y_m) \le 0$. So $Q(x_1,\ldots,x_l)$ is not positive definite, a contradiction.

■

We now return to the proof of 10.6.

Suppose Y is some graph satisfying conditions. (i)-(iii). By 10.7 and 10.8 we know that no graph of the form $\widetilde{A},\widetilde{B},\widetilde{C},\widetilde{D},\widetilde{E},\widetilde{F}$ or $\widetilde{G}$ can be obtained as a subgraph of Y.

(a) Y contains no cycles, for otherwise Y would have a subgraph of the form $\widetilde{A}_l$.

(b) If Y has a triple edge then $Y = G_2$, for otherwise Y would have $\widetilde{G}_2$ as a subgraph.

(c) Suppose Y has no triple edge. Then Y can have no more than one double edge, for otherwise Y has a subgraph of type $\widetilde{C}_l$.

(d) Suppose Y has one double edge. Then Y has no branch point, for otherwise Y has $\widetilde{B}_l$ as a subgraph.

(e) If the double edge is not at one end then $Y = F_4$, for otherwise Y has a subgraph $\widetilde{F}_4$. If the double edge is at one end, $Y = B_l$.

(f) Now suppose Y has only single edges. Then Y cannot have a branch point with four or more branches, for otherwise Y has $\widetilde{D}_4$ as a subgraph.

(g) Y can have no more than one branch point, for otherwise Y has a subgraph $\widetilde{D}_l$, $l \geq 5$.

(h) If Y has no branch points, $Y = A_l$. So suppose Y has just one branch point with three branches of lengths $l_1 \leq l_2 \leq l_3$, $l_1 + l_2 + l_3 + 1 = l$. Then $l_1 = 1$, for otherwise Y would have $\widetilde{E}_6$ as a subgraph.

(i) If $l_1 = l_2 = 1$, $Y = D_l$. Also, $l_2 \leq 2$, for otherwise Y has $\widetilde{E}_7$ as a subgraph.

(j) So assume $l_1 = 1$, $l_2 = 2$. Then $l_3 \leq 4$, for otherwise Y has $\widetilde{E}_8$ as a subgraph.

$$l_3 = 2 \Rightarrow Y = E_6$$
$$l_3 = 3 \Rightarrow Y = E_7$$
$$l_3 = 4 \Rightarrow Y = E_8$$

■

Corollary 10.9. *Every Dynkin diagram of a semisimple Lie algebra has connected components of type* $A_l, B_l, D_l, E_6, E_7, E_8, F_4$ *and* G_2.

The Cartan matrix $A = (a_{ij})$ determines the Dynkin diagram since $n_{ij} = a_{ij}a_{ji}$. However, the Dynkin diagram does not always determine the Cartan matrix. Recall that the a_{ij} satisfy

$$a_{ii} = 2$$
$$a_{ij} \in \{0,1,2,3\} \text{ for } i \neq j$$

If $n_{ij} = 1$ then $a_{ij} = a_{ji} = 1$. If $n_{ij} = 2$ then $(a_{ij}, a_{ji}) = (-2,-1)$ or $(-1,-2)$. If $n_{ij} = 3$ then $(a_{ij}, a_{ji}) = (-3,-1)$ or $(-1,-3)$. In this last case the Dynkin diagram is G_2. We have

$$A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \text{ or } \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

and one is obtained from the other by re-labeling the vertices.

Suppose $n_{ij} = 2$. If $l = 2$ the possibilities for the Cartan matrix are

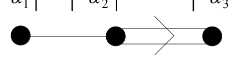
$$A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \text{ or } \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix},$$

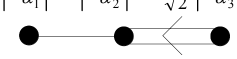
again obtainable from one another by re-labeling. If $l \geq 3$ there are two possible Cartan matrices:

$$B_l = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -2 & 2 \end{pmatrix} \qquad C_l = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -2 \\ & & & -1 & 2 \end{pmatrix}$$

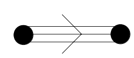
$$B_3 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix} \qquad C_3 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$

$$|h_{\alpha_1}| = |h_{\alpha_2}| = \sqrt{2}|h_{\alpha_3}| \qquad |h_{\alpha_1}| = |h_{\alpha_2}| = \tfrac{1}{\sqrt{2}}|h_{\alpha_3}|$$





We place an arrow on the Dynkin diagram when we have a double or triple edge; the arrow points from the longer root to the shorter one. For example, with G_2:



long short

Theorem 10.10. *The possible Cartan matrices with connected Dynkin diagrams are (up to permutation of the numbering of the vertices):*

$$A_l = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \qquad B_l = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -2 & 2 \end{pmatrix}$$

$$C_l = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -2 \\ & & & -1 & 2 \end{pmatrix} \qquad D_l = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & 2 & -1 & -1 \\ & & & -1 & 2 & 0 \\ & & & -1 & 0 & 2 \end{pmatrix}$$

$$E_6 = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & 0 & -1 \\ & & -1 & 2 & -1 & \\ & & 0 & -1 & 2 & \\ & & -1 & & & 2 \end{pmatrix} \qquad E_7 = \begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & -1 & 2 & -1 & 0 & -1 \\ & & & -1 & 2 & -1 & \\ & & & 0 & -1 & 2 & \\ & & & -1 & & & 2 \end{pmatrix}$$

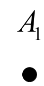
$$E_8 = \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & 0 & -1 \\ & & & & -1 & 2 & -1 & \\ & & & & 0 & -1 & 2 & \\ & & & & -1 & & & 2 \end{pmatrix}$$

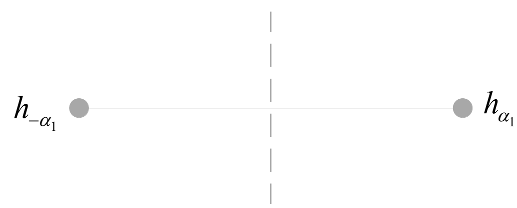
$$F_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \qquad G_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

11. THE INDECOMPOSABLE ROOT SYSTEMS

A root system is called *indecomposable* if it has a connected Dynkin diagram.

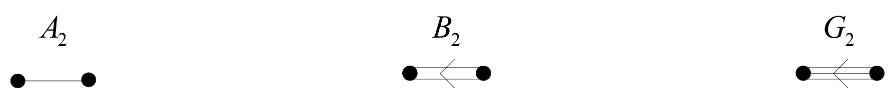
Case $l = 1$. We have only one possibility, A_1:



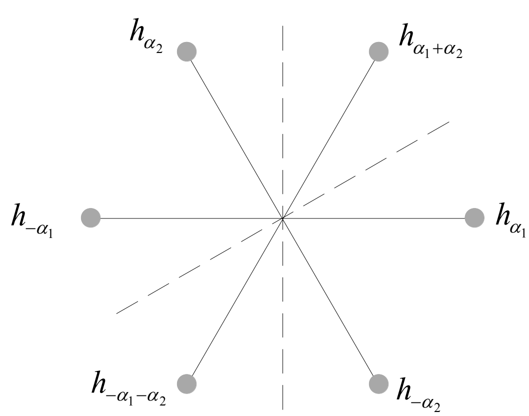


$$\Phi = \{\pm \alpha_1\}$$
$$W = \langle s_{\alpha_1} \rangle;\ |W| = 2$$

Case $l = 2$. Here we have three possibilities

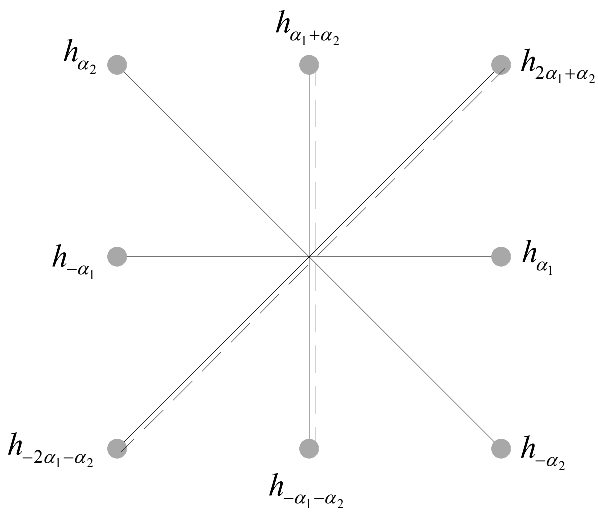


Type A_2.



$$\Phi = \{\pm \alpha_1, \pm \alpha_2, \pm(\alpha_1 + \alpha_2)\}$$

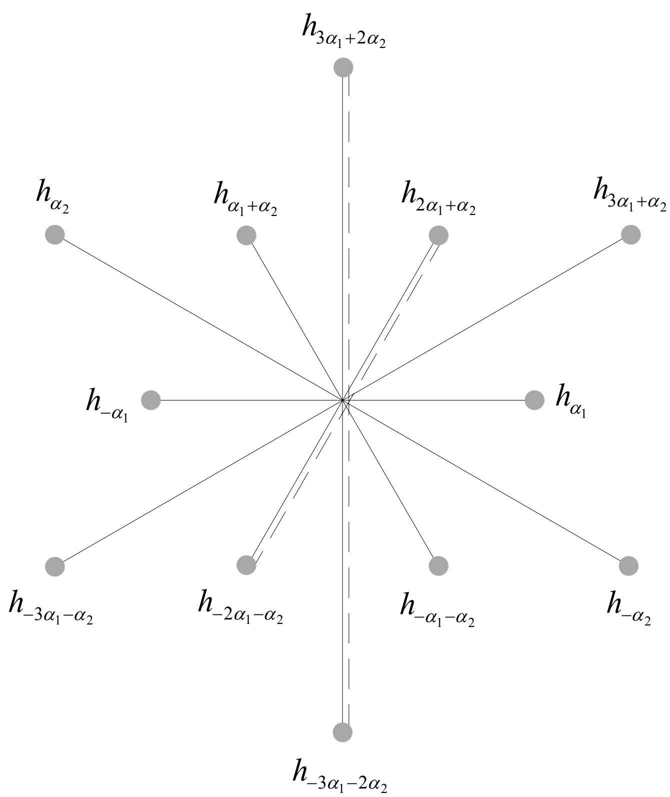
Type B_2.



$$\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2), \pm(2\alpha_1+\alpha_2)\}$$

(In the above diagram, the dashed lines indicating the reflection axes are shown slightly offset for clarity.)

Type G_2.

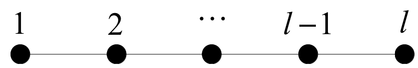


$$\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2), \pm(2\alpha_1+\alpha_2), \pm(3\alpha_1+\alpha_2), \pm(3\alpha_1+2\alpha_2)\}$$

(Again, the dashed lines indicating the reflection axes are shown slightly offset.)

Case $l \geq 3$. *Type* A_l. It is convenient to describe the root system of type A_l in a Euclidean space of dimension $l+1$.

Let V be an $(l+1)$-dimensional Euclidean space. Let $\{\varepsilon_0, \dots, \varepsilon_n\}$ be an orthogonal basis of vectors of the same length, so $\langle \varepsilon_i, \varepsilon_j \rangle = K\delta_{ij}$ for some $K > 0$.



Define $h_{\alpha_1} = \varepsilon_0 - \varepsilon_1$, $h_{\alpha_2} = \varepsilon_1 - \varepsilon_2$, …, $h_{\alpha_l} = \varepsilon_{l-1} - \varepsilon_l$. The h_{α_i} are linearly independent.

$$\langle h_{\alpha_i}, h_{\alpha_i} \rangle = 2K$$
$$\langle h_{\alpha_i}, h_{\alpha_j} \rangle = 0 \text{ if } j \neq i-1, i, i+1$$
$$\langle h_{\alpha_i}, h_{\alpha_{i+1}} \rangle = -K$$
$$a_{i,i+1} = 2\frac{\langle h_{\alpha_i}, h_{\alpha_{i+1}} \rangle}{\langle h_{\alpha_i}, h_{\alpha_i} \rangle} = \frac{-2K}{2K} = -1$$

Thus for suitable K the h_{α_i} form a fundamental system of roots of type A_l. Let V_0 be the subspace spanned by these vectors; $\dim(V_0) = l$. Consider the map $V \to V$ given by $\varepsilon_0 \leftrightarrow \varepsilon_1$, $\varepsilon_i \mapsto \varepsilon_i$ for $i \geq 2$.

$$h_{\alpha_1} \mapsto -h_{\alpha_1}$$
$$h_{\alpha_2} \mapsto h_{\alpha_1} + h_{\alpha_2}$$
$$h_{\alpha_i} \mapsto h_{\alpha_i} \text{ for } i \geq 2$$

This is s_{α_1}. Similarly, the linear map $V \to V$ such that $\varepsilon_{i-1} \leftrightarrow \varepsilon_i$, all others fixed, is s_{α_i}.

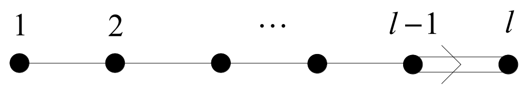
W is generated by $s_{\alpha_1}, \dots, s_{\alpha_l}$. The group generated by all transpositions $(\varepsilon_{i-1}\varepsilon_i)$ is isomorphic to S_{l+1}. So we have a homomorphism $S_{l+1} \to V$. This map is surjective; it is also injective, since any permutation of $\{\varepsilon_0, \dots, \varepsilon_n\}$ that fixes each h_{α_i} is the identity. Hence, $W \cong S_{l+1}$.

Each h_α has the form $h_\alpha = w(h_{\alpha_i})$ for some $w \in W$ and some i. Hence

$$\Phi = \{\varepsilon_i - \varepsilon_j \mid 0 \leq i \neq j \leq l\}.$$

So $|\Phi| = l(l+1)$.

Type B_l.



Let V be a Euclidean space of dimension l with basis $\{\varepsilon_1, \ldots, \varepsilon_l\}$ such that $\langle \varepsilon_i, \varepsilon_j \rangle = K\delta_{ij}$. Define

$$\begin{aligned}
h_{\alpha_1} &= \varepsilon_1 - \varepsilon_2 \\
h_{\alpha_2} &= \varepsilon_2 - \varepsilon_3 \\
&\vdots \\
h_{\alpha_{l-1}} &= \varepsilon_{l-1} - \varepsilon_l \\
h_{\alpha_l} &= \varepsilon_l
\end{aligned}$$

These form a fundamental system of vectors of type B_l.

$$\left|h_{\alpha_1}\right| = \ldots = \left|h_{\alpha_{l-1}}\right| = \sqrt{2}\left|h_{\alpha_l}\right|$$

$$\left\langle h_{\alpha_i}, h_{\alpha_l} \right\rangle = 0 \text{ for } 1 \le i \le l-2$$

$$\left\langle h_{\alpha_{l-1}}, h_{\alpha_l} \right\rangle = -K$$

$$2\frac{\left\langle h_{\alpha_{l-1}}, h_{\alpha_l} \right\rangle}{\left\langle h_{\alpha_{l-1}}, h_{\alpha_{l-1}} \right\rangle} = \frac{-2K}{2K} = -1$$

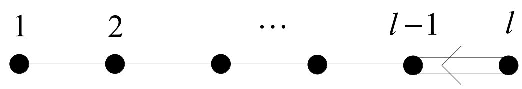
$$\begin{aligned}
s_{\alpha_1} &: \varepsilon_1 \leftrightarrow \varepsilon_2 \text{ and leaves others fixed,} \\
s_{\alpha_2} &: \varepsilon_2 \leftrightarrow \varepsilon_3 \text{ and leaves others fixed,} \\
&\vdots \\
s_{\alpha_{l-1}} &: \varepsilon_{l-1} \leftrightarrow \varepsilon_l \text{ and leaves others fixed,} \\
s_{\alpha_l} &: \varepsilon_l \mapsto -\varepsilon_l \text{ and leaves others fixed.}
\end{aligned}$$

$W = \left\langle s_{\alpha_1}, \ldots, s_{\alpha_l} \right\rangle$; for $w \in W$, $w(\varepsilon_i) = \pm\varepsilon_j$. $|W| = 2^l l!$. Each h_α has the form $h_\alpha = w(h_{\alpha_i})$ for some $w \in W$ and some i. Hence,

$$\Phi = \left\{ \pm\varepsilon_i \pm \varepsilon_j \;\middle|\; 1 \le i \ne j \le l \right\} \cup \left\{ \pm\varepsilon_i \;\middle|\; 1 \le i \le l \right\}$$

$$|\Phi| = 2^2\binom{l}{2} + 2l = 2l(l-1) + 2l = 2l^2$$

Type C_l.



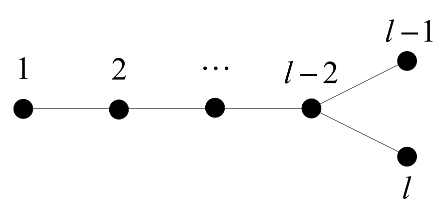
Let V be a Euclidean space of dimension l with basis $\{\varepsilon_1,\ldots,\varepsilon_l\}$ such that $\langle\varepsilon_i,\varepsilon_j\rangle = K\delta_{ij}$. Define

$$\begin{aligned} h_{\alpha_1} &= \varepsilon_1 - \varepsilon_2 \\ &\vdots \\ h_{\alpha_{l-1}} &= \varepsilon_{l-1} - \varepsilon_l \\ h_{\alpha_l} &= 2\varepsilon_l \end{aligned}$$

W is the same as for B_l. For $w \in W$, $w(\varepsilon_i) = \pm\varepsilon_j$. $|W| = 2^l l!$. The h_α are the vectors $\pm\varepsilon_i \pm \varepsilon_j$ (for $i \neq j$) and $\pm 2\varepsilon_i$. Hence,

$$|\Phi| = 2^2\binom{l}{2} + 2l = 2l(l-1) + 2l = 2l^2$$

Type D_l.



Let V be a Euclidean space of dimension l with basis $\{\varepsilon_1,\ldots,\varepsilon_l\}$ such that $\langle\varepsilon_i,\varepsilon_j\rangle = K\delta_{ij}$. Define

$$\begin{aligned} h_{\alpha_1} &= \varepsilon_1 - \varepsilon_2 \\ &\vdots \\ h_{\alpha_{l-1}} &= \varepsilon_{l-1} - \varepsilon_l \\ h_{\alpha_l} &= \varepsilon_{l-1} + \varepsilon_l \end{aligned}$$

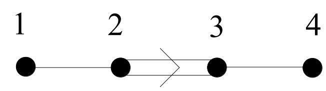
This is a fundamental system of roots of type D_l.

$$
\begin{gathered}
s_{\alpha_1} : \varepsilon_1 \leftrightarrow \varepsilon_2 \text{ and leaves others fixed,} \\
\vdots \\
s_{\alpha_{l-1}} : \varepsilon_{l-1} \leftrightarrow \varepsilon_l \text{ and leaves others fixed,} \\
s_{\alpha_l} : \pm\varepsilon_{l-1} \mapsto \mp\varepsilon_l \text{ and leaves others fixed.}
\end{gathered}
$$

For $w \in W$, $w(\varepsilon_i) = \pm\varepsilon_j$. There will be an even number of sign changes, so $|W| = 2^{l-1} l!$. The h_α have the form $\pm\varepsilon_i \pm \varepsilon_j$ for $i \neq j$. So

$$|\Phi| = 2^2 \binom{l}{2} = 4\,\tfrac{l(l-1)}{2} = 2l(l-1)$$

Type F_4.



Let V be a 4-dimensional Euclidean space with basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}, \langle \varepsilon_i, \varepsilon_j \rangle = K\delta_{ij}$.

$$
\begin{aligned}
h_{\alpha_1} &= \varepsilon_1 - \varepsilon_2 \\
h_{\alpha_2} &= \varepsilon_2 - \varepsilon_3 \\
h_{\alpha_3} &= \varepsilon_3 \\
h_{\alpha_4} &= \tfrac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4)
\end{aligned}
$$

This is a fundamental system of vectors of type F_4. $s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_3}$ permute $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and change signs arbitrarily.

$$
s_{\alpha_4} : \begin{cases}
\varepsilon_1 \mapsto \tfrac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4) = \varepsilon_1 + h_{\alpha_4} \\
\varepsilon_2 \mapsto \tfrac{1}{2}(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4) = \varepsilon_2 + h_{\alpha_4} \\
\varepsilon_3 \mapsto \tfrac{1}{2}(-\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4) = \varepsilon_3 + h_{\alpha_4} \\
\varepsilon_4 \leftrightarrow \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4) \\
\varepsilon_1 + \varepsilon_2 \mapsto -\varepsilon_3 + \varepsilon_4
\end{cases}
$$

Let $S = \{h_\alpha \mid \alpha \in \Phi\}$.

$$
\begin{gathered}
\{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \le i \neq j \le 3\} \subseteq S \\
\{\pm\varepsilon_i \mid 1 \le i \le 3\} \subseteq S
\end{gathered}
$$

$$\tfrac{1}{2}\left(\pm\varepsilon_1 \pm\varepsilon_2 \pm\varepsilon_3 \pm\varepsilon_4\right)\in S$$
$$\left\{\pm\varepsilon_i \pm\varepsilon_4 \mid 1\le i\le 3\right\}\subseteq S$$
$$\pm\varepsilon_4 \in S$$

So S contains

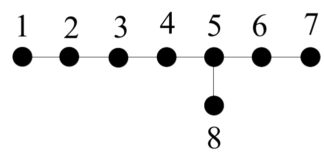
$$\pm\varepsilon_i \pm\varepsilon_j \text{ for } 1\le i\ne j\le 4$$
$$\pm\varepsilon_i \text{ for } 1\le i\le 4$$
$$\tfrac{1}{2}\left(\pm\varepsilon_1 \pm\varepsilon_2 \pm\varepsilon_3 \pm\varepsilon_4\right)$$

This collection of vectors is closed under the actions of $s_{\alpha_1},\dots,s_{\alpha_4}$.

$$|\Phi| = 2^2\binom{4}{2} + 2.4 + 2^4 = 48$$

(We have 24 short roots and 24 long ones.)

Type E_8.



Let V be a Euclidean space of dimension 8 with basis $\{\varepsilon_1,\dots,\varepsilon_8\}$ such that $\langle\varepsilon_i,\varepsilon_j\rangle = K\delta_{ij}$. Define

$$\left.\begin{aligned} h_{\alpha_1} &= \varepsilon_1 - \varepsilon_2 \\ &\vdots \\ h_{\alpha_6} &= \varepsilon_6 - \varepsilon_7 \\ h_{\alpha_7} &= \varepsilon_6 + \varepsilon_7 \end{aligned}\right\}\text{type } D_7$$
$$h_{\alpha_8} = -\tfrac{1}{2}\left(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6+\varepsilon_7+\varepsilon_8\right)$$
$$\left|h_{\alpha_7}\right| = \left|h_{\alpha_8}\right|$$
$$\left\langle h_{\alpha_7}, h_{\alpha_8}\right\rangle = -K$$
$$2\frac{\left\langle h_{\alpha_7}, h_{\alpha_8}\right\rangle}{\left\langle h_{\alpha_7}, h_{\alpha_7}\right\rangle} = \frac{-2K}{2K} = -1$$

These vectors form a fundamental system of type E_8. $s_{\alpha_1},\dots,s_{\alpha_7}$ permute $\varepsilon_1,\dots,\varepsilon_7$ and change an even number of signs.

$$s_{\alpha_8}:\varepsilon_i \mapsto \tfrac{1}{4}\left(-\varepsilon_1+\dots+3\varepsilon_i+\dots-\varepsilon_8\right)=\varepsilon_i+\tfrac{1}{2}h_{\alpha_8}$$

Let $S=\{h_\alpha \mid \alpha\in\Phi\}$. S contains $\pm\varepsilon_i\pm\varepsilon_j$ for $1\le i\ne j\le 7$. S contains $\frac{1}{2}\sum_{i=1}^{8}\varepsilon_i$ so S contains $\frac{1}{2}\left(-\varepsilon_1-\dots-\varepsilon_6+\varepsilon_7+\varepsilon_8\right)=s_{\alpha_8}\left(\varepsilon_7+\varepsilon_8\right)$. So $\varepsilon_7+\varepsilon_8\in S$. So S contains $\pm\varepsilon_i\pm\varepsilon_8$ for $1\le i\le 7$. S also contains $\frac{1}{2}\left(\pm\varepsilon_1\pm\dots\pm\varepsilon_8\right)$ with an even number of negative signs. So S contains

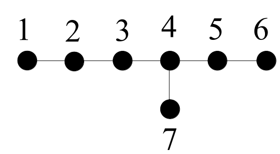
$$\pm\varepsilon_i\pm\varepsilon_j \text{ for } 1\le i\ne j\le 8$$
$$\tfrac{1}{2}\left(\pm\varepsilon_1\pm\dots\pm\varepsilon_8\right) \text{ with an even number of negative signs}$$

This is the whole of S, for it is invariant under $s_{\alpha_1},\dots,s_{\alpha_8}$: this is clear for $s_{\alpha_1},\dots,s_{\alpha_7}$ but requires a little work to check for s_{α_8}. So the roots of E_8 are

$$\pm\varepsilon_i\pm\varepsilon_j \text{ for } 1\le i\ne j\le 8$$
$$\tfrac{1}{2}\sum_{i=1}^{8}\pm\varepsilon_i \text{ where } \Pi(\pm)=1$$

$$|\Phi|=2^2\binom{8}{2}+2^7=240$$

Type E_7.



Take V as before – $\dim(V)=8$. Take V_0 to be the subspace of V perpendicular to $\varepsilon_1-\varepsilon_8$. $\dim(V_0)=7$ and $h_{\alpha_2},\dots,h_{\alpha_8}$ form a basis of V_0. This is a fundamental system of type E_7. Consider $S=\{h_\alpha \mid \alpha\in\Phi(E_7)\}$. This set lies in $\{h_\alpha \mid \alpha\in\Phi(E_8)\}\cap V_0$. This intersection is

$$\pm\varepsilon_i\pm\varepsilon_j \text{ for } 2\le i\ne j\le 7$$
$$\pm\left(\varepsilon_1+\varepsilon_8\right)$$
$$\tfrac{1}{2}\left(\varepsilon_1\pm\varepsilon_2\pm\dots\pm\varepsilon_7+\varepsilon_8\right) \text{ where } \Pi(\pm)=1$$
$$\tfrac{1}{2}\left(-\varepsilon_1\pm\varepsilon_2\pm\dots\pm\varepsilon_7-\varepsilon_8\right) \text{ where } \Pi(\pm)=1$$

All of these can be obtained from $h_{\alpha_2},\ldots,h_{\alpha_8}$ by means of $s_{\alpha_2},\ldots,s_{\alpha_8}$. This is obvious except for $\pm(\varepsilon_1+\varepsilon_8)$.

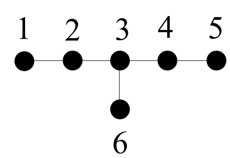
$$s_{\alpha_8} : \varepsilon_1+\varepsilon_2 \leftrightarrow \tfrac{1}{2}(\varepsilon_1-\varepsilon_2-\ldots-\varepsilon_7+\varepsilon_8)$$

So $\pm(\varepsilon_1+\varepsilon_8)\in S$. So S is

$$\begin{gathered}\pm\varepsilon_i\pm\varepsilon_j \text{ for } 2\le i\ne j\le 7\\ \pm(\varepsilon_1+\varepsilon_8)\\ \tfrac{1}{2}(\varepsilon_1\pm\varepsilon_2\pm\ldots\pm\varepsilon_7+\varepsilon_8) \text{ where } \Pi(\pm)=1\\ \tfrac{1}{2}(-\varepsilon_1\pm\varepsilon_2\pm\ldots\pm\varepsilon_7-\varepsilon_8) \text{ where } \Pi(\pm)=1\end{gathered}$$

$$|\Phi(E_7)| = 2^2\binom{6}{2}+2+2^5+2^5$$

Type E_6.



We proceed as before. $h_{\alpha_3},\ldots,h_{\alpha_8}$ form a fundamental system of vectors of type E_6. Let V_0 be the subspace of V for E_8 that is orthogonal to $\varepsilon_1-\varepsilon_8$ and $\varepsilon_2-\varepsilon_8$. $\dim(V_0)=6$ and $h_{\alpha_3},\ldots,h_{\alpha_8}$ form a basis of this space.

$$\{h_\alpha \mid \alpha\in\Phi(E_6)\}\subseteq\{h_\alpha \mid \alpha\in\Phi(E_8)\}\cap V_0$$

The h_α in V_0 are

$$\begin{gathered}\pm\varepsilon_i\pm\varepsilon_j \text{ for } 3\le i\ne j\le 7\\ \tfrac{1}{2}(\varepsilon_1+\varepsilon_2\pm\varepsilon_3\pm\ldots\pm\varepsilon_7+\varepsilon_8) \text{ where } \Pi(\pm)=1\\ \tfrac{1}{2}(-\varepsilon_1-\varepsilon_2\pm\varepsilon_3\pm\ldots\pm\varepsilon_7-\varepsilon_8) \text{ where } \Pi(\pm)=1\end{gathered}$$

All of these are obtainable from $h_{\alpha_3},\ldots,h_{\alpha_8}$ by $s_{\alpha_3},\ldots,s_{\alpha_8}$.

$$|\Phi(E_6)| = 2^2\binom{5}{2}+2^4+2^4 = 72$$

Theorem 11.1. *The number of roots in each of the indecomposable root systems is*

A_l	B_l	C_l	D_l	E_6	E_7	E_8	F_4	G_2
$l(l+2)$	$l(2l+1)$	$l(2l+1)$	$l(2l-1)$	78	133	248	52	14

12. THE SEMISIMPLE LIE ALGEBRAS

Theorem 12.1. *(a) If a semisimple Lie algebra L has connected Dynkin diagram Δ then L is simple.*
(b) If L is a semisimple Lie algebra whose Dynkin diagram Δ has connected components $\Delta_1,\ldots,\Delta_r$ then $L = L_1 \oplus \cdots \oplus L_r$ where L_i is a simple Lie algebra with Dynkin diagram Δ_i.

Proof. (a) Let $L = H \oplus (\oplus_{\alpha\in\Phi} L_\alpha)$ be a Cartan decomposition with connected Dynkin diagram Δ. Let $0 \neq I \lhd L$. We first show that $I \cap H \neq 0$. Suppose not, i.e. that $I \cap H = 0$. Let $0 \neq x \in I$ with $x = h + \sum_\alpha \mu_\alpha e_\alpha$ and the number of non-zero μ_α as small as possible. Let $\mu_\beta \neq 0$.

$$[xh_\beta] = \sum_\alpha \mu_\alpha [e_\alpha h_\beta] = \sum_\alpha \mu_\alpha \alpha(h_\beta) e_\alpha$$

By 8.7 we can choose $e_{-\beta}$ with $[e_{-\beta}e_\beta] = h_\beta$.

$$[[xh_\beta]e_{-\beta}] = -\mu_\beta \beta(h_\beta) h_\beta + \sum_{\substack{\alpha\in\Phi \\ \alpha-\beta\in\Phi}} \mu_\alpha \alpha(h_\beta) N_{\alpha,-\beta} e_{\alpha-\beta}$$

$[[xh_\beta]e_{-\beta}] \in I$ is non-zero since

$$-\mu_\beta \beta(h_\beta) h_\beta = -\underbrace{\mu_\beta}_{\neq 0} \underbrace{\langle h_\beta, h_\beta \rangle}_{\neq 0} h_\beta \neq 0$$

The number of non-zero μ_α with $\alpha - \beta \in \Phi$ is less than before, a contradiction. Hence, $I \cap H \neq 0$.

We next show that $I \supseteq H$. Suppose not. Then $0 \subset I \cap H \subset H$. $I \cap H$ is not orthogonal to all h_{α_i}, $\alpha_i \in \Pi$. For suppose $I \cap H$ is not orthogonal to h_{α_i}. Let $x \in I \cap H$ be such that $\langle x, h_{\alpha_i} \rangle \neq 0$. Then

$$[e_{\alpha_i} x] = \alpha_i(x) e_{\alpha_i} = \langle h_{\alpha_i}, x \rangle e_{\alpha_i} \in I$$

So $e_{\alpha_i} \in I$. So $[e_{-\alpha_i} e_{\alpha_i}] = h_{\alpha_i} \in I$. So for each $\alpha_i \in \Pi$ either $\langle I \cap H, h_{\alpha_i} \rangle = 0$ or $h_{\alpha_i} \in I$. Both classes are non-empty. Choose $h_{\alpha_j} \notin I$; then $\langle I \cap H, h_{\alpha_j} \rangle = 0$. This means Δ is disconnected, a contradiction. Hence, $H \subseteq I$.

Now let $\alpha \in \Phi$.

$$[e_\alpha h_\alpha] = \alpha(h_\alpha)e_\alpha = \underbrace{\langle h_\alpha, h_\alpha \rangle}_{\neq 0} e_\alpha$$

So $e_\alpha \in I$. Hence I contains H and all e_α. So $I = L$; L is simple.

(b) Suppose Δ is the disjoint union of connected components $\Delta_1, \ldots, \Delta_r$. Then Π is the union of orthogonal components $\Pi_1, \ldots, \Pi_r$. Let H_i be the subspace spanned by $\{h_\alpha \mid \alpha \in \Pi_i\}$. Then $H = H_1 \oplus \ldots \oplus H_r$ and the H_i are mutually orthogonal. Now consider s_α for some $\alpha \in \Pi_i$. Then α transforms H_i into itself and fixes each vector in H_j for $j \neq i$; $s_\alpha(H_j) = H_j$. Since the s_α for $\alpha \in \Pi$ generate W, $w(H_j) = H_j$ for each $w \in W$.

For each $\alpha \in \Phi$, $h_\alpha = w(h_{\alpha_i})$ for some $w \in W$ and some i. So $h_\alpha \in H_i$ for some i. Let $\Phi_i = \{\alpha \in \Phi \mid h_\alpha \in H_i\}$. Then $\Phi = \Phi_1 \cup \ldots \cup \Phi_r$. Let L_i be the subspace of L spanned by H_i and the L_α with $\alpha \in \Phi_i$. We see that $L = L_1 \oplus \ldots \oplus L_r$ as a direct sum of vector spaces.

To see that L_i is a subalgebra of L it is sufficient to show that $\alpha, \beta \in \Phi_i \Rightarrow [e_\alpha e_\beta] \in L_i$. If $\alpha + \beta \notin \Phi$ then $[e_\alpha e_\beta] = 0$. If $\beta = -\alpha$ then $[e_\alpha e_{-\alpha}] = -h_\alpha \in H_i$. If $\alpha + \beta \in \Phi$ then $\alpha + \beta \in \Phi_i$ and $h_{\alpha+\beta} = h_\alpha + h_\beta \in H_i$. So L_i is a subalgebra.

We next check that $i \neq j \Rightarrow [L_i L_j] = 0$. Let $\alpha \in \Phi_i$, $\beta \in \Phi_j$.

$$[h_\alpha h_\beta] = 0$$
$$[h_\alpha e_\beta] = 0 \text{ since } \langle h_\alpha, h_\beta \rangle = 0$$
$$[e_\alpha h_\beta] = 0 \text{ since } \langle h_\alpha, h_\beta \rangle = 0$$
$$[e_\alpha e_\beta] = 0 \text{ since } \alpha + \beta \notin \Phi$$

$h_\alpha + h_\beta$ does not lie in any H_k, $h_\alpha \in H_i$, $h_\beta \in H_j$. So $[L_i L_j] = 0$ for $i \neq j$:

$$[x_1 + \ldots + x_r, y_1 + \ldots + y_r] = [x_1 y_1] + \ldots + [x_r y_r]$$

So $L = L_1 \oplus \ldots \oplus L_r$ as a direct sum of Lie algebras. We now see that each L_i is semisimple. Let $I \lhd L_i$ be soluble. $[IL_j] = 0$ if $i \neq j$, so $I \lhd L$. I is a soluble ideal of L, but L is semisimple, so $I = 0$, so L_i is semisimple.

We now show that H_i is a Cartan subalgebra of L_i. H is a Cartan subalgebra of L, so there is a regular element $x \in L$ such that H is the 0-(generalized) eigenspace of $\operatorname{ad} x$. Since $x \in H$ and $H = H_1 \oplus \ldots \oplus H_r$ we can write $x = x_1 + \ldots + x_r$ with $x_i \in H_i$.

$$\operatorname{ad} x_i : L_j \to \begin{cases} L_j & i = j \\ 0 & i \neq j \end{cases}$$

So the 0-eigenspace of $\operatorname{ad} x$ on L is the direct sum of the 0-eigenspaces of the $\operatorname{ad} x_i$ on the L_i. x is regular in L if and only if each x_i is regular in L_i. So each x_i is regular in L_i and the 0-eigenspace of $\operatorname{ad} x_i$ in L_i is H_i. So H_i is a Cartan subalgebra of L_i.

$$L_i = H_i \oplus \left(\oplus_{\alpha \in \Phi_i} L_\alpha\right)$$

is a Cartan decomposition of L_i. So Φ_i is the root system of L_i; Π_i is a fundamental root system of L_i; the Dynkin diagram of L_i is Δ_i. But Δ_i is connected, so L_i is simple by (a).

■

We next consider simple Lie algebras with a given indecomposable Cartan matrix A.

Existence Problem: Is there a simple Lie algebra with given Cartan matrix A?

Isomorphism Problem: Are any two such Lie algebras isomorphic?

Let L be a simple Lie algebra and H a Cartan subalgebra of L:

$$L = H \oplus \left(\oplus_{\alpha \in \Phi} L_\alpha\right)$$
$$\Phi = \Phi^+ \cup \Phi^-$$

For each $\alpha \in \Phi^+$ choose $0 \neq e_\alpha \in L_\alpha$; $L_\alpha = \mathbb{C} e_\alpha$. Choose $e_{-\alpha} \in L_{-\alpha}$ such that $[e_{-\alpha} e_\alpha] = h_\alpha$. If $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ then the h_{α_i} and e_α form a basis of L. $[L_\alpha L_\beta] \subseteq L_{\alpha+\beta}$ so $[e_\alpha e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$ if $\alpha + \beta \in \Phi$, $\alpha \neq -\beta$.

$$[h_{\alpha_i} h_{\alpha_j}] = 0$$
$$[e_\alpha h_{\alpha_i}] = \langle h_\alpha, h_{\alpha_i} \rangle e_\alpha$$
$$[e_{-\alpha} e_\alpha] = h_\alpha$$
$$[e_\alpha e_\beta] = \begin{cases} N_{\alpha,\beta} e_{\alpha+\beta} & \alpha + \beta \in \Phi \\ 0 & \text{otherwise} \end{cases}$$

The $N_{\alpha,\beta}$ are called the *structure constants*.

Proposition 12.2. *The structure constants* $N_{\alpha,\beta}$ *satisfy*
(i) $N_{\alpha,\beta} = -N_{\beta,\alpha}$*;*
(ii) if $\alpha,\beta,\gamma \in \Phi$ *have* $\alpha+\beta+\gamma=0$ *then* $N_{\alpha,\beta} = N_{\beta,\gamma} = N_{\gamma,\alpha}$*;*
(ii) if $\alpha,\beta,\gamma,\delta \in \Phi$ *have* $\alpha+\beta+\gamma+\delta=0$ *and no pair have sum zero then*

$$N_{\alpha,\beta}N_{\gamma,\delta} + N_{\beta,\gamma}N_{\alpha,\delta} + N_{\gamma,\alpha}N_{\beta,\delta} = 0$$

(if $\xi,\eta \in \Phi$*,* $\eta \neq -\xi$*,* $\eta+\xi \notin \Phi$ *take* $N_{\xi,\eta}=0$*);*
(iv) if $\alpha,\beta \in \Phi$ *have* $\alpha+\beta \in \Phi$ *then*

$$N_{\alpha,\beta}N_{-\alpha,-\beta} = -\frac{(p+1)q}{2}\langle h_\alpha, h_\alpha\rangle$$

where the α *-chain of roots through* β *is*

$$-p\alpha+\beta,\ldots,\beta,\ldots,q\alpha+\beta.$$

In particular, $N_{\alpha,\beta} \neq 0$*, so* $[L_\alpha L_\beta] = L_{\alpha+\beta}$.

Proof. (i) $[e_\alpha e_\beta] = -[e_\beta e_\alpha]$ so $N_{\alpha,\beta} = -N_{\beta,\alpha}$.

(ii) Suppose $\alpha+\beta+\gamma=0$.

$$\begin{aligned}
& [[e_\alpha e_\beta]e_\gamma] + [[e_\beta e_\gamma]e_\alpha] + [[e_\gamma e_\alpha]e_\beta] = 0 \\
\Rightarrow\quad & N_{\alpha,\beta}[e_{\alpha+\beta}e_\gamma] + N_{\beta,\gamma}[e_{\beta+\gamma}e_\alpha] + N_{\gamma,\alpha}[e_{\gamma+\alpha}e_\beta] = 0 \\
\Rightarrow\quad & N_{\alpha,\beta}h_\gamma + N_{\beta,\gamma}h_\alpha + N_{\gamma,\alpha}h_\beta = 0 \\
\Rightarrow\quad & \left(-N_{\alpha,\beta} + N_{\beta,\gamma}\right)h_\alpha + \left(-N_{\alpha,\beta} + N_{\gamma,\alpha}\right)h_\beta = 0
\end{aligned}$$

Since h_α and h_β are linearly independent, $N_{\alpha,\beta} = N_{\beta,\gamma} = N_{\gamma,\alpha}$.

(iii) Take $\alpha,\beta,\gamma,\delta \in \Phi$ with zero sum and no opposite pairs.

$$\begin{aligned}
& [[e_\alpha e_\beta]e_\gamma] + [[e_\beta e_\gamma]e_\alpha] + [[e_\gamma e_\alpha]e_\beta] = 0 \\
\Rightarrow\quad & N_{\alpha,\beta}[e_{\alpha+\beta}e_\gamma] + N_{\beta,\gamma}[e_{\beta+\gamma}e_\alpha] + N_{\gamma,\alpha}[e_{\gamma+\alpha}e_\beta] = 0 \\
\Rightarrow\quad & \left(N_{\alpha,\beta}N_{\alpha+\beta,\gamma} + N_{\beta,\gamma}N_{\beta+\gamma,\alpha} + N_{\gamma,\alpha}N_{\gamma+\alpha,\beta}\right)e_{\alpha+\beta+\gamma} = 0 \\
\Rightarrow\quad & N_{\alpha,\beta}N_{\gamma,\delta} + N_{\beta,\gamma}N_{\alpha,\delta} + N_{\gamma,\alpha}N_{\beta,\delta} = 0
\end{aligned}$$

(iv) Let $\alpha,\beta \in \Phi$ with $\alpha+\beta \in \Phi$.

$$
\begin{aligned}
& [[e_\alpha e_{-\alpha}]e_\beta]+[[e_{-\alpha}e_\beta]e_\alpha]+[[e_\beta e_\alpha]e_{-\alpha}]=0 \\
\Rightarrow\quad & -[h_\alpha e_\beta]+N_{-\alpha,\beta}[e_{-\alpha+\beta}e_\alpha]+N_{\beta,\alpha}[e_{\alpha+\beta}e_{-\alpha}]=0 \\
\Rightarrow\quad & \left(\beta(h_\alpha)+N_{-\alpha,\beta}N_{-\alpha+\beta,\alpha}+N_{\beta,\alpha}N_{\alpha+\beta,-\alpha}\right)e_\beta=0 \\
\Rightarrow\quad & N_{\alpha,\beta}N_{-\alpha+\beta,\alpha}+N_{\beta,\alpha}N_{\alpha+\beta,-\alpha}=-\left\langle h_\alpha,h_\beta\right\rangle \\
\Rightarrow\quad & N_{\alpha,\beta}N_{-\alpha,-\beta}-N_{\alpha,-\alpha+\beta}N_{-\alpha,\alpha-\beta}=\left\langle h_\alpha,h_\beta\right\rangle
\end{aligned}
$$

Take $M_{\alpha,\beta}=N_{\alpha,\beta}N_{-\alpha,-\beta}$, so

$$M_{\alpha,\beta}-M_{\alpha,-\alpha+\beta}=\left\langle h_\alpha,h_\beta\right\rangle$$

Let the α -chain of roots through β be

$$-p\alpha+\beta,\ldots,\beta,\ldots,q\alpha+\beta$$

So

$$
\begin{aligned}
M_{\alpha,\beta}-M_{\alpha,-\alpha+\beta} &= \left\langle h_\alpha,h_\beta\right\rangle \\
M_{\alpha,-\alpha+\beta}-M_{\alpha,-2\alpha+\beta} &= \left\langle h_\alpha,h_{-\alpha+\beta}\right\rangle \\
M_{\alpha,-2\alpha+\beta}-M_{\alpha,-3\alpha+\beta} &= \left\langle h_\alpha,h_{-2\alpha+\beta}\right\rangle \\
&\vdots \\
M_{\alpha,-p\alpha+\beta} &= \left\langle h_\alpha,h_{-p\alpha+\beta}\right\rangle
\end{aligned}
$$

So

$$
\begin{aligned}
M_{\alpha,\beta} &= (p+1)\left\langle h_\alpha,h_\beta\right\rangle-\left\langle h_\alpha,h_\alpha\right\rangle(1+2+\ldots+p) \\
&= (p+1)\left\langle h_\alpha,h_\beta\right\rangle-\tfrac{p(p+1)}{2}\left\langle h_\alpha,h_\alpha\right\rangle
\end{aligned}
$$

By, by 8.11, $2\left\langle h_\alpha,h_\beta\right\rangle\big/\left\langle h_\alpha,h_\alpha\right\rangle=p-q$, so

$$
\begin{aligned}
M_{\alpha,\beta} &= \left\langle h_\alpha,h_\alpha\right\rangle\left(\tfrac{(p+1)(p-q)}{2}-\tfrac{p(p+1)}{2}\right) \\
&= -\tfrac{(p+1)q}{2}\left\langle h_\alpha,h_\alpha\right\rangle
\end{aligned}
$$

So $N_{\alpha,\beta}\neq 0$; $[L_\alpha L_\beta]=L_{\alpha+\beta}$.

■

This result has certain consequences. Let $\alpha,\beta\in\Phi$, $\alpha+\beta\in\Phi$, $[e_\alpha e_\beta]=N_{\alpha,\beta}e_{\alpha+\beta}$. Let $\gamma=-\alpha-\beta$. Then $\alpha+\beta+\gamma=0$. We have the following ordered pairs of roots whose sum is a root:

$$\begin{array}{cccccc} (\alpha,\beta) & (\beta,\gamma) & (\gamma,\alpha) & (\beta,\alpha) & (\gamma,\beta) & (\alpha,\gamma) \\ (-\alpha,-\beta) & (-\beta,-\gamma) & (-\gamma,-\alpha) & (-\beta,-\alpha) & (-\gamma,-\beta) & (-\alpha,-\gamma) \end{array}$$

We have a total order $\alpha\prec\beta$. An ordered pair (α,β) such that $0\prec\alpha\prec\beta$ is called a *special pair*.

Either one or two of α,β,γ are positive; if one is positive two of $-\alpha,-\beta,-\gamma$ are positive. Of the twelve pairs above just one is special.

$N_{\alpha,\beta}$, for any ordered pair (α,β), can be expressed in terms of $N_{\xi,\eta}$ for (ξ,η) a special pair by using 12.2(i), (ii), (iv). So consider $N_{\alpha,\beta}$ when (α,β) is special; $\alpha+\beta\in\Phi^+\setminus\Pi$. This root may be expressible as $\alpha+\beta=\alpha'+\beta'$ where (α',β') is special and distinct from (α,β).

A special pair (α,β) is called *extra special* if for any special pair (α',β') with $\alpha+\beta=\alpha'+\beta'$ we have $\alpha\preceq\alpha'$.

The number of extra special pairs is $\left|\Phi^+\setminus\Pi\right|$.

Now let (α',β') be special but not extra special. Then $\alpha+\beta=\alpha'+\beta'$ where (α,β) is extra special – such an extra special pair exists because the set of special and extra special pairs is finite.

$$\alpha'+\beta'+(-\alpha)+(-\beta)=0$$
$$N_{\alpha',\beta'}N_{-\alpha,-\beta}+N_{\beta',-\alpha}N_{\alpha',-\beta}+N_{-\alpha,\alpha'}N_{\beta',-\beta}=0$$
$$0\prec\alpha\prec\alpha'\prec\beta'\prec\beta$$
$$N_{\alpha',\beta'}N_{-\alpha,-\beta}+N_{\beta-\alpha',\alpha'}N_{-\alpha,-(\beta'-\alpha)}+N_{\beta-\beta',\beta'}N_{-(\alpha'-\alpha),-\alpha}=0$$

We show that $N_{\alpha',\beta'}$ is determined by $N_{\xi,\eta}$'s for extra special pairs (ξ,η). We use induction on $\alpha'+\beta'$:

$N_{\alpha',\beta'}$ is determined by $N_{\alpha,\beta}$, $N_{\beta-\alpha',\alpha'}$, $N_{\alpha,\beta'-\alpha}$, $N_{\beta-\beta',\beta'}$, $N_{\alpha'-\alpha,\alpha}$. (α,β) is extra special.

Either $(\beta-\alpha',\alpha')$ or $(\alpha',\beta-\alpha')$ is special:

$$(\beta-\alpha')+\alpha'=\beta\prec\alpha+\beta=\alpha'+\beta'$$

So $N_{\beta-\alpha',\alpha'}$ can be expressed in terms of $N_{\xi,\eta}$ for extra special pairs (ξ,η).

Either $(\alpha,\beta'-\alpha)$ or $(\beta'-\alpha,\alpha)$ is special:

$$\alpha+(\beta'-\alpha)=\beta'\prec\alpha'+\beta'$$

Either $(\beta-\beta',\beta')$ or $(\beta',\beta-\beta')$ is special:

$$(\beta-\beta')+\beta'=\beta\prec\alpha'+\beta'$$

Either $(\alpha'-\alpha,\alpha)$ or $(\alpha,\alpha'-\alpha)$ is special:

$$(\alpha'-\alpha)+\alpha=\alpha'\prec\alpha'+\beta'$$

Hence, $N_{\alpha',\beta'}$ can be expressed in terms of $N_{\xi,\eta}$'s for extra special pairs (ξ,η). So relations 12.2(i)-(iv) expresses all $N_{\alpha',\beta'}$'s in terms of $N_{\xi,\eta}$'s for extra special pairs (ξ,η).

Theorem 12.3. *There is a unique simple Lie algebra, up to isomorphism, with a given indecomposable Cartan matrix.*

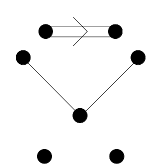
$$L=H\oplus\left(\oplus_{\alpha\in\Phi}L_\alpha\right)$$
$$\dim(L)=l+|\Phi|$$

Thus, the simple Lie algebras and their dimensions are given by

$$\dim(A_l)=l(l+2)\quad(l\geq 1)\qquad \dim(B_l)=l(2l+1)\quad(l\geq 2)\qquad \dim(C_l)=l(2l+1)\quad(l\geq 3)\qquad \dim(D_l)=l(2l-1)\quad(l\geq 4)$$
$$\dim(E_6)=78\qquad \dim(E_7)=133\qquad \dim(E_8)=248\qquad \dim(F_4)=52\qquad \dim(G_2)=14$$

Note. The following are isomorphic:

$$B_2\cong C_2$$
$$A_3\cong D_3$$
$$D_2\cong A_1\oplus A_1$$



Proof. *Uniqueness.* Let L,L' be simple Lie algebras with indecomposable Cartan matrix $A=(a_{ij})$. L has a Cartan decomposition $L=H\oplus\left(\oplus_{\alpha\in\Phi}L_\alpha\right)$. If $\Pi=\{\alpha_1,\ldots,\alpha_l\}$ then H

has basis $\{h_{\alpha_1},\dots,h_{\alpha_l}\}$; L has basis $\{h_{\alpha_1},\dots,h_{\alpha_l}\}\cup\{e_\alpha \mid \alpha\in\Phi\}$. Multiplication of basis elements:

$$[h_{\alpha_i}h_{\alpha_j}]=0$$
$$[e_\alpha h_{\alpha_i}]=\langle h_\alpha,h_{\alpha_i}\rangle e_\alpha$$
$$[e_{-\alpha}e_\alpha]=h_\alpha$$
$$[e_\alpha e_\beta]=\begin{cases}N_{\alpha,\beta}e_{\alpha+\beta} & \alpha+\beta\in\Phi\\ 0 & 0\neq\alpha+\beta\notin\Phi\end{cases}$$

All scalar products $\langle h_\alpha,h_\beta\rangle$ are determined by A. Also, all of the h_α (as linear combinations of the h_{α_i}) are determined by A.

$$s_{\alpha_i}(h_{\alpha_j})=h_{\alpha_j}-a_{ij}h_{\alpha_i}$$

So the s_{α_i} are determined by A. The Weyl group W is generated by $s_{\alpha_1},\dots,s_{\alpha_l}$. So W is determined by A. $h_\alpha=w(h_{\alpha_i})$ for some i and some $w\in W$. So the h_α are determined by A.

$$s_\alpha(h_\beta)=h_\beta-2\frac{\langle h_\alpha,h_\beta\rangle}{\langle h_\alpha,h_\alpha\rangle}h_\alpha$$

So $2\langle h_\alpha,h_\beta\rangle/\langle h_\alpha,h_\alpha\rangle$ is determined by A. But

$$\frac{1}{\langle h_\alpha,h_\alpha\rangle}=\sum_{\beta\in\Phi}\left(\frac{\langle h_\alpha,h_\beta\rangle}{\langle h_\alpha,h_\alpha\rangle}\right)^2$$

by 8.13. So $\langle h_\alpha,h_\alpha\rangle$ is determined by A. So $\langle h_\alpha,h_\beta\rangle$ is determined by A.

Suppose a basis $\{h'_{\alpha_1},\dots,h'_{\alpha_l}\}\cup\{e'_\alpha \mid \alpha\in\Phi\}$ of L' is given. We describe how to choose a basis of L. The h_{α_i} are uniquely determined. Choose $e_\alpha\neq0$ in L_α for each $\alpha\in\Pi$. For each $\alpha\in\Phi^+\setminus\Pi$ there is a unique extra special pair (β,γ) such that $\alpha=\beta+\gamma$, $\beta,\gamma\prec\alpha$.

Assume by induction that e_β,e_γ are already chosen. Choose e_α by $e_\alpha=N_{\beta,\gamma}[e_\beta e_\gamma]$ where $N_{\beta,\gamma}=N'_{\beta,\gamma}$, the structure constant for L'. Having thus chosen e_α for $\alpha\in\Phi^+$, we choose $e_{-\alpha}$ by $[e_{-\alpha}e_\alpha]=h_\alpha$.

The $N_{\alpha,\beta}$ for arbitrary α,β are determined by the $N_{\xi,\eta}$, where (ξ,η) is extra special, by 12.2. Since $N_{\xi,\eta} = N'_{\xi,\eta}$ for all extra special (ξ,η) it follows that $N_{\alpha,\beta} = N'_{\alpha,\beta}$ for all $\alpha,\beta \in \Phi$ with $\alpha + \beta \in \Phi$.

This shows that L and L' are isomorphic.

Existence. (Sketch proof.) Begin with Cartan matrix $A = (a_{ij})$. Let H be an l-dimensional vector space over $\mathbb{C}$ with basis $h_{\alpha_1},\ldots,h_{\alpha_l}$. We define $s_{\alpha_i} : H \to H$ by $s_{\alpha_i}(h_{\alpha_j}) = h_{\alpha_j} - a_{ij}h_{\alpha_i}$, a self-inverse map. Let W be the group of all non-singular linear maps $H \to H$ generated by $s_{\alpha_1},\ldots,s_{\alpha_l}$. W is finite. Correspondingly,

$$\left\{ h_\alpha = w\left(h_{\alpha_i}\right) \mid w \in W, 1 \le i \le l \right\}$$

is also finite. (The h_α were determined in Chapter 11.) We now define a bilinear map

$$\begin{gathered} H \times H \to \mathbb{C} \\ (x,y) \mapsto \langle x,y \rangle \end{gathered}$$

This form is uniquely determined by A. Define $\alpha \in H^*$ by $\alpha(x) = \langle h_\alpha, x \rangle$; let Φ be the set of all such α.

Let L be a vector space over $\mathbb{C}$ with $\dim(L) = \dim(H) + |\Phi|$ with basis

$$\left\{ h_{\alpha_1},\ldots,h_{\alpha_l} \right\} \cup \left\{ e_\alpha \mid \alpha \in \Phi \right\}$$

Define a bilinear map

$$\begin{gathered} L \times L \to L \\ (x,y) \mapsto [xy] \end{gathered}$$

We define $[\ ]$ on the basis elements by

$$\begin{gathered} [h_{\alpha_i} h_{\alpha_j}] = 0 \\ [e_\alpha h_{\alpha_i}] = -[h_{\alpha_i} e_\alpha] = \left\langle h_\alpha, h_{\alpha_i} \right\rangle e_\alpha \\ [e_{-\alpha} e_\alpha] = h_\alpha \\ [e_\alpha e_\beta] = \begin{cases} N_{\alpha,\beta} e_{\alpha+\beta} & \alpha + \beta \in \Phi \\ 0 & 0 \ne \alpha + \beta \notin \Phi \end{cases} \end{gathered}$$

The $N_{\alpha,\beta}$ can be chosen arbitrarily if (α,β) is extra special, e.g. $N_{\alpha,\beta}=1$. $N_{\alpha,\beta}$ is determined for all other pairs by 12.2. So multiplication of basis elements is determined by A. We make various checks:

Check $[xx]=0$ for all $x\in L$. (Easy.) Check $[[xy]z]+[[yz]x]+[[zx]y]+0$. (Most are easy, but $x=e_\alpha$, $y=e_\beta$, $z=e_\gamma$ is difficult.) Then L is a Lie algebra, $L=H\oplus\left(\oplus_{\alpha\in\Phi}L_\alpha\right)$, $L_\alpha=\mathbb{C}e_\alpha$. Check that H is a Cartan subalgebra of L. (Difficult.) Then $L=H\oplus\left(\oplus_{\alpha\in\Phi}L_\alpha\right)$ is a Cartan decomposition of L with respect to H. (Easy.) Then Φ is the set of roots of L with respect to H. $\Pi=\{\alpha_1,\dots,\alpha_l\}$ is a fundamental system of roots inside Φ. We have

$$2\frac{\left\langle h_{\alpha_i},h_{\alpha_j}\right\rangle}{\left\langle h_{\alpha_i},h_{\alpha_i}\right\rangle}=a_{ij}$$

so A is the Cartan matrix. Finally, the argument of 12.1(a) proves that L is simple.

■

Review.

$$L\text{ semisimple}\xrightarrow[\substack{\text{choose}\\ \text{Cartan}\\ \text{subalgebra}\\ H}]{}\Phi\xrightarrow[\substack{\text{choose}\\ \text{fundamental}\\ \text{roots}\\ \Pi}]{}A=(a_{ij})$$

If we choose a different Cartan subalgebra and a different fundamental system do we get a different A?

Theorem 12.4. *(i) Let L be a Lie algebra and H_1,H_2 Cartan subalgebras. Then there exists an automorphism $\theta:L\to L$ such that $\theta(H_1)=H_2$.*
(ii) A subalgebra H of L is a Cartan subalgebra if and only if H is nilpotent and $H=\mathcal{N}(H)$.

Theorem 12.5. *Let Φ be the root system of a semisimple Lie algebra and let Π_1,Π_2 be two fundamental systems in Φ. Then there is a $w\in W$ such that $w(\Pi_1)=\Pi_2$.*

12.4 and 12.5 imply that the Cartan matrix is uniquely determined by L. So the simple Lie algebras on our list are pairwise non-isomorphic.

We have four infinite families of simple Lie algebras and five exceptional ones:

Classical				Exceptional				
A_l	B_l	C_l	D_l	E_6	E_7	E_8	F_4	G_2
$l(l+2)$	$l(2l+1)$	$l(2l+1)$	$l(2l-1)$	78	133	248	52	14
$l \geq 1$	$l \geq 2$	$l \geq 3$	$l \geq 4$					

Type A_l. We can write $\dim(A_l) = l(l+2) = (l+1)^2 - 1$. The set $\mathfrak{sl}_{l+1}(\mathbb{C})$ of all $(l+1)\times(l+1)$ matrices of trace zero forms a Lie algebra of type A_l. The diagonal subalgebra is a Cartan subalgebra.

Type B_l. The set $\mathfrak{so}_{2l+1}(\mathbb{C})$ of all $(2l+1)\times(2l+1)$ matrices X satisfying

$$X^{\mathrm{T}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{pmatrix} = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{pmatrix} X$$

forms a simple Lie algebra of type B_l. The diagonal subalgebra is a Cartan subalgebra. $\mathfrak{so}_{2l+1}(\mathbb{C})$ is isomorphic to the Lie algebra of all $(2l+1)\times(2l+1)$ skew-symmetric matrices. Elements of $\mathfrak{so}_{2l+1}(\mathbb{C})$ have the block form

$$X = \begin{pmatrix} 0 & X_{01} & X_{02} \\ -X_{02}^{\mathrm{T}} & X_{11} & X_{12} \\ -X_{01}^{\mathrm{T}} & X_{22} & -X_{11}^{\mathrm{T}} \end{pmatrix}$$

where X_{11} is an arbitrary $l \times l$ matrix, X_{12} and X_{21} are $l \times l$ symmetric matrices and X_{01} and X_{02} are arbitrary $1 \times l$ matrices (row vectors).

Type C_l. The set $\mathfrak{sp}_{2l}(\mathbb{C})$ of all $2l \times 2l$ matrices X satisfying

$$X^{\mathrm{T}}\begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix} = -\begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix} X$$

forms a simple Lie algebra of type C_l. The diagonal subalgebra is a Cartan subalgebra. Elements of $\mathfrak{sp}_{2l}(\mathbb{C})$ have the block form

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & -X_{11}^{\mathrm{T}} \end{pmatrix}$$

where X_{11} is an arbitrary $l \times l$ matrix and X_{12} and X_{21} are $l \times l$ symmetric matrices.

Type D_l. The set $\mathfrak{so}_{2l}(\mathbb{C})$ of all $2l \times 2l$ matrices X such that

$$X^{\mathrm{T}}\begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix} = -\begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix} X$$

forms a simple Lie algebra of type D_l. The diagonal subalgebra is a Cartan subalgebra. $\mathfrak{so}_{2l}(\mathbb{C})$ is isomorphic to the Lie algebra of all $2l \times 2l$ skew-symmetric matrices. Elements of $\mathfrak{so}_{2l}(\mathbb{C})$ have the block form

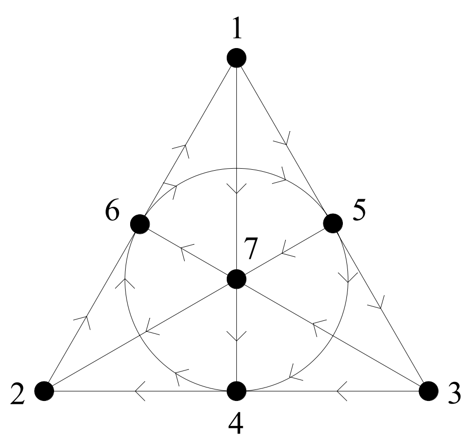
$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & -X_{11}^{\mathrm{T}} \end{pmatrix}$$

where X_{11} is an arbitrary $l \times l$ matrix and X_{12} and X_{21} are $l \times l$ skew-symmetric matrices.

$\mathfrak{sl}_m(\mathbb{C})$ is the Lie algebra of $\mathrm{SL}_m(\mathbb{C}) = \{X \in \mathrm{GL}_m(\mathbb{C}) \mid \det(X) = 1\}$; $\mathfrak{so}_m(\mathbb{C})$ is the Lie algebra of $\mathrm{SO}_m(\mathbb{C}) = \{X \in \mathrm{GL}_m(\mathbb{C}) \mid X^{\mathrm{T}}X = I_m \text{ and } \det(X) = 1\}$.

Type G_2. $\dim(G_2) = 14$. Consider the algebra of *octonians* (a.k.a. *Cayley numbers*), $\mathcal{O}$. $\dim(\mathcal{O}) = 8$. $\mathcal{O}$ has basis $1, e_1, e_2, \ldots, e_7$:

1 is the multiplicative identity;
$e_i^2 = -1$ for $1 \le i \le 7$;
$e_i e_j = \pm e_k$ for $1 \le i \ne j \le 7$.



The projective plane over the 2-field.

$e_i e_j = e_k$ if $i \to j$; $e_i e_j = -e_k$ if $i \leftarrow j$.

$\mathcal{O}$ is a non-associative algebra. The set of all derivations of $\mathcal{O}$, i.e. linear maps $D:\mathcal{O}\to\mathcal{O}$ such that $D(xy)=D(x)y+xD(y)$, forms a Lie algebra of type G_2.

Type F_4. Define the *octonian conjugate*:

$$x = a_0 1 + \sum_{i=1}^{7} a_i e_i$$
$$\bar{x} = a_0 1 - \sum_{i=1}^{7} a_i e_i$$
$$x = \bar{x} \Leftrightarrow x = a_0 1$$

A matrix M over $\mathcal{O}$ is called *Hermitian* if $M^{\mathrm{T}} = \overline{M}$. Let $\mathcal{J}$ be the $\mathbb{C}$-vector space of all 3×3 Hermitian matrices over $\mathcal{O}$. Such matrices have the form

$$M = \begin{pmatrix} a1 & x & y \\ \bar{x} & b1 & z \\ \bar{y} & \bar{z} & c1 \end{pmatrix}$$

where $a,b,c\in\mathbb{C}$ and $x,y,z\in\mathcal{O}$. $\dim(\mathcal{J})=27$. We define multiplication on $\mathcal{J}$ by

$$M_1 \times M_2 = \tfrac{1}{2}(M_1M_2 + M_2M_1)$$
$$M_1 \times M_2 \in \mathcal{J} \text{ for } M_1, M_2 \in \mathcal{J}$$

$\mathcal{J}$ is a commutative non-associative algebra; it is an example of a *Jordan algebra*, the axioms for which are that

$$X \times Y = Y \times X$$
$$(X^2 \times Y)\times X = X^2 \times (Y \times X)$$

The derivations of $\mathcal{J}$ form a simple Lie algebra of type F_4.

E_6, E_7 and E_8 can all be described in terms of $\mathcal{O}$ and $\mathcal{J}$.

There is an alternative approach to the existence theorem, which proceeds (in outline) as follows:

Let L be a simple Lie algebra with Cartan matrix $A=(a_{ij})$, $L = H \oplus (\oplus_{\alpha\in\Phi} L_\alpha)$. H has basis $h_{\alpha_1},\ldots,h_{\alpha_l}$. Let

$$h_i = \frac{2h_{\alpha_i}}{\langle h_{\alpha_i}, h_{\alpha_i}\rangle}.$$

$h_1,\dots,h_l$ also form a basis of H. Choose $0 \neq e_i \in L_{\alpha_i}$ and $0 \neq f_i \in L_{-\alpha_i}$.

$$\begin{aligned}[e_j h_i] &= \alpha_j(h_i)e_j \\ &= \langle h_{\alpha_j}, h_i \rangle e_j \\ &= \frac{2\langle h_{\alpha_i}, h_{\alpha_j}\rangle}{\langle h_{\alpha_i}, h_{\alpha_i}\rangle} \\ &= a_{ij}e_j \\ [f_j h_i] &= -a_{ij} f_j\end{aligned}$$

Choose f_i with $[f_i e_i] = h_i$; $e_1,\dots,e_l$ generate $\oplus_{\alpha\in\Phi^+} L_\alpha$; $f_1,\dots,f_l$ generate $\oplus_{\alpha\in\Phi^-} L_\alpha$; $h_1,\dots,h_l$ generate H. So $G = \{e_i, f_i, h_i \mid 1 \le i \le l\}$ generates L. We have relations $\mathcal{R}$:

$$\begin{gathered}[h_i h_j] = 0 \\ [e_j h_i] = a_{ij} e_j \\ [f_j h_i] = -a_{ij} f_j \\ [f_i e_i] = h_i \\ [f_j e_i] = 0 \text{ if } i \neq j \\ [e_i \dots e_i [e_i e_j]] = 0 \text{ if } i \neq j \ (1 - a_{ij} \ \ e_i\text{'s}) \\ [f_i \dots f_i [f_i f_j]] = 0 \text{ if } i \neq j \ (1 - a_{ij} \ \ f_i\text{'s})\end{gathered}$$

(The requirements for $1 - a_{ij}$ e_i's and f_i's arise from consideration of the α_i-chain of roots through α_j.) The Lie algebra generated by G with relations $\mathcal{R}$ is a finite-dimensional Lie algebra with Cartan matrix A.

L is constructed as follows: let R be the polynomial ring $\mathbb{C}\langle e_1,\dots,e_l, f_1,\dots,f_l, h_1,\dots,h_l\rangle$ with non-commutative variables. $[R]$ is the Lie algebra obtained from R. Let M be the subalgebra generated by $e_1,\dots,e_l, f_1,\dots,f_l, h_1,\dots,h_l$. Let I be the ideal of M generated by

$$[h_i h_j],\ [e_j h_i] - a_{ij}e_j,\ [f_j h_i] + a_{ij} f_j,\ [f_j e_i] - \delta_{ij} h_i,\ [e_i \dots e_i [e_i e_j]],\ [f_i \dots f_i [f_i f_j]].$$

Then $L = M/I$. We can show that L is finite-dimensional and has Cartan matrix A.