UNIVERSITY OF WARWICK

MA453 LIE ALGEBRAS

Autumn Term 2002

Lectured by Professor Roger Carter

Typed by Tim Sullivan

INDEX

1. DEFINITIONS AND BASIC PRINCIPLES

An *algebra* over $\mathbb{C}$ is a vector space A over $\mathbb{C}$ with a bilinear multiplication $A \times A \to A$, $(x, y) \mapsto xy$. Bilinearity means that for all $x, y \in A$ and $\lambda \in \mathbb{C}$

$$(x_1 + x_2)y = x_1 y + x_2 y$$
$$x(y_1 + y_2) = xy_1 + xy_2$$
$$(\lambda x)y = x(\lambda y) = \lambda(xy)$$

All the algebras in this course will be over $\mathbb{C}$. The main advantage of $\mathbb{C}$ is that it is algebraically closed.

An *associative algebra* is an algebra A such that for all $x, y, z \in A$, $x(yz) = (xy)z$.

A *Lie algebra*[†] is an algebra L with multiplication $L \times L \to L$, $(x, y) \mapsto [xy]$ such that

$$[xx] = 0 \text{ for all } x \in L,$$
$$[[xy]z] + [[yz]x] + [[zx]y] = 0 \text{ for all } x, y, z \in L \text{ – the \textit{Jacobi identity}.}$$

Unless otherwise specified, L shall be an arbitrary Lie algebra. Where dictated by requirements of clarity, we shall write $[x, y]$ for $[xy]$.

Lemma 1.1. *For all* $x, y \in L$, $[xy] = -[yx]$.

Proof.

$$0 = [x + y, x + y] = [xx] + [xy] + [yx] + [yy] = [xy] + [yx]$$

■

We say that Lie multiplication is *anticommutative*.

Lemma 1.2. *Suppose* A *is an associative algebra. Then* A *can be made into a Lie algebra by defining* $[xy] = xy - yx$.

Proof. $[xx] = 0$ is clear.

$$[[xy]z] = [xy - yx, z] = xyz - yxz - zxy + zyx$$
$$[[yz]x] = yzx - zyx - xyz + xzy$$
$$[[zx]y] = zxy - xzy - yxz + yxz$$

[†] After Marius Sophus Lie (1842-99).

All terms cancel, so we have the Jacobi identity.

■

The Lie algebra obtained in this way is denoted $[A]$.

We can also multiply subspaces in the following way: let L be a Lie algebra and H, K subspaces of L. We define $[HK]$ to be the smallest subspace containing all the Lie products $[hk]$ for $h \in H, k \in K$. So

$$[HK] = \{[h_1k_1] + \ldots + [h_rk_r] \mid h_i \in H, k_i \in K\}.$$

Lemma 1.3. *If H, K are subspaces of L then $[HK] = [KH]$.*

Proof.

$$[HK] \ni [h_1k_1] + \ldots + [h_rk_r] = -[k_1h_1] - \ldots - [k_rh_r] \in [KH]$$

■

Multiplication of subspaces is commutative.

$H \subseteq L$ is called a *subalgebra* of L if H is a subspace of L and $[HH] \subseteq H$. That is, a subalgebra of L is a subset of L that is itself a Lie algebra under the same operations as L.

A subset $I \subseteq L$ is called an *ideal* of L if I is a subspace of L and $[IL] \subseteq I$. We will write $I \lhd L$.

Note. Since $[IL] = [LI]$, $[IL] \subseteq I \Leftrightarrow [LI] \subseteq I$.

Every ideal of L is also a subalgebra of L, but the converse is not true.

Example. Consider $M_2 = \{2 \times 2 \text{ matrices over } \mathbb{C}\}$. $[M_2]$ is a Lie algebra. The subset T of elements of $[M_2]$ of trace zero form an ideal of $[M_2]$. The subset U of elements of $[M_2]$ with upper-right element zero form a subalgebra of $[M_2]$, but not an ideal.

Proposition 1.4. *(i) If H, K are subalgebras then $H \cap K$ is a subalgebra.*
(ii) If $H, K \lhd L$ then $H \cap K \lhd L$.
(iii) If $H \lhd L$ and K is a subalgebra then $H + K$ is a subalgebra.
(iv) If $H, K \lhd L$ then $H + K \lhd L$.

Proof. (i) $H \cap K$ is certainly a subspace.

$$[H \cap K, H \cap K] \subseteq [HH] \subseteq H$$
$$[H \cap K, H \cap K] \subseteq [KK] \subseteq K$$

(ii)

$$[H \cap K, L] \subseteq [HL] \subseteq H$$
$$[H \cap K, L] \subseteq [KL] \subseteq K$$

(iii)

$$[H + K, H + K] \subseteq [HH] + [HK] + [KH] + [KK] \subseteq H + H + H + K \subseteq H + K$$

(iv)

$$[H + K, L] \subseteq [HL] + [KL] \subseteq H + K$$

■

Note. The sum of two subalgebras need not be a subalgebra.

We can form *factor algebras*: let $I \triangleleft L$. In particular, I is an additive subgroup so we can form the factor group L/I; the elements of L/I are the cosets $I + x$ for $x \in L$.

$$(I + x) + (I + y) = I + (x + y)$$
$$\lambda(I + x) = I + \lambda x$$

We define $[I + x, I + y] = I + [xy]$. We do need to check that this is well-defined, i.e. that if $I + x = I + x'$ and $I + y = I + y'$ then $I + [xy] = I + [x'y']$. We can find $i_1, i_2 \in I$ such that $x' = i_1 + x$ and $y' = i_2 + y$. So

$$\begin{aligned}[x'y'] &= [i_1 + x, i_2 + y] \\ &= [i_1 i_2] + [i_1 y] + [x i_2] + [xy] \\ &\in I + [xy]\end{aligned}$$

So the coset containing $[xy]$ is the same as that containing $[x'y']$.

It is easy to verify that L/I is a Lie algebra.

A *homomorphism* of Lie algebras is a linear map $\theta : L_1 \to L_2$ such that for all $x, y \in L_1$, $\theta([xy]) = [\theta(x), \theta(y)]$. If θ is bijective it is called an *isomorphism* and we write $L_1 \cong L_2$.

Proposition 1.5. *Let $\theta : L_1 \to L_2$ be a homomorphism with kernel K. Then $K \triangleleft L_1$, $\mathrm{im}(\theta)$ is a subalgebra of L_2 and $L_1/K \cong \mathrm{im}(\theta)$.*

Proof. Let $x, y \in L_1$. $[\theta(x), \theta(y)] = \theta[xy] \in \theta(L_1)$, so $\mathrm{im}(\theta)$ is a subalgebra of L_2.

Now let $x \in K$ and $y \in L_1$. Then

$$\theta[xy] = [\theta(x), \theta(y)] = [0, \theta(y)] = 0$$

so $y \in K$. Hence, $K \lhd L_1$.

Now let $x, y \in L_1$.

$$\theta(x) = \theta(y) \Leftrightarrow \theta(x - y) = 0 \Leftrightarrow x - y \in K \Leftrightarrow K + x = K + y$$

So $\theta(x) \mapsto K + x$ is a bijection between $\mathrm{im}(\theta)$ and L_1/K. We now check that this bijection is an isomorphism of Lie algebras: let $x, y, z \in L_1$.

$$\begin{aligned}[\theta(x), \theta(y)] = \theta(z) &\Leftrightarrow \theta[xy] = \theta(z) \\ &\Leftrightarrow K + [xy] = K + z \\ &\Leftrightarrow [K + x, K + y] = K + z\end{aligned}$$

So $\mathrm{im}(\theta) \cong L_1/K$.

■

Proposition 1.6. *Let $I \lhd L$ and H a subalgebra of L. Then $I + H$ and $I \cap H$ are subalgebras (by 1.4) and*
(i) $I \lhd I + H$,
(ii) $I \cap H \lhd H$,
(iii) $(I + H)/I \cong H/(I \cap H)$.

Proof. (i) $[I, I + H] \subseteq [I, L] \subseteq I$.

(ii)

$$[I \cap H, H] \subseteq [IH] \subseteq I$$
$$[I \cap H, H] \subseteq [HH] \subseteq H$$

(iii) We can form $(I + H)/I$ and $H/(I \cap H)$. Elements of $(I + H)/I$ have the form $I + i + h = I + h$ for $h \in H$. Define a map $\theta : H \to (I + H)/I$ by $\theta(h) = I + h$. This map is a homomorphism since $[I + h, I + h'] = I + [hh']$. It is surjective: $\mathrm{im}(\theta) = (I + H)/I$. Consider $\ker(\theta)$: $h \in \ker(\theta) \Leftrightarrow I + h = I \Leftrightarrow h \in I$. So $\ker(\theta) = H \cap I$. By 1.5 $H/\ker(\theta) \cong \mathrm{im}(\theta)$, so $(I + H)/I \cong H/(I \cap H)$.

■

Note. In this course we shall consider only finite-dimensional Lie algebras over $\mathbb{C}$. In this case,

$$\dim(L/I) = \dim(L) - \dim(I)$$

To prove this, select a basis $e_1, \ldots, e_r$ of I and extend to a basis $e_1, \ldots, e_n$ of L. Each element of L has the form $\lambda_1 e_1 + \ldots + \lambda_n e_n$; each element of L/I has the form $I + \lambda_{r+1} e_{r+1} + \ldots + \lambda_n e_n = (I + \lambda_{r+1} e_{r+1}) + \ldots + (I + \lambda_n e_n)$. $I + e_{r+1}, \ldots, I + e_n$ form a basis for L/I. So $\dim(L/I) = n - r = \dim(L) - \dim(I)$.

Examples. If $\dim(L) = 1$, L has basis x. $[xx] = 0$, so $[LL] = 0$.

If $\dim(L) = 2$ let x, y be a basis for L. $[xx] = [yy] = 0$, but $[xy] = -[yx] = ?$ Possibly $[LL] = 0$. If $[LL] \neq 0$ then $\dim([LL]) = 1$. Let x' be a basis for $[LL]$ and x', y' a basis for L itself. Then we have $[x'y'] = \lambda x'$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. Re-choose $y'' = \lambda^{-1} y'$: we then have that $[xy] = x$. Hence, we have two Lie algebras of dimension 2.

2. REPRESENTATIONS AND MODULES OF LIE ALGEBRAS

Recall that $M_n = \{n \times n \text{ matrices over } \mathbb{C}\}$ and that $[M_n]$ is the Lie algebra of such matrices with $[AB] = AB - BA$.

A *representation* of a Lie algebra L is a homomorphism $\rho : L \to [M_n]$ for some $n \in \mathbb{N}$. I.e.,

$$\rho[xy] = [\rho(x), \rho(y)] = \rho(x)\rho(y) - \rho(y)\rho(x).$$

If ρ is a representation of L then so is ρ' given by $\rho'(x) = T^{-1}\rho(x)T$ where T is a non-singular $n \times n$ matrix independent of x. We say that two representations are *equivalent* if there is a non-singular T such that $\rho'(x) = T^{-1}\rho(x)T$ holds for all $x \in L$.

An L-*module* is a vector space V over $\mathbb{C}$ with a map $V \times L \to V$ such that
(i) $(v, x) \mapsto vx$ is linear in both v and x;
(ii) $v[xy] = (vx)y - (vy)x$ for all $v \in V$ and $x, y \in L$.

We shall only deal with finite-dimensional L-modules in this course.

A *submodule* W of V is a subspace of V such that $wx \in W$ for all $w \in W, x \in L$ i.e. a subspace closed under the right action of the element of L.

Proposition 2.1. *Let V be an L-module with basis $e_1, \ldots, e_n$. Let $x \in L$ and let $e_i x = \sum_{i=1}^{n} \rho_{ij}(x) e_j$. Let $\rho(x)$ be the matrix with ij th entry $\rho_{ij}(x)$. Then $x \mapsto \rho(x)$ is a representation of L and a different choice of basis gives an equivalent representation.*

Proof. The linear transformation $v \mapsto vx$ has matrix $\rho(x)$; $v \mapsto (vx)y$ has matrix $\rho(x)\rho(y)$; $v \mapsto (vy)x$ has matrix $\rho(y)\rho(x)$; $v \mapsto (vx)y - (vy)x$ has matrix $\rho(x)\rho(y) - \rho(y)\rho(x)$. That is, the linear transformation $v \mapsto v[xy]$ has matrices $\rho[xy]$ and $\rho(x)\rho(y) - \rho(y)\rho(x)$. So $\rho[xy] = \rho(x)\rho(y) - \rho(y)\rho(x)$, so ρ is a representation of L.

Now take a new basis $f_1, \ldots, f_n$ of V. The linear transformation $v \mapsto vx$ is represented by a matrix $T^{-1}\rho(x)T$, where $e_i = \sum_{i=1}^{n} T_{ij} f_j$. So we get a representation $x \mapsto T^{-1}\rho(x)T$ that is equivalent to ρ.

■

An L-module is called *irreducible* if it has no submodules except itself and 0; otherwise it is said to be *reducible*.

An L-module V is called *decomposable* if there are submodules $V_1, V_2 \neq 0$ of V such that $V = V_1 \oplus V_2$; otherwise it is said to be *indecomposable*.

Proposition 2.2. *L is itself an L-module under the map $L \times L \to L : (x, y) \mapsto [xy]$.*

This is the *adjoint* L-module; we define $\operatorname{ad} y : L \to L$ by $(\operatorname{ad} y)x = [xy]$.

Proof. It is sufficient to show that for all $x, y, z \in L$, $[z[xy]] = [[zx]y] - [[zy]x]$. This follows immediately from the Jacobi identity and the anticommutativity of Lie multiplication.

■

A *derivation* of a Lie algebra L is a linear map $D : L \to L$ such that for all $x, y \in L$,

$$D[xy] = [Dx, y] + [x, Dy].$$

Proposition 2.3. *Let $x \in L$. Then $\operatorname{ad} x$ is a derivation of L.*

Proof. Linearity is clear. We need to check that $(\operatorname{ad} x)[yz] = [(\operatorname{ad} x)y, z] + [y, (\operatorname{ad} x)z]$. This is true if and only if the Jacobi identity is true.

■

Let V be an L-module, W a subspace of V and H a subspace of L. We define WH to be the subspace spanned by wh for all $w \in W, h \in H$.

If W is a submodule of V, V/W is itself an L-module under the action $(W + v)x = W + vx$ for $v \in V, x \in L$. This action is well-defined because

$$(W + v)x = (W + v')x \Rightarrow v - v' \in W \Rightarrow (v - v')x \in W \Rightarrow W + vx = W + v'x.$$

3. ABELIAN, NILPOTENT AND SOLUBLE LIE ALGEBRAS

A Lie algebra L is *abelian* if $[LL]=0$, i.e. $[xy]\equiv 0$.

Define $L^1=L$ and inductively define $L^{n+1}=[L^nL]$.

Proposition 3.1. *For each $n\in\mathbb{N}$, $L^n\lhd L$.*

Proof. It is sufficient to show that if $H,K\lhd L$ then $[HK]\lhd L$. Let $x\in H, y\in K, z\in L$. Is $[[xy]z]\in[HK]$?

$$[[xy]z]=-[[yz]x]-[[zx]y]$$

But $y,[yz]\in K$ and $x,[zx]\in H$ so $[[HK]L]\subseteq[HK]$.

Clearly $L^1=L\lhd L$. If we assume inductively that $L^n\lhd L$ then the above workings show that $L^{n+1}=[L^nL]\lhd L$, and the result follows.

■

Proposition 3.2. $L=L^1\supseteq L^2\supseteq L^3\supseteq\ldots$

Proof. For each n, $L^{n+1}=[L^nL]\subseteq L^n$ since $L^n\lhd L$.

■

L is *nilpotent* if there is an $n\in\mathbb{N}$ such that $L^n=0$.

Clearly every abelian Lie algebra is nilpotent as $L^2=0$.

Example. Let L be the Lie algebra of upper-triangular $n\times n$ matrices with zeroes on the principal diagonal; $\dim(L)=\frac{1}{2}n(n-1)$; L is a subalgebra of $[M_n]$. Define subspaces H_i by requiring that elements of H_i have zeroes on and below the $(i-1)$th diagonal above the principal diagonal.

Lie multiplication shows that $[H_i,L]\subseteq H_{i-1}$. We show that $L^i\subseteq H_i$ by induction on i. If $i=1$ then $L=L^1=H_1$. Assume $L^i\subseteq H_i$ for $i=r$. Then

$$\begin{aligned}L^{r+1}&=[L^rL]\\&\subseteq[H^rL]\\&\subseteq H_{r+1}\end{aligned}$$

In particular, $L^n\subseteq H_n=0$, so L is nilpotent.

Proposition 3.3. *For all* $m,n \geq 1$, $\left[L^m L^n\right] \subseteq L^{m+n}$.

Proof. Use induction on n. If $n=1$ then $\left[L^m L^1\right] = \left[L^m L\right] = L^{m+1}$. Assume for $n=r$ and consider $n=r+1$:

$$\begin{aligned}\left[L^m L^{r+1}\right] &= \left[L^m \left[L^r L\right]\right] \\ &= \left[\left[L^r L\right] L^m\right] \\ &\subseteq \left[\left[L L^m\right] L^r\right] + \left[\left[L^m L^r\right] L\right] \text{by the Jacobi identity}\end{aligned}$$

So

$$\begin{aligned}\left[L^m L^{r+1}\right] &\subseteq \left[L^{m+1} L^r\right] + \left[\left[L^m L^r\right] L\right] \\ &\subseteq L^{m+1+r} + \left[L^{m+r} L\right] \\ &\subseteq L^{m+r+1}\end{aligned}$$

■

We now inductively define another sequence of subspaces of L:

$$\begin{aligned}L^{(0)} &= L \\ L^{(i+1)} &= \left[L^{(i)} L^{(i)}\right]\end{aligned}$$

The $L^{(i)}$ are all ideals of L, so $L^{(i+1)} \subseteq L^{(i)}$, so $L = L^{(0)} \supseteq L^{(1)} \supseteq L^{(2)} \supseteq \dots$.

We say that L is *soluble* if there is an $n \in \mathbb{N}$ such that $L^{(n)} = 0$.

Proposition 3.4. *(i)* $L^{(n)} \subseteq L^{2^n}$.
(ii) Every nilpotent Lie algebra is soluble.

Proof. (i) Induction on n: if $n=0$ $L^{(0)} = L = L^1$. Assume for $n=r$:

$$\begin{aligned}L^{(r+1)} &= \left[L^{(r)} L^{(r)}\right] \\ &\subseteq \left[L^{2^r} L^{2^r}\right] \\ &\subseteq L^{2^{r+1}}\end{aligned}$$

(ii) Suppose L is nilpotent. Then there is an n such that $L^{2^n} = 0$. By (i) $L^{(n)} = 0$ also, so L is soluble.

■

Example. Let L be the set of all $n \times n$ matrices with zeroes below the principal diagonal. $[LL] \subseteq L$; L is a subalgebra of $[M_n]$. Define subspaces H_i by requiring that elements of H_i have zeroes on and below the i th diagonal above the principal diagonal.

We have that $[H_i H_i] \subseteq H_{i+1}$. We show that $L^{(i)} \subseteq H_i$ by induction on i. If $i = 0$ then $L^{(0)} = L = H_0$. Assume for $i = r$. Then $L^{(r+1)} = [L^{(r)} L^{(r)}] \subseteq [H_r H_r] \subseteq H_{r+1}$. In particular, $L^{(n)} \subseteq H_n$ so L is soluble.

However, L is not nilpotent. To see this, consider

$$A = \begin{pmatrix} 0 & \mu_1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & \mu_{n-1} \\ 0 & & & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$AB = \begin{pmatrix} 0 & \mu_1\lambda_2 & & 0 \\ & 0 & \ddots & \\ & & \ddots & \mu_{n-1}\lambda_n \\ 0 & & & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} 0 & \mu_1\lambda_1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & \mu_{n-1}\lambda_{n-1} \\ 0 & & & 0 \end{pmatrix}$$

$$[AB] = AB - BA = \begin{pmatrix} 0 & \mu_1(\lambda_2 - \lambda_1) & & 0 \\ & 0 & \ddots & \\ & & \ddots & \mu_{n-1}(\lambda_n - \lambda_{n-1}) \\ 0 & & & 0 \end{pmatrix}$$

By choosing the λ_i all unequal and the μ_j suitably we can get any desired matrix

$$[AB] = \begin{pmatrix} 0 & v_1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & v_{n-1} \\ 0 & & & 0 \end{pmatrix}$$

Let K be the subspace consisting of all matrices of this form.

$$K \subseteq [KL] \subseteq [[KL]L] \subseteq \ldots \subseteq [L \ldots L] = L^i$$

So $K \subseteq L^i$ for all i and $K \neq 0$, so $L^i \neq 0$ for all i. Hence, L is not nilpotent.

Proposition 3.5. *(i) Every subalgebra of a soluble Lie algebra is soluble.*
(ii) Every factor algebra of a soluble Lie algebra is soluble.
(iii) If $I \lhd L$ and $I, L/I$ are soluble then L is soluble.

Proof. (i) Let H be a subalgebra of L. L is soluble so $L^{(n)} = 0$ for some n. We show that $H^{(i)} \subseteq L^{(i)}$ for all i. This is true for $i = 0$. Assume it is true for $i = r$.

$$H^{(r+1)} = \left[H^{(r)}H^{(r)}\right] \subseteq \left[L^{(r)}L^{(r)}\right] = L^{(r+1)}$$

So $H^{(n)} = 0$ and H is soluble.

(ii) $I \lhd L$. We need to show that $(L/I)^{(i)} = \left(I + L^{(i)}\right)/I$. This is true for $i = 0$; assume it for $i = r$:

$$\begin{aligned}(L/I)^{(r+1)} &= [(L/I)^{(r)}, (L/I)^{(r)}] \\ &= [\tfrac{I+L^{(r)}}{I}, \tfrac{I+L^{(r)}}{I}] \\ &= \tfrac{\left[I+L^{(r)}, I+L^{(r)}\right]}{I} \\ &= \tfrac{I+\left[L^{(r)}L^{(r)}\right]}{I} \\ &= \tfrac{I+L^{(r+1)}}{I}\end{aligned}$$

If $L^{(n)} = 0$ then $(L/I)^{(n)} = I/I$, the zero subspace of L/I.

(iii) Suppose I and L/I are soluble. L/I soluble $\Leftrightarrow (L/I)^{(m)} = I/I$ for some m. $\frac{I+L^{(m)}}{I} = \frac{I}{I}$, so $L^{(m)} \subseteq I$. I is soluble so $I^{(n)} = 0$ for some n.

$$L^{(m+n)} = \left(L^{(m)}\right)^{(n)} \subseteq I^{(n)} = 0$$

So $L^{(m+n)} = 0$; L is soluble.

■

Proposition 3.6. *Let H, K be soluble ideals of L. Then $H + K$ is a soluble ideal of L.*

Proof. We know that $H + K \lhd L$. By 1.6, $\frac{H+K}{H} \cong \frac{K}{H \cap K}$. K is soluble, so $K/(H \cap K)$ is soluble. Hence, $(H + K)/H$ is soluble. By the previous proposition, since H is soluble, $H + K$ is soluble.

■

Corollary 3.7. Any Lie algebra L has a unique maximal soluble ideal.

Proof. Since $\dim(L) < \infty$ L certainly has a maximal soluble ideal. Let H, K be two maximal soluble ideals of L. Then $H + K$ is a soluble ideal. $H \subseteq H + K$ and H is maximal, so $H = H + K$. Similarly $K = H + K$, so $H = K$.

■

This maximal soluble ideal of L is called the *soluble radical* L, usually denoted R. If $R = 0$ we say that L is *semisimple.*

L/R is semisimple. For if R'/R is the soluble radical of L/R then since R'/R and R are both soluble, so is R'. Hence $R' \subseteq R$. So $R' = R$ and $R'/R = R/R$ is the zero subspace of L/R.

A Lie algebra L is called *simple* if it has no ideals other than 0 and L.[†]

If $\dim(L) = 1$ then L is certainly simple. There are other simple Lie algebras. If L is abelian and simple then $\dim(L) = 1$, since $[LL] = 0$. If L is soluble and simple and $\dim(L) = 1$ then $L \neq [LL]$, so $[LL] = 0$.

If $I \lhd L$ then the ideals of L/I have the form J/I for $J \lhd L$, $J \supseteq I$. So L/I is simple if and only if I is maximal.

A *composition series* of L is a sequence of subalgebras

$$L = K_0 \supset K_1 \supset \ldots \supset K_r = 0$$

where $K_{i+1} \lhd K_i$ is maximal. The factor algebras K_{i-1}/K_i are all simple Lie algebras and are known as the *composition factors* of L.

Proposition 3.8. *L is soluble if and only if all composition factors in a composition series of L are 1-dimensional.*

Proof. Let $L = K_0 \supset K_1 \supset \ldots \supset K_r = 0$ be a composition series. L is soluble, so K_i is soluble, so K_i/K_{i+1} is soluble. So $\dim(K_i/K_{i+1}) = 1$. Conversely, suppose that $\dim(K_i/K_{i+1}) = 1$. Then certainly K_i/K_{i+1} is soluble (even abelian). K_{r-1} is soluble and K_{r-2}/K_{r-1} is soluble, so K_{r-2} is soluble. K_{r-3}/K_{r-2} is soluble, so K_{r-3} is soluble. Eventually we see that $K_0 = L$ is soluble.

■

[†] The classification of the simple Lie algebras was completed in the 1890's by Élie Cartan and Wilhelm Killing, working independently.

4. REPRESENTATIONS OF NILPOTENT LIE ALGEBRAS

We shall first discuss representations of abelian Lie algebras.

Proposition 4.1. *Let L be abelian. Then every irreducible L-module has dimension 1. Every linear map $L \to \mathbb{C}$ is a 1-dimensional representation of L.*

Proof. A 1-dimensional representation of L is by definition a linear map $\lambda : L \to \mathbb{C}$ such that $\lambda([xy]) = \lambda(x)\lambda(y) - \lambda(y)\lambda(x)$. But the RHS of this equation is zero; since L is abelian the LHS is always zero, too. So every linear map $L \to \mathbb{C}$ is a representation of L.

Let V be an irreducible L-module and let $x \in L$; consider the linear map $V \to V$, $v \mapsto vx$. Let w be an eigenvector of this map; i.e. $w \neq 0$ and $wx = \lambda x$ for some $\lambda \in \mathbb{C}$, where λ is the eigenvalue. Let $W = \{v \in V \mid vx = \lambda v\}$, the eigenspace. W is a subspace of V. Since $w \neq 0$, $W \neq 0$. We shall show that W is a submodule of V.

Let $v \in W, y \in L$.

$$(vy)x = (vx)y + \underbrace{v[yx]}_{\substack{=0 \\ \because L \text{ abelian}}} = (vx)y = (\lambda v)y = \lambda(vy)$$

So $vy \in W$, which shows that W is a submodule of V. But V is irreducible so $V = W$. So $vx = \lambda v$ for all $v \in V$. Hence each $x \in L$ acts on V by scalar multiplication. So every subspace of V is a submodule. Hence $\dim(V) = 1$.

■

We now recall some linear algebra.

Let $A \in M_n$. Then the *characteristic polynomial* of A is $\chi(t) = \det(tI_n - A)$. For non-singular $T \in M_n$, A and $T^{-1}AT$ have the same characteristic polynomial:

$$\begin{aligned}\det(tI_n - T^{-1}AT) &= \det(T^{-1}(tI_n - A)T) \\ &= \det(T^{-1})\det(tI_n - A)\det(T) \\ &= \det(tI_n - A)\end{aligned}$$

If V is an n-dimensional vector space over $\mathbb{C}$ and $\theta : V \to V$ is a linear map we define the characteristic polynomial of θ to be the characteristic polynomial of any matrix representing θ.

$\chi(t) \in \mathbb{C}[t]$ factorizes into linear factors:

$$\chi(t) = (t-\lambda_1)^{m_1} \dots (t-\lambda_r)^{m_r} \text{ with } \lambda_i \neq \lambda_j \text{ for } i \neq j .$$

The λ_i are the *eigenvalues*, each with multiplicity m_i.

Question 1: Is there a decomposition of V into a direct sum of subspaces, one for each λ_i?

Answer 1: Yes.

There is an *eigenvector* $v_i \in V$ with eigenvalue λ_i, i.e. $\theta(v_i) = \lambda_i v_i$. The *eigenspace* for θ with respect to the eigenvalue λ_i is

$$\begin{aligned} \mathrm{ES}(\theta, \lambda_i) &= \{ v \in V \mid \theta(v) = \lambda_i v \} \\ &= \{ v \in V \mid (\theta - \lambda_i I)v = 0 \} \end{aligned}$$

Question 2: Is $\dim(\mathrm{ES}(\theta, \lambda_i)) = m_i$?

Question 3: Is V the direct sum of the eigenspaces of the λ_i?

Example. Let $\dim(V) = 2$. Let $\{e_1, e_2\}$ be a basis for V and take θ such that $\theta : e_1 \mapsto e_2 \mapsto 0$. The matrix of θ is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\chi(t) = \begin{vmatrix} t & -1 \\ 0 & t \end{vmatrix} = t^2$$

θ has eigenvalues $0,0$. The eigenspace of θ with eigenvalue 0 is $\mathbb{C}e_2$, so

$$\dim(\mathrm{ES}(\theta, 0)) = 1 \neq 2 = \text{multiplicity of } 0 .$$

So

Answer 2: No.

Answer 3: No.

The *generalized eigenspace* of θ with respect to the eigenvalue λ_i is

$$\begin{aligned}\mathrm{GES}(\theta,\lambda_i) &= \{ v \in V \mid v \text{ is annihilated by some power of } (\theta - \lambda_i I) \} \\ &= \{ v \in V \mid \exists N \in \mathbb{N} \text{ s.t. } (\theta - \lambda_i I)^N v = 0 \}\end{aligned}$$

So, in the above example, $\mathrm{GES}(\theta,0) = V$.

Proposition 4.2. (The Decomposition Theorem) *Let V be a vector space of dimension n over $\mathbb{C}$ and let $\theta : V \to V$ be a linear map with characteristic polynomial*

$$\chi(t) = \prod_{i=1}^{r} (t - \lambda_i)^{m_i}$$

with λ_i distinct and $\sum_{i=1}^{r} m_i = n$. Let $V_i = \mathrm{GES}(\theta,\lambda_i)$. Then
(i) $V = V_1 \oplus \ldots \oplus V_r$;
(ii) $\dim(V_i) = m_i$;
(iii) $\theta(V_i) \subseteq V_i$;
(iv) The characteristic polynomial of $\theta|_{V_i}$ is $(t-\lambda_i)^{m_i}$;
(v) $V_i = \{ v \in V \mid (\theta - \lambda_i I)^{m_i} v = 0 \}$.

Proof. The proof (omitted) uses the Cayley-Hamilton Theorem, i.e. that $\chi(\theta) : V \to V$ satisfies $\chi(\theta) = 0$.

■

Theorem 4.3. *Let L be a nilpotent Lie algebra and V an L-module. Let $y \in L$ and $\rho(y) : V \to V : v \mapsto vy$. Then the generalized eigenspaces V_i of V with respect to $\rho(y)$ are all submodules of V.*

Note. This does not hold for arbitrary Lie algebras: we need the nilpotency condition.

Recall. Leibnitz's formula for differentiation:

$$D^n(fg) = \sum_{i=0}^{n} \binom{n}{i} (D^{n-i} f)(D^i g)$$

where D denotes the action of differentiation (once).

We first prove

Proposition 4.4. *Let L be a Lie algebra and V an L-module. Let $v \in V$, $x, y \in L$ and $\alpha, \beta \in \mathbb{C}$. Then*

$$(vx)(\rho(y)-(\alpha+\beta)I)^n = \sum_{i=0}^{n}\binom{n}{i} v(\rho(y)-\alpha I)^{n-i}\left(x(\operatorname{ad} y-\beta I)^i\right)$$

Note. If $\alpha=\beta=0$ then

$$(vx)(\rho(y))^n = \sum_{i=0}^{n}\binom{n}{i} v\left(\rho(y)^{n-i}\right)\left(x(\operatorname{ad} y)^i\right)$$

Proof. We use induction on n.

If $n=1$ then $\text{LHS} = (vx)y-(\alpha+\beta)vx$ and

$$\begin{aligned}\text{RHS} &= v(\rho(y)-\alpha I)+v(x(\operatorname{ad} y-\beta I))\\ &= (vy)x-avx+v[xy]-\beta vx\end{aligned}$$

By the module axioms, $\text{LHS}=\text{RHS}$.

Now assume the result for $n=r$.

$$\begin{aligned}(vx)(\rho(y)-(\alpha+\beta)I)^{r+1} &= \left(\sum_{i=0}^{r}\binom{r}{i} v(\rho(y)-\alpha I)^{r-i}\left(x(\operatorname{ad} y-\beta I)^i\right)\right)(\rho(y)-(\alpha+\beta)I)\\ &= \sum_{i=0}^{r}\binom{r}{i} v(\rho(y)-\alpha I)^{r-i}\rho(x_i)(\rho(y)-(\alpha+\beta)I)\end{aligned}$$

where $x_i = x(\operatorname{ad} y-\beta I)^i$. Now

$$\begin{aligned}\rho(x_i)\rho(y) &= \rho(y)\rho(x_i)+\rho[x_i y]\\ \rho(x_i)(\rho(y)-(\alpha+\beta)I) &= (\rho(y)-\alpha I)\rho(x_i)+\rho(x_i(\operatorname{ad} y-\beta I))\\ &= (\rho(y)-\alpha I)\rho(x_i)\rho(x_{i+1})\end{aligned}$$

So

$$\begin{aligned}(vx)(\rho(y)-(\alpha+\beta)I)^{r+1} &= \sum_{i=0}^{r}\binom{r}{i} v(\rho(y)-\alpha I)^{r-i+1}\rho(x_i)+\sum_{i=0}^{r}\binom{r}{i} v(\rho(y)-\alpha I)^{r-i}\rho(x_{i+1})\\ &= \sum_{i=0}^{r}\binom{r}{i} v(\rho(y)-\alpha I)^{r-i+1}\rho(x_i)+\sum_{i=1}^{r+1}\binom{r}{i-1} v(\rho(y)-\alpha I)^{r-i+1}\rho(x_i)\\ &= \sum_{i=0}^{r+1}\left(\binom{r}{i}+\binom{r}{i-1}\right) v(\rho(y)-\alpha I)^{r-i+1}\rho(x_i)\end{aligned}$$

which is

$$\sum_{i=0}^{r+1}\binom{r+1}{i}v(\rho(y)-\alpha I)^{r-i+1}\left(x(\operatorname{ad} y-\beta I)^i\right)$$

■

We shall call the formula of Proposition 4.4 the *Leibnitz formula for Lie algebras*.

Proof. (of 4.3.) Consider the map $\rho(y): V \to V$. Let V_i be the generalized eigenspace of this map with eigenvalue λ_i. Let $v \in V_i$ and $x \in L$. To show that $vx \in V_i$ we require that $(vx)(\rho(y)-\lambda_i I)^N = 0$ for suitably large N. Apply Leibnitz with $\alpha = \lambda_i$, $\beta = 0$:

$$(vx)(\rho(y)-\lambda_i I)^N = \sum_{i=0}^{N}\binom{N}{i}v(\rho(y)-\lambda_i I)^{N-i}\left(x(\operatorname{ad} y)^i\right)$$

$v \in V_i$ so $v(\rho(y)-\lambda_i I)^{N-i} = 0$ if $N-i$ is sufficiently large. Since L is nilpotent $x(\operatorname{ad} y)^i = 0$ if i is suitably large. Thus, if N is suitably large, $(vx)(\rho(y)-\lambda_i I)^N = 0$, and so $vx \in V_i$. Thus, each generalized eigenspace is a submodule of V.

■

Corollary 4.5. *If L is a nilpotent Lie algebra and V is an indecomposable L-module then for all $y \in L$ the linear map $v \mapsto vy$ has only one eigenvalue.*

Proof. We know that $V = V_1 \oplus \ldots \oplus V_r$ for generalized eigenspaces V_i of $\rho(y)$. These are all submodules. Since V is indecomposable, $r = 1$.

■

Proposition 4.6. *Let L be a nilpotent Lie algebra and V an indecomposable L-module. Let $y \in L$ have a single eigenvalue $\lambda(y)$ on V. Then the map $y \mapsto \lambda(y)$ is a 1-dimensional representation of L.*

Proof. Let $\dim(V) = n$. It is clear that $y \mapsto \lambda(y)$ is linear. We must also show that

$$\lambda[xy] = \lambda(x)\lambda(y) - \lambda(y)\lambda(x).$$

The RHS is clearly zero, so we need to show that the LHS is zero as well. Consider the trace function:

$$\begin{aligned}\operatorname{tr}(A) &= \sum_i \alpha_{ii}\\ &= \sum \text{eigenvalues of } A\\ &= -\left(\text{coefficient of } t^{n-1} \text{ in } \chi(t)\right)\\ \operatorname{tr}(AB) &= \operatorname{tr}(BA)\end{aligned}$$

Consider $\rho[xy]:V\to V$; this has only one eigenvalue, $\lambda[xy]$.

$$\operatorname{tr}(\rho[xy]) = n\lambda[xy]$$
$$\begin{aligned}\operatorname{tr}(\rho[xy]) &= \operatorname{tr}(\rho(x)\rho(y)-\rho(y)\rho(x))\\ &= \operatorname{tr}(\rho(x)\rho(y))-\operatorname{tr}(\rho(y)\rho(x))\\ &= 0\end{aligned}$$

So $n\lambda[xy]=0$ and $\lambda[xy]=0$, as required.

■

Proposition 4.7. *Let L be a nilpotent Lie algebra and V an indecomposable L-module. Let $y\in L$ and let $\lambda(y)$ be the unique eigenvalue of $\rho(y)$. Define $\sigma(y):V\to V$ by*

$$\sigma(y)=\rho(y)-\lambda(y)I\ .$$

Then
(i) σ is a representation of L;
(ii) $\sigma(y)$ is a nilpotent linear map for all $y\in L$.

Proof. (i) We must show that

$$\sigma[xy]=\sigma(x)\sigma(y)-\sigma(y)\sigma(x)$$

$$\begin{aligned}\text{RHS} &= (\rho(x)-\lambda(x)I)(\rho(y)-\lambda(y)I)-(\rho(y)-\lambda(y)I)(\rho(x)-\lambda(x)I)\\ &= \rho(x)\rho(y)-\rho(y)\rho(x)\\ &= \rho[xy]\\ &= \sigma[xy]+\lambda[xy]I\\ &= \sigma[xy]\end{aligned}$$

(ii) $\rho(y)$ has characteristic polynomial $(t-\lambda(y))^n$, so $\sigma(y)=\rho(y)-\lambda(y)I$ has characteristic polynomial

$$\begin{aligned}\det(tI-\sigma(y)) &= \det((t+\lambda(y))I-\rho(y))\\ &= (t+\lambda(y)-\lambda(y))^n\\ &= t^n\end{aligned}$$

So, by the Cayley-Hamilton Theorem, $\sigma(y)$ satisfies $\sigma(y)^n=0$.

■

A representation $\sigma:L\to[M_n]$ is called a *nil representation* if each matrix $\sigma(y)$ for $y\in L$ is nilpotent.

Proposition 4.8. *Let L be a nilpotent Lie algebra and σ a nil representation of L. Then σ is equivalent to a representation under which each $x \in L$ is represented by a matrix with zeroes on and below the principal diagonal.*

Proof. Let V be an L-module giving representation σ. Suppose V is irreducible. L is nilpotent, so $L^m = 0$ for some m. So $VL^m = 0$. We show that $VL = 0$ by descending induction, i.e. that $VL^i = 0 \Rightarrow VL^{i-1} = 0$.

Let $x \in L^{i-1}$. $\sigma(x)$ is nilpotent so $\sigma(x)^k = 0$ for some k, i.e. $((vx)x)\dots x = 0$ (with k x's). So there is a $v \in V$ such that $v \neq 0$ and $vx = 0$. Let U be the set of all such v; we claim U is a submodule of V. Let $u \in U$, $y \in L$.

$$(uy)x = \underbrace{(ux)}_{=0}y + u\underbrace{[yx]}_{\in L^i} = 0$$

So $uy \in U$; hence U is a submodule of V. $U \neq 0$ and V is irreducible, so $U = V$. Hence, $Vx = 0$ for all $x \in L^{i-1}$, i.e. $VL^{i-1} = 0$. $VL^m = 0$ and $VL^i = 0 \Rightarrow VL^{i-1} = 0$, so $VL = 0$. But in this situation every subspace of V is a submodule. Since V is irreducible we have that $\dim(V) = 1$. So $x \mapsto (0) \in [M_1]$.

If the module V is not irreducible then

$$V = V_0 \supset V_1 \supset \dots \supset V_n = 0$$

where each V_{i+1} is a maximal proper submodule of V_i. V gives a nil representation, so V_i gives a nil representation; V_i/V_{i+1} gives a nil representation of L. But V_i/V_{i+1} is irreducible, so $\dim(V_i/V_{i+1}) = 1$ and $(V_i/V_{i+1})L = 0$, i.e. $V_iL \subset V_{i+1}$.

Choose a basis $e_1, \dots, e_n$ of V adapted to the chain of subspaces, i.e.

$$\begin{gathered} V_0 \text{ has basis } e_1, \dots, e_n, \\ V_1 \text{ has basis } e_2, \dots, e_n \\ \vdots \end{gathered}$$

$V_ix \subset V_{i+1}$ so the matrix representing x with respect to this basis has the required upper triangular form.

■

Corollary 4.9. *Let L be a nilpotent Lie algebra and V an indecomposable L-module. Then we can choose a basis for V such that the matrix representation of x has the form*

$$\begin{pmatrix} \lambda(x) & & * \\ & \ddots & \\ 0 & & \lambda(x) \end{pmatrix}$$

i.e. zeroes below the principal diagonal, and all elements on the principal diagonal equal.

Proof. Follows from 4.5, 4.6, 4.7 and 4.8.

■

Corollary 4.10. *Let L be a nilpotent Lie algebra and V an irreducible L-module. Then* $\dim(V)=1$.

We now consider arbitrary L-modules.

Let L be a nilpotent Lie algebra and V any L-module. A *weight* of V is a 1-dimensional representation $\lambda : L \to \mathbb{C}$ such that there is a $v \in V \setminus \{0\}$ annihilated by some power of $\rho(x) - \lambda(x)I$ for all $x \in L$, where $\rho(x) : V \to V : v \mapsto vx$.

If λ is a weight of V the corresponding *weight space* V_λ of V is

$$V_\lambda = \{ v \in V \mid v \text{ annihilated by some power of } \rho(x) - \lambda(x)I \ \forall x \in L \}$$

V_λ is a subspace of V.

Theorem 4.11. (The Weight Space Decomposition Theorem) *Let L be a nilpotent Lie algebra and V an L-module. Then*
(i) V has only finitely many weights;
(ii) V is the direct sum of its weight spaces;
(iii) each weight space is a submodule of V;
(iv) a basis can be chosen for each λ-weight space V_λ such that the matrix representation on V_λ has the form

$$x \mapsto \begin{pmatrix} \lambda(x) & & * \\ & \ddots & \\ 0 & & \lambda(x) \end{pmatrix}$$

Proof. V may be expressed as a direct sum of indecomposable submodules. Each indecomposable submodule determines a weight λ by 4.6. Let W_λ be the direct sum of all indecomposable components with weight λ. Then $V = \oplus_\lambda W_\lambda$. We need to show that $V_\lambda = W_\lambda$.

Certainly $W_\lambda \subseteq V_\lambda$. Take $v \in V_\lambda$; since $V = \oplus_\mu W_\mu$ we can write $v = \sum_\mu v_\mu$ for $v_\mu \in W_\mu$. For some N, $v(\rho(x) - \lambda(x)I)^N = 0$. So $\sum_\mu v_\mu(\rho(x) - \lambda(x)I)^N = 0$ and each $v_\mu(\rho(x) - \lambda(x)I)^N \in W_\mu$. So each $v_\mu(\rho(x) - \lambda(x)I)^N = 0$. Suppose $\lambda \neq \mu$, so there is an $x \in L$ such that $\lambda(x) \neq \mu(x)$. By 4.9, $\rho(x)$ is represented on W_μ by a matrix of the form

$$\begin{pmatrix} \mu(x) & & * \\ & \ddots & \\ 0 & & \mu(x) \end{pmatrix}$$

so $\rho(x) - \lambda(x)I$ is represented on W_μ by

$$\begin{pmatrix} \mu(x) - \lambda(x) & & * \\ & \ddots & \\ 0 & & \mu(x) - \lambda(x) \end{pmatrix}$$

Choose $x \in L$ such that $\lambda(x) \neq \mu(x)$. Then the matrix of $\rho(x) - \lambda(x)I$ is non-singular on W_μ. So the matrix on $(\rho(x) - \lambda(x)I)^N$ on W_μ is non-singular. So

$$v_\mu(\rho(x) - \lambda(x)I)^N = 0 \Rightarrow v_\mu = 0.$$

So $v_\mu = 0$ for all $\mu \neq \lambda$. So $v = \sum_\mu v_\mu = v_\lambda$. Hence $v \in W_\lambda$, so $V_\lambda \subseteq W_\lambda$, so $V_\lambda = W_\lambda$.

(i) $V_\lambda \neq 0 \Rightarrow W_\lambda \neq 0$ so λ is one of the finite number of weights in our decomposition of V. So there are only finitely many weights.

(ii) $V = \oplus_\lambda W_\lambda$ and $V_\lambda = W_\lambda$ so $V = \oplus_\lambda V_\lambda$.

(iii) $V_\lambda = W_\lambda$ is a submodule of V.

(iv) Follows from 4.9.

■

The decomposition $V = \oplus_\lambda V_\lambda$ is called the *weight space decomposition* of V.

5. CARTAN SUBALGEBRAS

Proposition 5.1. *Let L be a nilpotent Lie algebra. Then the adjoint representation of L is a nil representation.*

Proof. The adjoint representation comes from the L-module L itself, $\operatorname{ad} x : y \mapsto [yx]$. If $y \in L$ then we have $[yx] \in L^2$, $[[yx]x] \in L^3$ and so on. But L is nilpotent, so $L^m = 0$ for some $m \in \mathbb{N}$. I.e., $[[yx]x \ldots x]$ ($m-1$ x's) is zero. So $(\operatorname{ad} x)^{m-1} = 0$; $\operatorname{ad} x$ is nil.

■

The converse is also true.

Theorem 5.2. (Engel's Theorem) *If L is a Lie algebra for which the adjoint representation is a nil representation then L is nilpotent.*

Proof. Suppose not. Choose a maximal nilpotent subalgebra N of L. $[LN] \subseteq L$, so we can regard L as an N-module. $[NN] \subseteq N$, so N is an N-submodule of L. Let M be an N-submodule of L containing N such that M/N is an irreducible N-module. Since N is nilpotent, $\dim(M/N) = 1$ by 4.10. So $\dim(M) = \dim(N) + 1$.

L gives a nil representation of L, and so L gives a nil representation of N. So M gives a nil representation of N. So M/N gives a nil representation of N, $n \mapsto (\alpha)$, nil if and only if $\alpha = 0$. So $(M/N)x \subseteq N/N$ for all $x \in N$. So $[MN] \subseteq N$. Since $\dim(M) = \dim(N) + 1$, $M = N + \mathbb{C}m$. Hence,

$$\begin{aligned}[MM] &= [N + \mathbb{C}m, N + \mathbb{C}m] \\ &\subseteq [NN] + [N\mathbb{C}m] \\ &\subseteq N + N \\ &= N \\ &\subseteq M\end{aligned}$$

Thus, M is a subalgebra of L. Since $[NM] \subseteq N$, we also have that $N \lhd M$.

We know $M^2 \subseteq N$. We shall show that for each $i > 0$ there exists an integer n_i such that $M^{n_i} \subseteq N^i$. We use induction on i. If $i = 1$ take $n_i = 2$, since $M^2 \subseteq N$. Assume the statement is true for $i = r$. Then we have an n_r such that $M^{n_r} \subseteq N^r$.

$$\begin{aligned} M^{n_r+1} &= \left[M^{n_r} M\right] \\ &= \left[M^{n_r}, N + \mathbb{C}m\right] \\ &\subseteq \left[M^{n_r} N\right] + M^{n_r} \operatorname{ad} m \\ &\subseteq \left[N^r N\right] + M^{n_r} \operatorname{ad} m \\ &\subseteq N^{r+1} + M^{n_r} \operatorname{ad} m \end{aligned}$$

We now show by induction on j that $M^{n_r+j} \subseteq N^{r+1} M^{n_r} (\operatorname{ad} m)^j$. This is true for $j = 1$; assume it for $j = k$:

$$\begin{gathered} M^{n_r+k} \subseteq N^{r+1} + M^{n_r} (\operatorname{ad} m)^k \\ M^{n_r+k+1} \subseteq \left[N^{r+1}, M\right] + \left[M^{n_r} (\operatorname{ad} m)^k, M\right] \\ N \lhd M \Rightarrow N^{r+1} \lhd M \Rightarrow \left[N^{r+1}, M\right] \subseteq N^{r+1} \end{gathered}$$

So $M^{n_r+j} \subseteq N^{r+1} M^{n_r} (\operatorname{ad} m)^j$.

The adjoint representation of L is nil, so $(\operatorname{ad} m)^j = 0$ for large j. So, for such j, $M^{n_{r+1}} \subseteq N^{r+1}$, so there exists an n_r such that $M^{n_r} \subseteq N^r$.

N is nilpotent, so $N^r = 0$ for some r, hence $M^{n_r} = 0$. So M is nilpotent, which is a contradiction. So L is nilpotent.

■

We now consider arbitrary Lie algebras. Consider elements $x \in L$ "as far as possible" from 0, in that $\operatorname{ad} 0$ has all eigenvalues 0.

We say that $x \in L$ is *regular* if $\operatorname{ad} x : L \to L$ has as few eigenvalues zero as possible.

Example. Let $L = \{ A \in [M_2] \mid \operatorname{tr}(A) = 0 \}$. $\dim(L) = 3$. Basis of L:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$[he] = 2e,\ [hf] = -2f,\ [ef] = h$$

$x \in L$ has the form $ae + bh + cf$ for $a, b, c \in \mathbb{C}$.

$$\begin{aligned} [ex] &= -2be + ch \\ [hx] &= 2ae - 2cf \\ [fx] &= -ah + 2bf \end{aligned}$$

The matrix of $\operatorname{ad} x$ with respect to the basis (e, f, h) is

$$\begin{pmatrix} -2b & c & 0 \\ 2a & 0 & -2c \\ 0 & -a & 2b \end{pmatrix}$$

This has characteristic polynomial

$$\begin{vmatrix} t+2b & -c & 0 \\ -2a & t & 2c \\ 0 & a & t-2b \end{vmatrix} = (t+2b)(t^2-2bt-2ac)+2a(-ct+2bc)$$
$$= t^3-2bt^2-2act+2bt^2-4b^2t-4abc-2acb+4abc$$
$$= t^3-4t(b^2+ac)$$

So the multiplicity of zero as an eigenvalue is

$$1 \text{ if } b^2+ac \neq 0$$
$$3 \text{ is } b^2+ac=0$$

So $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ is regular if and only if $\begin{vmatrix} a & b \\ c & -a \end{vmatrix} \neq 0$.

Lemma 5.3. *Let M be a subalgebra of L. Then the set of all $x \in L$ such that $[Mx] \subseteq M$ is a subalgebra $\mathcal{N}(M)$ containing M, and $M \triangleleft \mathcal{N}(M)$. Moreover, $\mathcal{N}(M)$ is the largest subalgebra in which M is an ideal.*

Proof. Easy – see Exercise Sheet 1.

■

We call $\mathcal{N}(M)$ the *idealizer* (or *normalizer*) of M.

Theorem 5.4. *Let x be a regular element of L. Let $H = \mathrm{GES}(\mathrm{ad}\,x, 0)$. Then*
(i) H is a subalgebra of L;
(ii) H is nilpotent;
(iii) $H = \mathcal{N}(H)$.

Proof. (i) Let $y, z \in H$; we need to show that $[yz] \in H$. By Leibnitz,

$$[yz](\mathrm{ad}\,x)^n = \sum_{i=0}^{n} \binom{n}{i} \left[y(\mathrm{ad}\,x)^{n-i}, z(\mathrm{ad}\,x)^i\right]$$

$y \in H$, so $y(\operatorname{ad} x)^{n-i} = 0$ if $n-i$ is large. $z \in H$, so $z(\operatorname{ad} x)^{i} = 0$ if i is large. Hence, $[yz](\operatorname{ad} x)^{n} = 0$ for large n. So $[yz] \in H$, and H is a subalgebra.

(iii) We show that $H = \mathcal{N}(H)$. If $z \in \mathcal{N}(H)$ then $[Hz] \subseteq H$. Now $x \in H$ since $[xx] = 0$. So $[xz], [zx] \in H$. So $[zx]$ is annihilated by some power of $\operatorname{ad} x$. So $z \in H$, so $H \supseteq \mathcal{N}(H)$, hence $H = \mathcal{N}(H)$.

(ii) We show that H is nilpotent by Engel's Theorem, i.e. we show that the adjoint representation of H is nil. Let $\dim(L) = n$, $\dim(H) = l$. Choose a basis $e_1, \ldots, e_l$ of H and extend to a basis $e_1, \ldots, e_n$ of L. Let $y \in H$, $y = \lambda_1 e_1 + \ldots + \lambda_l e_l$, $\lambda_i \in \mathbb{C}$. Consider $\operatorname{ad} y : L \to L$. H is invariant under $\operatorname{ad} y$ since H is a subalgebra. Hence we also have $\operatorname{ad} y : H \to H$ and $\operatorname{ad} y : L/H \to L/H$.

Let $\chi_L(t)$ be the characteristic polynomial of $\operatorname{ad} y$ on L; let $\chi_H(t)$ be the characteristic polynomial of $\operatorname{ad} y$ on H; let $\chi_{L/H}(t)$ be the characteristic polynomial of $\operatorname{ad} y$ on L/H. We claim that $\chi_L(t) = \chi_H(t)\chi_{L/H}(t)$. $\operatorname{ad} y : L \to L$ has a matrix of the block form

$$A = \begin{pmatrix} B_{l\times l} & 0_{l\times(n-l)} \\ D_{(n-l)\times l} & C_{(n-l)\times(n-l)} \end{pmatrix}$$

$$\begin{aligned} \chi_L(t) &= \det(tI_n - A) \\ &= \det\begin{pmatrix} tI_l - B & 0 \\ B & tI_{n-l} - C \end{pmatrix} \\ &= \det(tI_l - B)\det(tI_{n-l} - C) \\ &= \chi_H(t)\chi_{L/H}(t) \end{aligned}$$

Given that $y = \lambda_1 e_1 + \ldots + \lambda_l e_l$, how do the coefficients of $\chi_L(t)$, $\chi_H(t)$ and $\chi_{L/H}(t)$ depend on the λ_i? The entries in A are linear functions of the λ_i. The coefficients in $\chi_L(t)$ etc. are polynomial functions of the λ_i.

Let $\chi_{L/H}(t) = b_0 + b_1 t + b_2 t^2 + \ldots$. We claim that b_0 is not the zero polynomial, for in the special case $y = x$ b_0 is non-zero. Let $\chi_H(t) = t^m(a_0 + a_1 t + a_2 t^2 + \ldots)$, where a_0 is not the zero polynomial. We know that $m \le l$ since $\chi_H(t)$ has degree l. So $\chi_L(t) = t^m(a_0 b_0 + \ldots)$ and $a_0 b_0$ is not the zero polynomial. Choose $\lambda_1, \ldots, \lambda_l$ such that $a_0 b_0 \neq 0$. For this y we have that $\operatorname{ad} y$ has eigenvalue 0 with multiplicity m. So, by the regularity of x, $m \ge l$; hence $m = l$.

So $\chi_H(t) = t^l(a_0 + \ldots)$ has degree l, and so is a multiple of t^l. Hence $\chi_H(t) = t^l$ since characteristic polynomials are monic. By the Cayley-Hamilton Theorem, $\operatorname{ad} y : H \to H$ satisfies $(\operatorname{ad} y)^l = 0$, so the adjoint representation of H is nil.

Hence, by Engel's Theorem, H is nilpotent.

■

The generalized eigenspace of $\operatorname{ad} x$ with eigenvalue zero where $x \in L$ is regular is called a *Cartan subalgebra* of L.

Any two Cartan subalgebras of L have equal dimension; this is called the *rank* of L.

Any Cartan subalgebra is nilpotent and is its own idealizer.

Example. Let $L = \{ A \in [M_2] \mid \operatorname{tr}(A) = 0 \}$.

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is a regular element of L. H will be the Cartan subalgebra given by $\operatorname{ad} h$. $\dim(H) = 1$. So

$$H = \mathbb{C}h = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \;\middle|\; a \in \mathbb{C} \right\}$$

In general, let H be a Cartan subalgebra of L. Then $[LH] \subseteq L$, so we can regard L as an H-module and decompose L as $L = \oplus_\lambda L_\lambda$, where the L_λ are the weight spaces of L as an H-module.

Consider the special case $\lambda = 0$, i.e. $0 : H \to \mathbb{C}$. L_0 is the 0-weight space.

Proposition 5.5. $L_0 = H$. *Thus, 0 is a weight of* H *on* L.

Proof. By definition,

$$L_0 = \left\{ y \in L \;\middle|\; (\operatorname{ad} x)^k y = 0 \text{ for some } k \text{ and all } x \in H \right\}.$$

But H is nilpotent, so $H^r = 0$, so $[[yx]x \ldots x]$ (with $r-1$ x's) is zero. So $H \subseteq L_0$.

Now suppose if possible that $H \neq L_0$. Then L_0/H is an H-module. By 4.11 the representation of H on L_0/H can be given by matrices

$$z \mapsto \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$$

So there is a non-zero element of L_0/H that is annihilated by all $z \in H$. So there exists a $y \in L_0 \setminus H$ such that $[yz] \in H$ for all $z \in H$. Hence $y \in \mathcal{N}(H)$. But $H = \mathcal{N}(H)$, so $y \in H$, a contradiction. Hence $H = L_0$.

■

Hence we have the *Cartan decomposition* of L as

$$L = H \oplus \left(\oplus_{\lambda \neq 0} L_\lambda\right)$$

The non-zero weights are called *roots*. Let Φ be the set of all roots – a finite set. Then

$$L = H \oplus \left(\oplus_{\alpha \in \Phi} L_\alpha\right)$$

By Lie multiplication we know that $[HH] \subseteq H$ and $[L_\alpha H] \subseteq L_\alpha$.

Proposition 5.6. *Let* $\alpha, \beta \in \Phi$. *Then*
(i) $[L_\alpha L_\beta] \subseteq L_{\alpha+\beta}$ *if* $\alpha + \beta \in \Phi$*;*
(ii) $[L_\alpha L_\beta] \subseteq H = L_0$ *if* $a + \beta = 0$*;*
(iii) $[L_\alpha L_\beta] = 0$ *if* $\alpha + \beta \notin \Phi$ *and* $\alpha + \beta \neq 0$.

Proof. (i) Let $y \in L_\alpha$, $z \in L_\beta$, $x \in H$. By Leibnitz,

$$[yz]\left(\operatorname{ad} x - (\alpha(x) + \beta(x))I\right)^N = \sum_{i=0}^{N} \binom{N}{i} \left[y\left(\operatorname{ad} x - \alpha(x)I\right)^{N-i}, z\left(\operatorname{ad} x - \beta(x)I\right)^i\right]$$

$y \in L_\alpha$, so y is annihilated by large powers of $(\operatorname{ad} x - \alpha(x)I)$; similarly z is annihilated by large powers of $(\operatorname{ad} x - \beta(x)I)$. So $[yz](\operatorname{ad} x - (\alpha(x) + \beta(x))I)^N = 0$ for large N. Hence, $[yz] \in L_{\alpha+\beta}$.

(ii) If $\alpha + \beta = 0$, $[yz] \in L_0 = H$ by 5.5, so $[L_\alpha L_{-\alpha}] \subseteq H$.

(iii) If $\alpha + \beta \notin \Phi \cup \{0\}$ then we deduce that $[yz] = 0$, otherwise there would be a non-zero element in the $(\alpha + \beta)$-weight space.

■

Consider $[L_\alpha L_{-\alpha}] \subseteq H$ for $\alpha \in \Phi$, and $\beta : H \to \mathbb{C}$. Consider the restriction $\beta : [L_\alpha L_{-\alpha}] \to \mathbb{C}$.

Proposition 5.7. *Let $\alpha \in \Phi$. Consider the subspace $[L_\alpha L_{-\alpha}] \subseteq H$. Let $\beta \in \Phi$. Then β restricted to $[L_\alpha L_{-\alpha}]$ is a rational multiple of α.*

Proof. If $-\alpha \notin \Phi$ then $L_{-\alpha} = 0$ and there is nothing to prove, so assume $-\alpha \in \Phi$. Let $\beta \in \Phi$ and consider the functions

$$\ldots, -2\alpha + \beta, -\alpha + \beta, \beta, \alpha + \beta, 2\alpha + \beta, \ldots,$$

all linear functions on H. Since Φ is finite there exist integers p, q such that

$$-p\alpha + \beta, \ldots, -\alpha + \beta, \beta, \alpha + \beta, \ldots, q\alpha + \beta$$

are roots but $-(p+1)\alpha + \beta$ and $(q+1)\alpha + \beta$ are not. If $-(p+1)\alpha + \beta = 0$ the result is clear; similarly if $(q+1)\alpha + \beta = 0$. So we can assume that $-(p+1)\alpha + \beta \neq 0$ and $(q+1)\alpha + \beta \neq 0$.

Let $M = L_{-p\alpha+\beta} \oplus \ldots \oplus L_{q\alpha+\beta}$. M is a subspace of L. Take $y \in L_\alpha$, $z \in L_{-\alpha}$. Then $[yz] \in L_0 = H$.

$M \operatorname{ad} y \subseteq M$ since $\operatorname{ad} y$ takes $L_{i\alpha+\beta}$ to $L_{(i+1)\alpha+\beta}$ and $L_{q\alpha+\beta}$ to 0.
$M \operatorname{ad} z \subseteq M$ since $\operatorname{ad} y$ takes $L_{i\alpha+\beta}$ to $L_{(i-1)\alpha+\beta}$ and $L_{-p\alpha+\beta}$ to 0.

Let $x = [yz] \in H$; $M \operatorname{ad} x \subseteq M$ by the above. We now calculate $\operatorname{tr}_M(\operatorname{ad} x)$ in two different ways:

$$\begin{aligned}
\operatorname{tr}_M(\operatorname{ad} x) &= \operatorname{tr}_M \operatorname{ad}[yz] \\
&= \operatorname{tr}_M(\operatorname{ad} y \operatorname{ad} z - \operatorname{ad} z \operatorname{ad} y) \\
&= \operatorname{tr}_M(\operatorname{ad} y \operatorname{ad} z) - \operatorname{tr}_M(\operatorname{ad} z \operatorname{ad} y) \\
&= 0
\end{aligned}$$

$\operatorname{ad} x$ acts on $L_{i\alpha+\beta}$ as

$$\begin{pmatrix} (i\alpha+\beta)(x) & & * \\ & \ddots & \\ 0 & & (i\alpha+\beta)(x) \end{pmatrix}$$

by 4.11. Hence,

$$\mathrm{tr}_{L_{i\alpha+\beta}}(\mathrm{ad}\,x) = (ia+\beta)(x)\dim(L_{i\alpha+\beta})$$

So

$$\begin{aligned}\mathrm{tr}_M(\mathrm{ad}\,x) &= \sum_{i=-p}^{q} \mathrm{tr}_{L_{i\alpha+\beta}}(\mathrm{ad}\,x) \\ &= \sum_{i=-p}^{q} (i\alpha+\beta)(x)\dim(L_{i\alpha+\beta}) \\ &= \alpha(x)\sum_{i=-p}^{q} i\dim(L_{i\alpha+\beta}) + \beta(x)\sum_{i=-p}^{q}\dim(L_{i\alpha+\beta})\end{aligned}$$

Equating the two traces gives

$$\beta(x)\underbrace{\sum_{i=-p}^{q}\dim(L_{i\alpha+\beta})}_{>0} = -\alpha(x)\sum_{i=-p}^{q} i\dim L_{i\alpha+\beta}$$

And so

$$\beta(x) = -\alpha(x)\frac{\sum_{i=-p}^{q} i\dim(L_{i\alpha+\beta})}{\sum_{i=-p}^{q}\dim(L_{i\alpha+\beta})}$$

So there exists an $r_{\beta,\alpha} \in \mathbb{Q}$ such that $\beta = r_{\beta,\alpha}\alpha$ on $[L_\alpha L_{-\alpha}]$.

■

6. THE KILLING FORM

We define a map $L \times L \to \mathbb{C}$ by $(x, y) \mapsto \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y)$. Define $\langle x, y\rangle = \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y)$. The map $(x, y) \mapsto \langle x, y\rangle$ is called the *Killing form*.

Proposition 6.1. *(i)* $\langle\cdot,\cdot\rangle$ *is bilinear;*
(ii) $\langle\cdot,\cdot\rangle$ *is symmetric;*
(iii) $\langle\cdot,\cdot\rangle$ *is invariant, i.e.* $\langle [xy], z\rangle = \langle x, [yz]\rangle$ *for all* $x, y, z \in L$.

Proof. (i) Easy.

(ii) Follows from the identity $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

(iii)

$$\begin{aligned}
\langle [xy], z\rangle &= \operatorname{tr}(\operatorname{ad}[xy] \operatorname{ad} z) \\
&= \operatorname{tr}((\operatorname{ad} x \operatorname{ad} y - \operatorname{ad} y \operatorname{ad} x)\operatorname{ad} z) \\
&= \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y \operatorname{ad} z) - \operatorname{tr}(\operatorname{ad} y \operatorname{ad} x \operatorname{ad} z) \\
&= \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y \operatorname{ad} z) - \operatorname{tr}(\operatorname{ad} x \operatorname{ad} z \operatorname{ad} y) \\
&= \operatorname{tr}(\operatorname{ad} x(\operatorname{ad} y \operatorname{ad} z - \operatorname{ad} z \operatorname{ad} y)) \\
&= \operatorname{tr}(\operatorname{ad} x \operatorname{ad}[yz]) \\
&= \langle x, [yz]\rangle
\end{aligned}$$

■

The Killing form is called *non-degenerate* if $\langle x, y\rangle = 0 \ \ \forall y \in L \ \Rightarrow x = 0$.

The Killing form is *identically zero* if $\langle x, y\rangle = 0 \ \ \forall x, y \in L$.

Proposition 6.2. *Let* $I \lhd L$ *and* $x, y \in I$. *Then* $\langle x, y\rangle_I = \langle x, y\rangle_L$. *Thus, the Killing form of* L *restricted to* I *is the Killing form of* I.

Proof. Choose a basis of I and extend to a basis of L. With respect to this basis, since I is an ideal, $\operatorname{ad} x$ is represented by a matrix of the block form

$$\begin{pmatrix} A_1 & 0 \\ A_2 & 0 \end{pmatrix}$$

Similarly, $\operatorname{ad} y$ is represented by

$$\begin{pmatrix} B_1 & 0 \\ B_2 & 0 \end{pmatrix}$$

So $\operatorname{ad} x \operatorname{ad} y$ is represented by

$$\begin{pmatrix} A_1B_1 & 0 \\ A_2B_1 & 0 \end{pmatrix}$$

Thus, $\operatorname{tr}(\operatorname{ad} x \operatorname{ad} y) = \operatorname{tr}(A_1B_1)$. But A_1 is the matrix of $\operatorname{ad} x$ on I and B_1 is the matrix of $\operatorname{ad} y$ on I. Thus,

$$\langle x, y\rangle_I = \operatorname{tr}(A_1B_1) = \langle x, y\rangle_L .$$

■

For any subspace M of L define the *perpendicular space* $M^\perp$ by

$$M^\perp = \{x \in L \mid \langle x, y\rangle = 0 \ \forall y \in M\}.$$

$M^\perp$ is also a subspace of L.

Lemma 6.3. $I \lhd L \Rightarrow I^\perp \lhd L$.

Proof. Let $x \in I^\perp$ and $y \in L$; we show that $[xy] \in I^\perp$. Let $z \in I$.

$$\langle [xy], z\rangle = \langle x, [yz]\rangle = 0$$

$[yz] \in I$, $x \in I^\perp$. Hence $[xy] \in I^\perp$.

■

In particular, $L^\perp \lhd L$:

$$L^\perp = \{x \in L \mid \langle x, y\rangle = 0 \ \forall y \in L\}$$

So $L^\perp = 0$ iff $\langle \cdot,\cdot\rangle$ is non-degenerate; $L^\perp = L$ iff $\langle \cdot,\cdot\rangle$ is identically zero.

Proposition 6.4. *Let L be a Lie algebra with $L \neq 0$, $L^2 = L$. Let H be a Cartan subalgebra of L. Then there is an $x \in H$ such that $\langle x, x\rangle \neq 0$.*

Proof. Consider the Cartan decomposition of L as an H-module:

$$L = \oplus_\lambda L_\lambda$$
$$L^2 = [LL] = [\oplus_\lambda L_\lambda, \oplus_\mu L_\mu] = \sum_{\lambda,\mu} [L_\lambda L_\mu]$$

Now $[L_\lambda L_\mu] \subseteq L_{\lambda+\mu}$ by 5.6, where $L_{\lambda+\mu} = 0$ if $\lambda + \mu$ is not a weight. Now $L_0 = H$, so, since $L^2 = L$, we have

$$H = \sum_\lambda [L_\lambda L_{-\lambda}] = [HH] + \sum_{\alpha\in\Phi} [L_\alpha L_{-\alpha}]$$

Now L is not nilpotent since $L^2 = L$, but H is nilpotent, so $H \neq L$. So there exists a root $\beta \in \Phi$. β is a 1-dimensional representation of H, $\beta \neq 0$. β vanishes on $[HH]$ since if $x, y \in H$ then $\beta[xy] = \beta(x)\beta(y) - \beta(y)\beta(x) = 0$. So $H = [HH]$.

Hence, there exists an $\alpha \in \Phi$ with $[L_\alpha L_{-a}] \neq 0$. In particular, $L_{-\alpha} \neq 0$. Also, β does not vanish on $[L_\alpha L_{-\alpha}]$. Choose $x \in [L_\alpha L_{-\alpha}]$ such that $\beta(x) \neq 0$. By definition, $\langle x, x\rangle = \mathrm{tr}(\mathrm{ad}\, x\, \mathrm{ad}\, x)$. $\mathrm{ad}\, x$ acts on L_λ by

$$\begin{pmatrix} \lambda(x) & & 0 \\ & \ddots & \\ 0 & & \lambda(x) \end{pmatrix}$$

by 4.11. So $(\mathrm{ad}\, x)^2$ acts on L_λ by

$$\begin{pmatrix} \lambda(x)^2 & & 0 \\ & \ddots & \\ 0 & & \lambda(x)^2 \end{pmatrix}$$

So $\langle x, x\rangle = \sum_\lambda \dim(L_\lambda)\lambda(x)^2$. However, by 5.7 there exists an $r_{\lambda,\alpha} \in \mathbb{Q}$ such that $\lambda(x) = r_{\lambda,\alpha}\alpha(x)$ since $x \in [L_\alpha L_{-\alpha}]$. So $\langle x, x\rangle = \alpha(x)^2 \sum_\lambda \dim(L_\lambda) r_{\lambda,\alpha}^2$. In particular, $\beta(x) = r_{\beta,\alpha}\alpha(x)$. Now $\beta(x) \neq 0$, so $\alpha(x) = 0$ and $r_{\beta,\alpha} \neq 0$. So

$$\langle x, x\rangle = \underbrace{\alpha(x)^2}_{\neq 0} \underbrace{\sum_\lambda \dim(L_\lambda) r_{\lambda,\alpha}^2}_{>0}$$

So $\langle x, x\rangle \neq 0$.

■

Theorem 6.5. *If the Killing form of L is identically zero then L is soluble.*

Proof. Use induction on $\dim(L)$. If $\dim(L)=1$ then L is certainly soluble. If $\dim(L)>1$, $L^2 \neq L$ by 6.4, and $L^2 \lhd L$. The Killing form on L^2 is the restriction of that on L, and so is identically zero. $\dim(L^2)<\dim(L)$; by induction L^2 is soluble. L/L^2 is abelian, and so soluble. Hence L is soluble.

■

Note. The converse is not true. Consider a Lie algebra of dimension 2, basis $\{x,y\}$, with $[xy]=x$.

$$\operatorname{ad} y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ (\operatorname{ad} y)^2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

So $\langle y,y\rangle = \operatorname{tr}\left((\operatorname{ad} y)^2\right)=1$.

For which Lie algebras is the Killing form non-degenerate?

Let R be the soluble radical of L. Then L is semisimple if and only if $R=0$.

Theorem 6.6. *The Killing form on L is non-degenerate if and only if L is semisimple.*

Proof. Suppose the Killing form on L is degenerate. Then $L^\perp \neq 0$. $L^\perp \lhd L$ by 6.3. So the Killing form of L restricted to $L^\perp$ is the Killing form of $L^\perp$. Hence, the Killing form of $L^\perp$ is identically zero, since $x,y\in L^\perp \Rightarrow \langle x,y\rangle=0$. Hence, by 6.5, $L^\perp$ is soluble. $L^\perp$ is a non-zero soluble ideal of L, and so L is not semisimple.

Conversely, suppose L is not semisimple. Then $R\neq 0$, so

$$R=R^{(0)} \supseteq R^{(1)} \supseteq \ldots R^{(k-1)} \supseteq R^{(k)}=0$$

Let $I=R^{(k-1)}$; $I\neq 0$ but $I^2=0$, and $I \lhd L$. Choose a basis of I and extend to a basis of L. Let $x\in I$ and $y\in L$. With respect to this basis, $\operatorname{ad} x : L \to L$ has matrix

$$\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}$$

$\operatorname{ad} y : L \to L$ has matrix

$$\begin{pmatrix} B_1 & 0 \\ B_2 & B_3 \end{pmatrix}$$

So $\operatorname{ad}x\operatorname{ad}y : L \to L$ has matrix

$$\begin{pmatrix} 0 & 0 \\ AB_1 & 0 \end{pmatrix}$$

Hence $\langle x, y\rangle = \operatorname{tr}(\operatorname{ad}x\operatorname{ad}y) = 0$. Thus $I \subseteq L^{\perp}$. Thus $L^{\perp} \neq 0$, which implies that the Killing form is degenerate.

■

Suppose L_1, L_2 are Lie algebras. We can define the *direct sum* $L_1 \oplus L_2$ to be the set of pairs $(x_1, x_2) \in L_1 \times L_2$ with $[(x_1, x_2)(y_1, y_2)] = ([x_1 y_1], [x_2 y_2])$. Similarly, we can define $L_1 \oplus \ldots \oplus L_k$.

The 1-dimensional Lie algebra is simple, and is called the *trivial* simple Lie algebra.

Theorem 6.7. *A Lie algebra is semisimple if and only if it is the direct sum of simple nontrivial Lie algebras.*

Proof. Let L be a semisimple Lie algebra. If L is simple L is nontrivial and there is nothing to prove. So assume L is not simple. Choose a minimal non-zero ideal $I \triangleleft L$, $I \neq L$. Consider $I^{\perp}$: $I^{\perp} \triangleleft L$ as well.

$$x \in I^{\perp} \Leftrightarrow \langle x, y\rangle = 0 \text{ for all } y \in I.$$

The Killing form is non-degenerate on L. This gives $\dim(I)$ linearly independent conditions on x, since the form is non-degenerate. So $\dim(I^{\perp}) = \dim(L) - \dim(I)$ by the Rank-Nullity Formula.

Now consider $I \cap I^{\perp}$. $I \cap I^{\perp}$ is an ideal, so the Killing form on $I \cap I^{\perp}$ is the restriction of that on L. But $x, y \in I \cap I^{\perp} \Rightarrow \langle x, y\rangle = 0$. So $I \cap I^{\perp}$ is soluble by 6.5. Since L is semisimple, $I \cap I^{\perp} = 0$.

$$\begin{aligned} \dim(I + I^{\perp}) &= \dim(I) + \dim(I^{\perp}) - \dim(I \cap I^{\perp}) \\ &= \dim(I) + \dim(I^{\perp}) \\ &= \dim(L) \end{aligned}$$

So $L = I + I^{\perp}$, and $I \cap I^{\perp} = 0$, so $L = I \oplus I^{\perp}$ as a direct sum of subspaces. Let $x \in I$ and $y \in I^{\perp}$. Then $[xy] \in [II^{\perp}] \subseteq I \cap I^{\perp} = 0$. So $L = I \oplus I^{\perp}$ as a direct sum of Lie algebras:

$$[a + b, a' + b'] = [aa'] + [bb']$$

We now show that I is simple. Let $J \lhd I$.

$$[JL] = [JI] + [JI^{\perp}] = [JI] \subseteq J$$

So J is an ideal of L contained in I. But I is minimal, so either $J = 0$ or $J = I$. Hence, I is simple.

We now show that $I^{\perp}$ is semisimple. Let J be a soluble ideal of $I^{\perp}$.

$$[JL] = [JI] + [JI^{\perp}] = [JI^{\perp}] \subseteq J$$

Thus, J is a soluble ideal of L. Since L is semisimple, $J = 0$, and so $I^{\perp}$ is semisimple.

So $L = I \oplus I^{\perp}$, I simple and $I^{\perp}$ semisimple. By induction, $I^{\perp}$ is the direct sum of simple nontrivial Lie algebras. Hence, L has this property.

Conversely, let $L = L_1 \oplus \ldots \oplus L_r$, where each L_i is simple and nontrivial. L_i is semisimple, and so its Killing form is non-degenerate by 6.6. For each i, $L_i \lhd L$. If $i \neq j$ and $x_i \in L_i$, $x_j \in L_j$, then $\langle x_i, x_j \rangle = 0$. For if $y \in L$ then $y \operatorname{ad} x_i = [yx_i] \in L_i$; $y \operatorname{ad} x_j = [yx_j] \in L_j$. So $y \operatorname{ad} x_i \operatorname{ad} x_j = [[yx_i]x_j] \in L_i \cap L_j = 0$. Thus $\operatorname{ad} x_i \operatorname{ad} x_j = 0$, so $\operatorname{tr}(\operatorname{ad} x_i \operatorname{ad} x_j) = 0$, so $\langle x_i, x_j \rangle = 0$.

We show $L^{\perp} = 0$. Let $x \in L^{\perp}$, $x = x_1 + \ldots + x_r$, $x_i \in L_i$. Let $y_i \in L_i$. Then $\langle x, y_i \rangle = \langle x_i, y_i \rangle = 0$, since $x \in L^{\perp}$. So $\langle x_i, y_i \rangle = 0$ for all $y_i \in L_i$. This implies $x_i = 0$ since the Killing form on L_i is non-degenerate. So $x = 0$, and so $L^{\perp} = 0$. Hence the Killing form on L is non-degenerate, and so L is semisimple.

■

7. THE LIE ALGEBRA $\mathfrak{sl}_n(\mathbb{C})$

$$\mathfrak{sl}_n(\mathbb{C}) = \{ A \in [M_n] \mid tr(A) = 0 \}$$

$\mathfrak{sl}_n(\mathbb{C})$ is an ideal of the Lie algebra $[M_n] = \mathfrak{gl}_n(\mathbb{C})$.

Theorem 7.1. $\mathfrak{sl}_n(\mathbb{C})$ *is a simple Lie algebra.*

Proof. Every ideal of $\mathfrak{sl}_n(\mathbb{C})$ is an ideal of $[M_n]$. We shall show that if $I \lhd [M_n]$ and $I \subseteq \mathfrak{sl}_n(\mathbb{C})$ then $I = \mathfrak{sl}_n(\mathbb{C})$ or $I = 0$.

Let I be a non-zero ideal of $[M_n]$ contained in $\mathfrak{sl}_n(\mathbb{C})$. Let $x \in I \setminus \{0\}$. Then

$$x \in \sum_{p,q} x_{pq} E_{pq} \text{ , } x_{pq} \in \mathbb{C} \text{ not all zero,}$$

where E_{pq} is an $n \times n$ matrix with 1 in the (p,q)th position and zeroes elsewhere.

Case 1: Suppose $\exists i \neq j$ such that $x_{ij} \neq 0$. Then

$$[E_{ii}x] = \sum_q x_{iq}E_{iq} - \sum_p x_{pi}E_{pi} \in I$$
$$[[E_{ii}x]E_{jj}] = x_{ij}E_{ij} + x_{ji}E_{ji} \in I$$
$$[E_{ii} - E_{jj}, x_{ij}E_{ij} - x_{ji}E_{ji}] = 2x_{ij}E_{ij} - 2x_{ji}E_{ji} \in I$$

So $4x_{ij}E_{ij} \in I$, $x_{ij} \neq 0$, so $E_{ij} \in I$.

Case 2: Suppose $x_{ij} = 0 \ \ \forall i \neq j$. Then $x = \sum_p x_{pp}E_{pp}$; $\sum_p x_{pp} = 0$, so not all the x_{pp} are equal. Suppose $x_{ii} \neq x_{jj}$.

$$[xE_{ij}] = (x_{ii} - x_{jj})E_{ij} \in I$$

So $E_{ij} \in I$.

So in either case $E_{ij} \in I$ for some $i \neq j$. Let $q \neq i, j$. Then $[E_{ij}, E_{iq}] = E_{iq} \in I$. So I contains all E_{iq} with $q \neq i$. Let $p \neq i, q$. Then $[E_{pi}, E_{iq}] = E_{pq} \in I$. So $E_{pq} \in I$ for all $p \neq q$. For $p \neq q$, $[E_{pq}, E_{qp}] = E_{pp} - E_{qq} \in I$. So $I = \mathfrak{sl}_n$. Hence, $\mathfrak{sl}_n(\mathbb{C})$ is simple.

■

It is easy to see that $\dim(\mathfrak{sl}_n(\mathbb{C})) = n^2 - 1$; assume $n \geq 2$. We now find a regular element of $\mathfrak{sl}_n(\mathbb{C})$.

Proposition 7.2. *Let*

$$x = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \in \mathfrak{sl}_n(\mathbb{C})$$

with $\sum_i \lambda_i = 0$ *and* $i \neq j \Rightarrow \lambda_i \neq \lambda_j$. *Then* x *is regular.*

Proof. Take a basis of $\mathfrak{sl}_n(\mathbb{C})$:

$$\{E_{ij} \mid i \neq j\} \cup \{E_{11} - E_{22}, E_{22} - E_{33}, \ldots, E_{n-1,n-1} - E_{nn}\}$$

Consider the matrix of $\operatorname{ad} x$ with respect to this basis:

$$[E_{ij} x] = (\lambda_j - \lambda_i) E_{ij} \text{ for } i \neq j$$
$$[E_{ii} - E_{i+1,i+1}, x] = 0$$

So the $(n^2-1) \times (n^2-1)$ matrix of $\operatorname{ad} x$ with respect to this basis is

$$\begin{pmatrix} \ddots & & & & 0 \\ & \lambda_j - \lambda_i & & & \\ & & \ddots & & \\ & & & 0 & \\ 0 & & & & \ddots \end{pmatrix}$$

The characteristic polynomial is $t^{n-1} \prod_{i \neq j} (t - \lambda_i + \lambda_j)$; the multiplicity of zero as an eigenvalue of $\operatorname{ad} x$ is $n-1$.

Now let $y \in \mathfrak{sl}_n(\mathbb{C})$. $\operatorname{ad} y$ is similar to a matrix in Jordan canonical form, say

$$\begin{pmatrix} J_{m_1}(\mu_1) & & 0 \\ & \ddots & \\ 0 & & J_{m_r}(\mu_r) \end{pmatrix} \qquad (*)$$

where $J_m(\mu)$ is the $m \times m$ matrix

$$\begin{pmatrix} \mu & 1 & & 0 \\ & \mu & \ddots & \\ & & \ddots & 1 \\ 0 & & & \mu \end{pmatrix}$$

$J_m(\mu) = \mu I_m + J_m$, where $J_m = J_m(0)$. J_m^k has 1's on the kth diagonal above the principal diagonal and zeroes elsewhere. We claim that $J_m(\mu)$ commutes with any matrix of the form $\alpha I_m + \beta J_m + \beta J_m^2 + \ldots$, so $(*)$ commutes with all matrices of the block form

$$\begin{pmatrix} \alpha_1 & \beta_1 & & 0 & & & & & \\ & \alpha_1 & \ddots & & \cdots & & 0 & & \\ & & \ddots & \beta_1 & & & & & \\ 0 & & & \alpha_1 & & & & & \\ & \vdots & & & \ddots & & \vdots & & \\ & & & & & \alpha_r & \beta_r & & 0 \\ & & & & & & \alpha_r & \ddots & \\ & 0 & & & \cdots & & & \ddots & \beta_r \\ & & & & & 0 & & & \alpha_r \end{pmatrix}$$

These matrices form a vector space of dimension $m_1 + m_2 + \ldots + m_r = n$.

Assume $m_1\alpha_1 + \ldots + m_r\alpha_r = 0$, i.e. the matrix is in $\mathfrak{sl}_n(\mathbb{C})$. We have a vector space of dimension $n-1$. All these matrices lie in the zero eigenspace of $\operatorname{ad} y$, so the multiplicity of zero as an eigenvalue of $\operatorname{ad} y$ is at least $n-1$.

Thus, x is regular.

■

Proposition 7.3. *The subalgebra of diagonal matrices in* $\mathfrak{sl}_n(\mathbb{C})$ *is a Cartan subalgebra.*

Proof. Let

$$x = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$\lambda_i \neq \lambda_j$ for $i \neq j$, $\sum_i \lambda_i = 0$. x is regular. Let

$$H = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \in \mathfrak{sl}_n(\mathbb{C}) \right\}$$

$\dim(H) = n-1$. If $y \in H$ then $[yx] = 0$, so $H \subseteq \mathrm{ES}(\mathrm{ad}\,x, 0) \subseteq \mathrm{GES}(\mathrm{ad}\,x, 0)$. But

$$\dim(H) = n-1 = \dim(\mathrm{GES}(\mathrm{ad}\,x, 0)).$$

Thus, H is a Cartan subalgebra of $\mathfrak{sl}_n(\mathbb{C})$.

■

Proposition 7.4. *Let $L = \mathfrak{sl}_n(\mathbb{C})$ and let H be the diagonal subalgebra. Then*

$$L = H \oplus \left(\oplus_{i \neq j} \mathbb{C}E_{ij}\right)$$

is the Cartan decomposition of L with respect to H.

Proof. Clearly $L = H \oplus \left(\oplus_{i \neq j} \mathbb{C}E_{ij}\right)$ as a direct sum of vector spaces. Let

$$h = \lambda_1 E_{11} + \ldots + \lambda_n E_{nn} \in H.$$

Then $[E_{ij}h] = (\lambda_j - \lambda_i)E_{ij}$. So $\mathbb{C}E_{ij}$ is an H-module giving a 1-dimensional representation

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \mapsto \lambda_j - \lambda_i \neq 0 \text{ since } i \neq j.$$

So this 1-dimensional representation of H is a weight and $\mathbb{C}E_{ij}$ lies in the weight space.

$$\dim(H) + \sum \dim(\text{weight space}) = \dim(L) = \dim(H) + \sum \dim(\mathbb{C}E_{ij})$$

So $\mathbb{C}E_{ij}$ is the full weight space, giving $L = H \oplus \left(\oplus_{i \neq j} \mathbb{C}E_{ij}\right)$ as a direct sum of Lie algebras.

■

Summary. $\mathfrak{sl}_n(\mathbb{C}) = [n \times n \text{ matrices of trace } 0]$. $\mathfrak{sl}_n(\mathbb{C})$ is simple.

$$H = \left\{ \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \middle| \sum_{i=1}^{n} \lambda_i = 0 \right\}$$

is a Cartan subalgebra. $L = H \oplus \left(\oplus_{i \neq j} \mathbb{C} E_{ij} \right)$. The roots are

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \mapsto \lambda_j - \lambda_i \text{ for } i \neq j .$$

8. THE CARTAN DECOMPOSITION

Throughout this chapter, let L be a semisimple Lie algebra and H a Cartan subalgebra of L.

$$L = \oplus_{\lambda} L_{\lambda}, \ \lambda \text{ weights – the Cartan decomposition}$$

$$H = L_0$$
$$L = H \oplus \left(\oplus_{\alpha \in \Phi} L_{\alpha}\right)$$

Proposition 8.1. *If* $x \in L_{\lambda}$, $y \in L_{\mu}$, $\lambda \neq -\mu$ *then* $\langle x, y \rangle = 0$.

Proof.

$$\langle x, y \rangle = \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y)$$
$$L_V \operatorname{ad} x \subseteq L_{V+\lambda}$$
$$L_V \operatorname{ad} x \operatorname{ad} y \subseteq L_{V+\lambda+\mu}$$

So if $\lambda + \mu \neq 0$ then $V + \lambda + \mu \neq V$. Choose a basis of L adapted to a Cartan decomposition. $\operatorname{ad} x \operatorname{ad} y$ is then represented by a block matrix with zero blocks on the diagonal:

$$\begin{pmatrix} 0 & & & * \\ & 0 & & \\ & & \ddots & \\ * & & & 0 \end{pmatrix}$$

So $\langle x, y \rangle = \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y) = 0$.

■

Proposition 8.2. *If* $\alpha \in \Phi$ *then* $-\alpha \in \Phi$.

Proof. Suppose if possible that $-\alpha \notin \Phi$. Then $L_{-\alpha} = 0$ and $L_{\alpha} \neq 0$. Let $x \in L_{\alpha}$. $\langle x, y \rangle = 0$ for all $y \in L_{\lambda}$, for all λ. Hence $\langle x, y \rangle = 0$ for all $y \in L$. But $\langle \cdot, \cdot \rangle$ is non-degenerate, so $x = 0$. Thus $L_{\alpha} = 0$, a contradiction.

■

Proposition 8.3. *The Killing form of* L *remains non-degenerate on restriction to* H*; i.e. if* $x \in H$ *and* $\langle x, y \rangle = 0$ *for all* $y \in H$ *then* $x = 0$.

Note. We are saying that the Killing form of L restricted to H is non-degenerate, *not* that the Killing form of H is non-degenerate. In fact, the Killing form of H is degenerate since H is nilpotent.

Proof. Let $x \in H$ satisfy $\langle x, y\rangle = 0$ for all $y \in H$. We also have $\langle x, y\rangle = 0$ for all $y \in L_\lambda$, $\lambda \neq 0$, by 8.1. So $\langle x, y\rangle = 0$ for all $y \in L$. So $x = 0$.

■

Proposition 8.4. *$[HH] = 0$; i.e. H is abelian.*

Proof. Suppose $x \in [HH]$ and let $y \in H$.

$$\langle x, y\rangle = \mathrm{tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y)$$

$$L = \oplus_\lambda L_\lambda, \text{ each } L_\lambda \text{ an } H\text{-module.}$$

On L_λ we have $\mathrm{ad}\, x$ represented by

$$\begin{pmatrix} \lambda(x) & & * \\ & \ddots & \\ 0 & & \lambda(x) \end{pmatrix}$$

and $\mathrm{ad}\, y$ represented by

$$\begin{pmatrix} \lambda(y) & & * \\ & \ddots & \\ 0 & & \lambda(y) \end{pmatrix}$$

Hence, $\mathrm{ad}\, x\, \mathrm{ad}\, y$ is represented by

$$\begin{pmatrix} \lambda(x)\lambda(y) & & * \\ & \ddots & \\ 0 & & \lambda(x)\lambda(y) \end{pmatrix}$$

So $\langle x, y\rangle = \mathrm{tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y) = \sum_\lambda \dim(L_\lambda)\lambda(x)\lambda(y)$. Let $h_1, h_2 \in H$. Then

$$\lambda[h_1 h_2] = \lambda(h_1)\lambda(h_2) - \lambda(h_2)\lambda(h_1) = 0$$

Since λ is a 1-dimensional representation. Hence $\lambda(x)=0$ for all $x\in[HH]$, so $\langle x,y\rangle=0$ for all $y\in H$. Since $\langle\cdot,\cdot\rangle$ is non-degenerate, $x=0$. So $[HH]=0$; i.e. H is abelian.

■

Let H^* denote the *dual space* of H, $\mathrm{Hom}(H,\mathbb{C})$. Then $\dim(H^*)=\dim(H)$. We can define a map $H\to H^*$, $h\mapsto h^*$, by $h^*(x)=\langle h,x\rangle$ for $x\in H$.

Lemma 8.5. *The map $h\mapsto h^*$ is an isomorphism of vector spaces.*

Proof. The map is clearly linear. Suppose h is in the kernel of this map, i.e. $h^*=0$. Then $\langle h,x\rangle=0$ for all $x\in H$. Since the Killing form is non-degenerate, $h=0$. So the kernel is trivial.

$$\dim(\text{image})=\dim(H)-\dim(\text{kernel})=\dim(H)=\dim(H^*)$$

■

Let $\alpha\in\Phi$. Then $\alpha\in H^*$, so there is a unique $h_\alpha\in H$ such that $\alpha(x)=\langle h_\alpha,x\rangle$ for all $x\in H$.

Proposition 8.6. *The h_α, as defined above, and taken over all $\alpha\in\Phi$, span H.*

Proof. Suppose not. Then there is an $x\in H$, $x\neq 0$, such that $\langle h_\alpha,x\rangle=0$ for each $\alpha\in\Phi$. Hence $\alpha(x)=0$ for all $\alpha\in\Phi$. Let $y\in H$.

$$\langle x,y\rangle=\mathrm{tr}(\mathrm{ad}\,x\,\mathrm{ad}\,y)=\sum\nolimits_\lambda \dim(L_\lambda)\lambda(x)\lambda(y)$$

Since $\lambda(x)=0$ for all weights λ, $\langle x,y\rangle=0$ for all $y\in H$. So $x=0$, a contradiction.

■

Proposition 8.7. *For $\alpha\in\Phi$, $h_\alpha\in[L_\alpha L_{-\alpha}]$, a subspace of H.*

Proof. Consider L_α as an H-module: it contains a 1-dimensional submodule $\mathbb{C}e_\alpha$, $e_\alpha\neq 0$. $[e_\alpha x]=\alpha(x)e_\alpha$ for $x\in H$. Let $y\in L_{-\alpha}$. We show that $[ye_\alpha]=\langle y,e_\alpha\rangle h_\alpha$. To see this let $z=[ye_\alpha]-\langle y,e_\alpha\rangle h_\alpha$ and let $x\in H$.

$$\begin{aligned}\langle z,x\rangle &= \langle [ye_\alpha],x\rangle - \langle y,e_\alpha\rangle\langle h_\alpha,x\rangle \\ &= \langle y,[e_\alpha x]\rangle - \langle y,e_\alpha\rangle\alpha(x) \\ &= \alpha(x)\langle y,e_\alpha\rangle - \langle y,e_\alpha\rangle\alpha(x) \\ &= 0\end{aligned}$$

So $\langle z,x\rangle = 0$ for all $x \in H$, so $z = 0$. So $[ye_\alpha] = \langle y,e_\alpha\rangle h_\alpha$. We can find $y \in L_{-\alpha}$ such that $\langle y,e_\alpha\rangle \neq 0$. If not, e_α is orthogonal to $L_{-\alpha}$ and to each L_λ with $\lambda \neq -\alpha$, so $e_\alpha \in L^\perp$, so $e_\alpha = 0$, a contradiction. Choose such a y, then

$$h_\alpha = \left[\frac{y}{\langle y,e_\alpha\rangle} e_\alpha\right] \in [L_{-\alpha}L_\alpha]$$

■

Proposition 8.8. *Let* $\alpha \in \Phi$. *Then* $\langle h_\alpha,h_\alpha\rangle \neq 0$.

Proof. Let $\beta \in \Phi$. We know that $h_\alpha \in [L_{-\alpha}L_\alpha]$. There exists $r_{\beta,\alpha} \in \mathbb{Q}$ such that $\beta = r_{\beta,\alpha}\alpha$ on $[L_{-\alpha}L_\alpha]$.

$$\langle h_\beta,h_\alpha\rangle = \beta(h_\alpha) = r_{\beta,\alpha}\alpha(h_\alpha) = r_{\beta,\alpha}\langle h_\alpha,h_\alpha\rangle$$

If $\langle h_\alpha,h_\alpha\rangle = 0$ then $\langle h_\beta,h_\alpha\rangle = 0$ for all $\beta \in \Phi$. But the set $\{h_\beta \mid \beta \in \Phi\}$ spans H. Hence $\langle x,h_\alpha\rangle = 0$ for all $x \in H$. This implies $h_\alpha = 0$, so $\alpha = 0$, a contradiction.

■

Theorem 8.9. *If* $\alpha \in \Phi$ *then* $\dim(L_\alpha) = 1$.

Note. Of course, $\dim(H) = \dim(L_0)$ is not generally 1.

Proof. Let M be the subspace of L given by

$$M = \mathbb{C}e_\alpha \oplus \mathbb{C}h_a \oplus L_{-\alpha} \oplus L_{-2\alpha} \oplus \ldots$$

where $\mathbb{C}e_\alpha$ is a 1-dimensional H-submodule of L_α. There are only finitely many summands since Φ is finite. Recall from the proof of 8.7 that there is an $e_{-\alpha} \in L_{-\alpha}$ such that $[e_{-\alpha}e_\alpha] = h_\alpha$. Also, for any $y \in L_{-\alpha}$, $[ye_\alpha] = \langle y,e_\alpha\rangle h_\alpha$.

We first show that $M \operatorname{ad} e_\alpha \subseteq M$:

$$[e_\alpha e_\alpha] = 0$$
$$[h_\alpha e_\alpha] = -[e_\alpha h_\alpha] = -\alpha(h_\alpha)e_\alpha$$

Let $y \in L_{-\alpha}$. Then

$$[ye_\alpha] = \langle y, e_\alpha \rangle h_\alpha \in \mathbb{C}h_\alpha$$
$$[L_{-i\alpha} e_\alpha] \subseteq L_{-(i-1)\alpha} \text{ for } i \geq 2$$

Secondly, we show that $M \operatorname{ad} e_{-\alpha} \subseteq M$.

$$[e_\alpha e_{-\alpha}] = -h_\alpha$$
$$[h_\alpha e_{-\alpha}] = -[e_{-\alpha} h_\alpha] = \alpha(h_\alpha)e_{-\alpha} \in L_{-\alpha}$$
$$[L_{-i\alpha} e_{-\alpha}] \subseteq L_{-(i+1)\alpha} \text{ for } i \geq 1$$

$[e_{-\alpha} e_\alpha] = h_\alpha$, so $M \operatorname{ad} h_a \subseteq M$. We now calculate $\operatorname{tr}_M(\operatorname{ad} h_\alpha)$ in two different ways.

$$\operatorname{tr}_M(\operatorname{ad} h_\alpha) = \operatorname{tr}_M(\operatorname{ad} e_{-a} \operatorname{ad} e_\alpha - \operatorname{ad} e_\alpha \operatorname{ad} e_{-\alpha}) = 0$$

$$\begin{aligned} \operatorname{tr}_M(\operatorname{ad} h_\alpha) &= \alpha(h_\alpha) + 0 - \dim(L_{-\alpha})a(h_\alpha) - 2\dim(L_{-2\alpha})\alpha(h_\alpha) - 3\dim(L_{-3\alpha})\alpha(h_\alpha) - \dots \\ &= \alpha(h_\alpha)(1 - \dim(L_{-\alpha}) - 2\dim(L_{-2\alpha}) - \dots) \\ &= \langle h_\alpha, h_\alpha \rangle (1 - \dim(L_{-\alpha}) - 2\dim(L_{-2\alpha}) - \dots) \end{aligned}$$

Since $\langle h_\alpha, h_\alpha \rangle \neq 0$, $1 - \dim(L_{-\alpha}) - 2\dim(L_{-2\alpha}) - \dots = 0$. Since $-\alpha \in \Phi$, $\dim(L_{-\alpha}) \geq 1$. Hence $\dim(L_{-\alpha}) = 1$ and $\dim(L_{-i\alpha}) = 0$ for $i \geq 2$.

So, interchanging $\alpha \leftrightarrow -\alpha$, we have that for each $\alpha \in \Phi$, $\dim(L_\alpha) = 1$ and $\dim(L_{i\alpha}) = 0$ for $i \geq 2$, $i \in \mathbb{N}$.

■

We have the following easy corollary:

Corollary 8.10. *If $\alpha \in \Phi$, $m\alpha \in \Phi$ and $m \in \mathbb{Z}$ then $m = \pm 1$.*

Now let $\alpha, \beta \in \Phi$, $\beta \neq \pm\alpha$. Consider

$$-p\alpha + \beta, \dots - \alpha + \beta, \beta, \alpha + \beta, 2\alpha + \beta, \dots, q\alpha + \beta$$

There are integers p, q such that all of the above are roots but $-(p+1)\alpha + \beta$ and $(q+1)\alpha + \beta$ are not roots. This collection is called the α*-chain of roots through* β.

Note that $\beta+(q+1)\alpha, \beta-(p+1)\alpha \neq 0$, so $L_{\beta+(q+1)\alpha}=0$, $L_{\beta-(p+1)\alpha}=0$. Let

$$M = L_{-p\alpha+\beta} \oplus \ldots \oplus L_{q\alpha+\beta}.$$

Choose $e_\alpha \in L_\alpha$, $e_{-\alpha} \in L_{-\alpha}$ such that $[e_{-\alpha}e_\alpha]=h_\alpha$. We claim that $M \operatorname{ad} e_\alpha \subseteq M$, $M \operatorname{ad} e_{-\alpha} \subseteq M$ and $M \operatorname{ad} h_\alpha \subseteq M$.

$$L_{i\alpha+\beta} \operatorname{ad} e_\alpha \subseteq L_{(i+1)\alpha+\beta}$$
$$L_{i\alpha+\beta} \operatorname{ad} e_{-\alpha} \subseteq L_{(i-1)\alpha+\beta}$$

So $M \operatorname{ad} h_\alpha \subseteq M$. We now calculate $\operatorname{tr}_M(\operatorname{ad} h_\alpha)$ in two different ways:

$$\operatorname{tr}_M(\operatorname{ad} h_\alpha) = \operatorname{tr}_M(\operatorname{ad} e_{-a} \operatorname{ad} e_\alpha - \operatorname{ad} e_\alpha \operatorname{ad} e_{-\alpha}) = 0$$

$$\operatorname{tr}_M(\operatorname{ad} h_\alpha) = \sum_{i=-p}^{q} (i\alpha+\beta)(h_\alpha) \text{ since } \dim(L_{i\alpha+\beta}) = 1.$$

So $\sum_{i=-p}^{q}(i\alpha+\beta)(h_\alpha)=0$; $\left(\sum_{i=-p}^{q} i\right)\alpha(h_\alpha)+\sum_{i=-p}^{q}\beta(h_\alpha)=0$.

$$\Rightarrow \quad \left(\tfrac{p(p+1)}{2}-\tfrac{q(q+1)}{2}\right)\langle h_\alpha,h_\alpha\rangle + (p+q+1)\langle h_\beta,h_\alpha\rangle = 0$$
$$\Rightarrow \quad \left(\tfrac{q-p}{2}\right)(p+q+1)\langle h_\alpha,h_\alpha\rangle + (p+q+1)\langle h_\beta,h_\alpha\rangle = 0$$
$$\Rightarrow \quad 2\langle h_\beta,h_\alpha\rangle / \langle h_\alpha,h_\alpha\rangle = p-q$$

Thus we have proved:

Proposition 8.11. *Let $\alpha,\beta\in\Phi$, $\beta\neq\pm\alpha$. Take the α-chain of roots through β,*

$$-p\alpha+\beta,\ldots-\alpha+\beta,\beta,\alpha+\beta,\ldots,q\alpha+\beta.$$

Then

$$\frac{2\langle h_\alpha,h_\beta\rangle}{\langle h_\alpha,h_\alpha\rangle} = p-q.$$

Corollary 8.12. *If $\alpha\in\Phi$, $\xi\alpha\in\Phi$ and $\xi\in\mathbb{C}$ then $\xi=\pm1$.*

Proof. If $\xi\neq\pm1$ set $\beta=\xi\alpha$. Then

$$\frac{2\langle h_\alpha, h_\beta\rangle}{\langle h_\alpha, h_\alpha\rangle} = 2\xi = p - q$$

So $2\xi \in \mathbb{Z}$. If $\xi \in \mathbb{Z}$ then $\xi = \pm 1$ by 8.10. If $\xi \notin \mathbb{Z}$ then $p \not\equiv q$ modulo 2. The α-chain of roots through β is

$$-p\alpha + \beta, \ldots -\alpha + \beta, \beta, \alpha + \beta, \ldots, q\alpha + \beta$$

But $\beta = \frac{p-q}{2}\alpha$, with p and q not both zero. So $\frac{\alpha}{2}$ appears in the α-chain. This implies $\alpha, \frac{\alpha}{2} \in \Phi$, which contradicts 8.10.

■

Proposition 8.13. *For all* $\alpha, \beta \in \Phi$, $\langle h_\alpha, h_\beta\rangle \in \mathbb{Q}$.

Proof. If $\beta \neq \pm\alpha$ then by 8.11 $\langle h_\alpha, h_\beta\rangle / \langle h_\alpha, h_\alpha\rangle \in \mathbb{Q}$. We show that $\langle h_\alpha, h_\alpha\rangle \in \mathbb{Q}$.

$$\begin{aligned}\langle h_\alpha, h_\alpha\rangle &= \operatorname{tr}(\operatorname{ad} h_\alpha \operatorname{ad} h_\alpha) \\ &= \sum_{\beta\in\Phi} \beta(h_\alpha)^2 \\ &= \sum_{\beta\in\Phi} \langle h_\alpha, h_\beta\rangle^2 \\ \frac{1}{\langle h_\alpha, h_\alpha\rangle} &= \sum_{\beta\in\Phi} \left(\frac{\langle h_\alpha, h_\beta\rangle}{\langle h_\alpha, h_\alpha\rangle}\right)^2 \in \mathbb{Q}\end{aligned}$$

So $\langle h_\alpha, h_\alpha\rangle \in \mathbb{Q}$; so $\langle h_\alpha, h_\beta\rangle \in \mathbb{Q}$.

■