9. The Root System and the Weyl Group

\{ h_\alpha \mid \alpha \in \Phi \} spans \ H, so we can find a subset \{ h_{\alpha_1}, \ldots, h_{\alpha_l} \} that forms a basis for \ H; \ \dim(H) = l.

**Proposition 9.1.** Let \( \alpha \in \Phi \). Then \( h_\alpha = \sum_{i=1}^l \mu_i h_{\alpha_i} \) for some \( \mu_i \in \mathbb{Q} \).

**Proof.** We know this for \( \mu_i \in \mathbb{C} \). Let \( \langle h_{\alpha_i}, h_{\alpha_i} \rangle = \xi_{ij} \in \mathbb{Q} \). The matrix \( \Xi = (\xi_{ij}) \) is non-singular, since if it were singular we would have \( \eta_1, \ldots, \eta_l \) not all zero such that \( \sum_{i=1}^l \eta_i \xi_{ij} = 0 \). Then

\[
\left\langle \sum_{i=1}^l \eta_i h_{\alpha_i}, h_{\alpha_i} \right\rangle = \sum_{i=1}^l \eta_i \xi_{ij} = 0
\]

So \( \left\langle \sum_{i=1}^l \eta_i h_{\alpha_i}, x \right\rangle = 0 \) for all \( x \in H \). \( \langle \cdot, \cdot \rangle \) is non-degenerate, so all \( \eta_i = 0 \), which is a contradiction.

\[
\left\langle h_{\alpha_i}, h_{\alpha_i} \right\rangle = \mu_{1,\alpha_1} \xi_{1,\alpha_1} + \cdots + \mu_{l,\alpha_l} \xi_{l,\alpha_l} \\
\quad \vdots \\
\left\langle h_{\alpha_i}, h_{\alpha_i} \right\rangle = \mu_{1,\alpha_l} \xi_{1,\alpha_l} + \cdots + \mu_{l,\alpha_l} \xi_{l,\alpha_l}
\]

We have \( l \) linear equations in \( l \) unknowns with a non-singular coefficient matrix, all the entries of which are rational. Hence, by Cramer’s Rule, there is a unique solution \( (\mu_i) \in \mathbb{Q}^l \)

\[
\text{Let } H_\mathbb{Q} \text{ be the set of all } \sum_{i=1}^l \mu_i h_{\alpha_i}, \mu_i \in \mathbb{Q}. \quad \dim_{\mathbb{Q}}(H_\mathbb{Q}) = l. \quad H_\mathbb{Q} \text{ is independent of the choice of basis; all } h_\alpha \in H_\mathbb{Q}.
\]

Let \( H_\mathbb{R} \) be the set of all \( \sum_{i=1}^l \mu_i h_{\alpha_i}, \mu_i \in \mathbb{R}. \quad \dim_{\mathbb{R}}(H_\mathbb{R}) = l \).

**Proposition 9.2.** Let \( x \in H_\mathbb{R} \). Then \( \langle x, x \rangle \in \mathbb{R}_{\geq 0} \) and \( \langle x, x \rangle = 0 \Leftrightarrow x = 0 \).

**Proof.** Let \( x \in H_\mathbb{R}, \ x = \sum_{i=1}^l \mu_i h_{\alpha_i}. \)
\[
\langle x, x \rangle = \sum_i \sum_j \mu_i \mu_j \langle h_{\alpha_i}, h_{\alpha_j} \rangle = \sum_i \sum_j \mu_i \mu_j \text{tr}(\text{ad} h_{\alpha_i} \text{ad} h_{\alpha_j}) = \sum_i \sum_j \mu_i \mu_j \sum_{\alpha \in \Phi} \alpha(h_{\alpha_i}) \alpha(h_{\alpha_j}) = \sum_\alpha \left( \sum_i \mu_i \alpha(h_{\alpha_i}) \right)^2
\]

So \( \langle x, x \rangle \in \mathbb{R} \) and \( \langle x, x \rangle \geq 0 \). Suppose \( \langle x, x \rangle = 0 \). Then for all \( \alpha \in \Phi \), \( \sum_i \mu_i \alpha(h_{\alpha_i}) = 0 \). In particular, \( \sum_i \mu_i \alpha(h_{\alpha_i}) = 0 \) for \( j = 1, \ldots, l \); \( \sum_i \mu_i \langle h_{\alpha_i}, h_{\alpha_j} \rangle = \sum_i \mu_i \varepsilon_{ij} = 0 \) for all \( j \). \( \Xi \) is non-singular, so \( \mu_i = 0 \) for all \( i \), so \( x = 0 \).

So all \( h_{\alpha} \in H_{\mathbb{R}} ; \) \( \dim_{\mathbb{R}}(H_{\mathbb{R}}) = l \). We introduce a total order on \( H_{\mathbb{R}} \): let \( x \in H_{\mathbb{R}} \), \( x = \sum_i \mu_i h_{\alpha_i} \). If \( x \neq 0 \) we say \( x > 0 \) if the first non-zero \( \mu_i \) is positive; if \( x \neq 0 \) we say \( x < 0 \) if the first non-zero \( \mu_i \) is negative. We have trichotomy: for each \( x \in H_{\mathbb{R}} \) precisely one of \( x = 0 \), \( x < 0 \), \( x > 0 \) is true.

So, for \( \alpha \in \Phi \), \( h_{\alpha} < 0 \) or \( h_{\alpha} > 0 \). Define \( \alpha < 0 \) if \( h_{\alpha} < 0 \) and \( \alpha > 0 \) if \( h_{\alpha} > 0 \). Define

\[
\Phi^+ = \{ \alpha \in \Phi \mid \alpha > 0 \}, \text{ the positive roots, and} \\
\Phi^- = \{ \alpha \in \Phi \mid \alpha < 0 \}, \text{ the negative roots.}
\]

Clearly, \( \Phi = \Phi^+ \cup \Phi^- \).

A \textit{fundamental root} is a positive root that is not the sum of two positive roots. Let \( \Pi \) be the set of fundamental roots.

**Proposition 9.3.** (i) Every positive root is a sum of fundamental roots.
(ii) \( \{ h_\alpha \mid \alpha \in \Pi \} \) is a basis of \( H_{\mathbb{R}} \).
(iii) If \( \alpha, \beta \in \Pi \) and \( \alpha \neq \beta \) then \( \langle h_\alpha, h_\beta \rangle \leq 0 \).

**Proof.** (i) Let \( \alpha \in \Phi^+ \). If \( \alpha \in \Pi \) we are done. If \( \alpha \notin \Pi \) the there exist \( \beta, \gamma \in \Phi^+ \) such that \( \alpha = \beta + \gamma \) with \( \beta, \gamma < \alpha \). Repeat to get the result.

(iii) Let \( \alpha, \beta \in \Pi \) with \( \alpha \neq \beta \). Then \( \alpha - \beta \notin \Phi \) since if not

\[
\alpha = (\alpha - \beta) + \beta \text{ or } \beta = (\beta - \alpha) + \alpha
\]
so either $\alpha$ or $\beta$ would be a sum of positive roots. Consider the $\alpha$-chain of roots through $\beta$:

$$\beta, \alpha + \beta, ..., q\alpha + \beta$$

$$\Rightarrow \quad 2\frac{\langle h_\alpha, h_\beta \rangle}{\langle h_\alpha, h_\alpha \rangle} = p - q = -q$$

by 8.11.

$$\Rightarrow \quad \langle h_\alpha, h_\beta \rangle \leq 0 \quad \text{since } \langle h_\alpha, h_\alpha \rangle \geq 0 \text{ by 9.2.}$$

(ii) By (i), the $h_\alpha$ for $\alpha \in \Phi$ span $H$. We show the $h_\alpha$ are linearly independent. Suppose not: then there exist $\mu_i \in \mathbb{R}$ not all zero such that

$$\sum_{\alpha, \in \Pi} \mu_i h_\alpha = 0$$

Rearrange this sum, taking all the positive $\mu_i$ to one side. Then

$$x = \mu_i h_{\alpha_i} + ... + \mu_i h_{\alpha_i} = \mu_j h_{\alpha_j} + ... + \mu_j h_{\alpha_j}$$

$$\mu_i, \mu_j > 0, \ i, j, \ \text{distinct for } 1 \leq u \leq r, 1 \leq v \leq s.$$ Then

$$\langle x, x \rangle = \langle \mu_i h_{\alpha_i} + ... + \mu_i h_{\alpha_i} + \mu_j h_{\alpha_j} + ... + \mu_j h_{\alpha_j} \rangle \leq 0$$

by (iii). So $x = 0$, a contradiction. 

Note. $\Phi^+$ can be chosen in many different ways. However, $\Pi$ is determined by $\Phi^+$ and $\Phi^+$ is determined by $\Pi$.

Example. Let $L = \mathfrak{sl}_n(\mathbb{C})$. The roots are

$$\begin{pmatrix} \lambda_1 & 0 \\ & \ddots \\ 0 & \lambda_n \end{pmatrix} \mapsto \lambda_j - \lambda_i \text{ for } j \neq i$$

Define $\Phi^+$ to be the roots with $j > i$. Then the fundamental roots are
\[
\begin{pmatrix}
\lambda_1 & 0 \\
\vdots & \ddots \\
0 & \lambda_n
\end{pmatrix} \mapsto \lambda_{i+1} - \lambda_i \text{ for } 1 \leq i \leq n-1
\]

\[
\lambda_j - \lambda_i = (\lambda_{i+1} - \lambda_i) + (\lambda_{i+2} - \lambda_{i+1}) + \ldots + (\lambda_j - \lambda_{j-1})
\]

\[
\dim(H) = n - 1 = l , \text{ the rank of } L .
\]

For each \( \alpha \in \Phi \) we define \( s_\alpha : H_\mathbb{R} \to H_\mathbb{R} \) by

\[
s_\alpha(x) = x - 2 \frac{\langle h_\alpha, x \rangle}{\langle h_\alpha, h_\alpha \rangle} h_\alpha
\]

\( s_\alpha \) is linear and \( s_\alpha(h_\alpha) = -h_\alpha \). The set of \( x \) such that \( \langle x, x \rangle = 0 \) forms a hyperplane i.e. a subspace of codimension 1. \( s_\alpha \) is the reflection of \( H_\mathbb{R} \) in the hyperplane orthogonal to \( h_\alpha \).

\[
s_\alpha^2 = \text{id}
\]

\( s_\alpha = s_{-\alpha} \)

Let \( W \) be the group of all non-singular linear maps \( H_\mathbb{R} \to H_\mathbb{R} \) generated by \( \{ s_\alpha \mid \alpha \in \Phi \} \). \( W \) is called the Weyl group.†

**Proposition 9.4.** (i) \( W \) is a finite group.
(ii) \( W \) is a group of isometries, i.e. for all \( x, y \in H_\mathbb{R} \), \( w \in W \), \( \langle w(x), w(y) \rangle = \langle x, y \rangle \).
(iii) For each \( \alpha \in \Phi \) and \( w \in W \) there is a \( \beta \in \Phi \) such that \( w(h_\alpha) = h_\beta \).

**Proof.** (ii) Let \( x, y \in H_\mathbb{R} \). Then

\[
\langle s_\alpha(x), s_\alpha(y) \rangle = \left( x - 2 \frac{\langle h_\alpha, x \rangle}{\langle h_\alpha, h_\alpha \rangle} h_\alpha, y - 2 \frac{\langle h_\alpha, y \rangle}{\langle h_\alpha, h_\alpha \rangle} h_\alpha \right)
\]

\[
= \langle x, y \rangle - 4 \frac{\langle h_\alpha, x \rangle \langle h_\alpha, y \rangle}{\langle h_\alpha, h_\alpha \rangle} - 4 \langle h_\alpha, x \rangle \langle h_\alpha, y \rangle \langle h_\alpha, h_\alpha \rangle
\]

\[
= \langle x, y \rangle
\]

So \( s_\alpha \) is an isometry; so \( w \) is an isometry for all \( w \in W \).

(iii) Now consider \( s_\alpha(h_\beta) \):

† After Hermann Weyl.
\[ s_\alpha(h_\beta) = h_{-\alpha} \]
\[ s_\alpha(h_{-\alpha}) = h_\alpha \]

So suppose \( \beta \neq \pm \alpha \). Consider the \( \alpha \)-chain of roots through \( \beta \),

\[ -p\alpha + \beta, \ldots, \beta, \ldots, q\alpha + \beta \]

\[ s_\alpha(h_\beta) = h_\beta - 2 \left( \frac{\langle h_\alpha, h_\beta \rangle}{\langle h_\alpha, h_\alpha \rangle} \right) h_\alpha \]
\[ = h_\beta - (p - q)h_\alpha \]
\[ = h_\beta + (q - p)\alpha \]

Now \( \beta + (q - p)\alpha \in \Phi \) since \( -p \leq p - q \leq q \). So \( s_\alpha \) permutes the \( h_\beta \) for \( \beta \in \Phi \). Hence \( w \in W \) permutes the \( h_\beta \) for \( \beta \in \Phi \). Note that

\[ \beta + ((q - p)\alpha + \beta) = (-p\alpha + \beta) + (q\alpha + \beta) \]

so \( s_\alpha \) inverts the \( h_\beta \) in a given \( \alpha \)-chain.

(i) We have a homomorphism from \( W \) to the group of permutations of the \( h_\alpha \) for \( \alpha \in \Phi \). \( \Phi \) is finite, so the image of this homomorphism is finite. If \( w \in W \) is in the kernel then \( w(h_\alpha) = h_\alpha \) for all \( \alpha \in \Phi \). Since the \( h_\alpha \) span \( H_\mathbb{R} \), \( w = \text{id} \). Hence, \( W \) is finite.

\[ \text{Proposition 9.5. Given any root } \alpha \in \Phi \text{ there exists a fundamental root } \alpha_i \in \Pi 	ext{ and a } w \in W \text{ such that } h_{\alpha_i} = w(h_{\alpha_i}). \]

**Proof.** Each \( \alpha \in \Phi \) has the form \( \alpha = n_1\alpha_1 + \ldots + n_i\alpha_i \), \( n_i \in \mathbb{Z} \). If \( \alpha \in \Phi^+ \) then all \( n_i \geq 0 \); if \( \alpha \in \Phi^- \) then all \( n_i \leq 0 \). We may assume \( \alpha \in \Phi^+ \) since if \( \alpha \in \Phi^- \) then use \( h_\alpha = s_{\alpha_i}(h_{-\alpha_i}) \). The quantity \( n_1 + \ldots + n_i \) is called the **height** of \( \alpha \), \( \text{ht}(\alpha) \). We use induction on \( \text{ht}(\alpha) \). If \( \text{ht}(\alpha) = 1 \) we are done, so assume \( \text{ht}(\alpha) > 1 \). By 8.12, at least two \( n_i > 0 \).

\[ 0 < \langle h_\alpha, h_\alpha \rangle = \sum_i n_i \langle h_i, h_i \rangle \]

All \( n_i \geq 0 \), so there exists \( i \) such that \( \langle h_i, h_i \rangle > 0 \). Let \( s_{\alpha_i}(h_\alpha) = h_\beta \).
\[ h_\beta = h_\alpha - 2 \frac{\langle h_\alpha, h_\alpha \rangle}{\langle h_\alpha, h_\alpha \rangle} h_\alpha, \]

\[ \beta = \alpha - 2 \frac{\langle h_\alpha, h_\alpha \rangle}{\langle h_\alpha, h_\alpha \rangle} \alpha. \]

So \( \text{ht}(\beta) < \text{ht}(\alpha) \). Passing from \( \alpha \) to \( \beta \) changes only one \( n_i \), hence \( \beta \) has at least one \( n_j > 0 \), so \( \beta \in \Phi^+ \). By induction, \( \beta = w'(h_{\alpha_j}) \) for some \( w' \in W \) and some \( \alpha_j \in \Pi \). Thus, taking \( w = s_{\alpha_i} w' \in W \),

\[ h_\alpha = s_{\alpha_i} (h_\beta) = s_{\alpha_i} w'(h_{\alpha_j}) = w(h_{\alpha_j}). \]

\[ \text{Proposition 9.6.} \text{ The Weyl group } W \text{ is generated by } s_{\alpha_i}, \ldots, s_{\alpha_i} \text{ for } \Pi = \{ \alpha_i, \ldots, \alpha_i \}. \]

\[ \text{Proof.} \text{ Suppose } W_0 \text{ is the subgroup generated by } \{ s_{\alpha_i} \mid \alpha_i \in \Pi \}. \text{ To show } W = W_0 \text{ we show } s_{\alpha} \in W_0 \text{ for all } \alpha \in \Phi. \text{ The proof of 9.5 shows that } h_\alpha = w(h_{\alpha_i}) \text{ for some } \alpha_i \in \Pi \text{ and some } w \in W. \text{ Consider } w s_{\alpha_i} w^{-1} \in W_0. \]

\[ w s_{\alpha_i} w^{-1}(h_\alpha) = w s_{\alpha_i}(h_{\alpha_i}) = w(-h_{\alpha_i}) = -h_\alpha \]

Let \( x \in H_{\mathbb{R}} \) be such that \( \langle h_\alpha, x \rangle = 0 \). Then

\[ \Rightarrow \quad \langle w^{-1}(h_\alpha), w^{-1}(x) \rangle = 0 \]

\[ \Rightarrow \quad \langle h_{\alpha_i}, w^{-1}(x) \rangle = 0 \]

\[ \Rightarrow \quad w s_{\alpha_i} w^{-1}(x) = w w^{-1}(x) = x \]

Hence, \( w s_{\alpha_i} w^{-1} = s_{\alpha} \). Then \( s_{\alpha} \in W_0 \), so \( W = W_0 \).

\[ \text{Example.} \ L = \mathfrak{sl}_3(\mathbb{C}); \ \dim(L) = 8. \]

\[ H = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{array}{l} \lambda_1 + \lambda_2 + \lambda_3 = 0 \end{array} \]

\[ \dim(H) = 2. \]
\[ L = H \oplus \mathbb{C}E_{12} \oplus \mathbb{C}E_{23} \oplus \mathbb{C}E_{13} \oplus \mathbb{C}E_{21} \oplus \mathbb{C}E_{32} \oplus \mathbb{C}E_{31} \]

Let

\[

h = \begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix}
\]

\[ [E_{ij}h] = (\lambda_j - \lambda_i)E_{ij} \]

The roots are

\[
\begin{align*}
\alpha_1 : h & \mapsto \lambda_2 - \lambda_1 \\
\alpha_2 : h & \mapsto \lambda_3 - \lambda_2 \\
-\alpha_1 : h & \mapsto \lambda_1 - \lambda_2 \\
-\alpha_2 : h & \mapsto \lambda_2 - \lambda_3 \\
\Phi & = \{ \pm \alpha_1, \pm \alpha_2, \pm (\alpha_1 + \alpha_2) \} \\
\Pi & = \{ \alpha_1, \alpha_2 \}
\end{align*}
\]

Consider the corresponding vectors \( h_{\alpha} \in H \). Let

\[

h = \begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix} \in H ,

h' = \begin{pmatrix}
\mu_1 & 0 & 0 \\
0 & \mu_2 & 0 \\
0 & 0 & \mu_3
\end{pmatrix} \in H
\]

\[
\langle h, h' \rangle = \text{tr(ad } h \text{ ad } h')
\]

\[
\begin{align*}
&= 2(\lambda_2 - \lambda_1)(\mu_2 - \mu_1) + 2(\lambda_3 - \lambda_2)(\mu_3 - \mu_2) + 2(\lambda_3 - \lambda_1)(\mu_3 - \mu_1) \\
&= 2(2\lambda_1\mu_1 + 2\lambda_2\mu_2 + 2\lambda_3\mu_3 - (\lambda_1\mu_1 + \lambda_2\mu_1 + \lambda_2\mu_3 + \lambda_3\mu_1 + \lambda_3\mu_3)) \\
&= 4(\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3) - 2(\lambda_1 + \lambda_2 + \lambda_3)(\mu_1 + \mu_2 + \mu_3) + 2(\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3) \\
&= 6(\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3) \\
&= 6 \text{tr}(hh')
\end{align*}
\]

\( h_{\alpha_1} \) satisfies \( \langle h_{\alpha_1}, h \rangle = \alpha_1(h) = \lambda_2 - \lambda_1 \), so

\[

h_{\alpha_1} = \frac{1}{6} \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Similarly,
\[ h_{\alpha_2} = \frac{1}{6} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

For \( x \in H_\mathbb{R} \) define \( |x| = \sqrt{\langle x, x \rangle} \). With this notation, \( |h_{\alpha_1}| = |h_{\alpha_2}| = 1/\sqrt{3} \) and

\[
\langle h_{\alpha_1}, h_{\alpha_2} \rangle = 6 \cdot \frac{1}{6} \cdot (-1) = -\frac{1}{6}
\]

The angle between \( h_{\alpha_1} \) and \( h_{\alpha_2} \) is given by the cosine formula:

\[
\angle(h_{\alpha_1}, h_{\alpha_2}) = \frac{|h_{\alpha_1}| |h_{\alpha_2}| \cos \theta}{\theta = 2\pi/3}
\]

\[ W = \{ \text{id}, s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_1} s_{\alpha_2}, s_{\alpha_2} s_{\alpha_1}, s_{\alpha_1} s_{\alpha_2} s_{\alpha_1} = s_{\alpha_2} s_{\alpha_1} s_{\alpha_2} = s_{\alpha_1} s_{\alpha_2} \} \]
10. THE DYNKIN DIAGRAM

We shall consider the geometrical properties of the $h_{\alpha}$ for $\alpha \in \Phi$.

**Proposition 10.1.** Let $\alpha, \beta \in \Phi$, $\beta \neq \pm \alpha$. Then

(i) the angle between $\alpha$ and $\beta$ is one of 

\[ \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{6}; \]

(ii) if the angle is $\frac{\pi}{3}$ or $2\pi/3$, $h_{\alpha}$ and $h_{\beta}$ have the same length;

(iii) if the angle is $\frac{\pi}{4}$ or $3\pi/4$, the ratio of the lengths of $h_{\alpha}$ and $h_{\beta}$ is $\sqrt{2}$;

(iv) if the angle is $\frac{\pi}{6}$ or $5\pi/6$, the ratio of the lengths of $h_{\alpha}$ and $h_{\beta}$ is $\sqrt{3}$.

**Proof.** Let $\theta_{\alpha\beta}$ be the angle between $h_{\alpha}$ and $h_{\beta}$. We have

\[ \langle h_{\alpha}, h_{\beta} \rangle = |h_{\alpha}| |h_{\beta}| \cos \theta_{\alpha\beta} \]

\[ \Rightarrow \]

\[ \langle h_{\alpha}, h_{\beta} \rangle^2 = \langle h_{\alpha}, h_{\alpha} \rangle \langle h_{\beta}, h_{\beta} \rangle \cos^2 \theta_{\alpha\beta} \]

\[ \Rightarrow \]

\[ 4 \cos^2 \theta_{\alpha\beta} = \frac{2 \langle h_{\alpha}, h_{\beta} \rangle}{\langle h_{\alpha}, h_{\alpha} \rangle} \frac{2 \langle h_{\beta}, h_{\alpha} \rangle}{\langle h_{\beta}, h_{\beta} \rangle} \]

By 8.11, both factors on the RHS are integers, so $4 \cos^2 \theta_{\alpha\beta} \in \mathbb{Z}$. $0 \leq \cos^2 \theta_{\alpha\beta} < 1$, so $0 \leq 4 \cos^2 \theta_{\alpha\beta} < 4$, so $4 \cos^2 \theta_{\alpha\beta} \in \{0,1,2,3\}$.

\[ \Rightarrow \]

\[ \cos^2 \theta_{\alpha\beta} \in \{0, \pm \frac{1}{2}, \pm \frac{1}{\sqrt{2}}, \pm \frac{\sqrt{3}}{2} \} \]

\[ \Rightarrow \]

\[ \theta_{\alpha\beta} \in \{\pi/2, \pi/3, 2\pi/3, \pi/4, 3\pi/4, \pi/6, 5\pi/6\} \]

\[ 4 \cos^2 \theta_{\alpha\beta} = \left( \frac{2 \langle h_{\alpha}, h_{\beta} \rangle}{\langle h_{\alpha}, h_{\alpha} \rangle} \right) \left( \frac{2 \langle h_{\beta}, h_{\alpha} \rangle}{\langle h_{\beta}, h_{\beta} \rangle} \right) \]

Suppose $\theta_{\alpha\beta}$ is $\pi/3$ or $2\pi/3$, so $4 \cos^2 \theta_{\alpha\beta} = 1$. $1 = 1.1 = (-1)(-1)$, so $|h_{\alpha}| = |h_{\beta}|$.

Suppose $\theta_{\alpha\beta}$ is $\pi/4$ or $3\pi/4$, so $4 \cos^2 \theta_{\alpha\beta} = 2$. $2 = 1.2 = 2.1 = (-1)(-2) = (-2)(-1)$. So one of $\langle h_{\alpha}, h_{\alpha} \rangle$ and $\langle h_{\beta}, h_{\beta} \rangle$ is twice the other, so one of $|h_{\alpha}|$, $|h_{\beta}|$ is $\sqrt{2}$ times the other.

Suppose $\theta_{\alpha\beta}$ is $\pi/6$ or $5\pi/6$, so $4 \cos^2 \theta_{\alpha\beta} = 3$. $3 = 1.3 = 3.1 = (-1)(-3) = (-3)(-1)$. So, as above, one of $|h_{\alpha}|$, $|h_{\beta}|$ is $\sqrt{3}$ times the other.

\[ \blacksquare \]
Proposition 10.2. Let \( \alpha \in \Phi \). Then every \( \alpha \)-chain of roots has at most four roots in it.

Proof. Consider the \( \alpha \)-chain of roots through \( \beta \) with \( \beta \) as the first root:

\[
\beta, \alpha + \beta, \ldots, q\alpha + \beta
\]

By 8.11, \( 2\langle h_\alpha, h_\beta \rangle / \langle h_\alpha, h_\alpha \rangle = -q \). The LHS is \( 0, -1, -2 \) or \(-3\) by 10.1. So \( q \leq 3 \). So the length of the \( \alpha \)-chain is at most 4.

Let

\[
a_{ij} = 2 \frac{\langle h_\alpha, h_\alpha \rangle}{\langle h_\alpha, h_\alpha \rangle}
\]

and \( A = (a_{ij}) \). \( A \) is called the Cartan matrix; the \( a_{ij} \) are the Cartan integers.

Proposition 10.3. The Cartan matrix has the following properties:

(i) for each \( i \), \( a_{ii} = 2 \);
(ii) for \( i \neq j \), \( a_{ij} \in \{0, -1, -2, -3\} \);
(iii) \( a_{ij} = -2 \Rightarrow a_{ji} = -1 \);
(iv) \( a_{ij} = 0 \Leftrightarrow a_{ji} = 0 \).

Proof. If \( i \neq j \) then \( 4\cos^2 \theta_{\alpha \beta} = a_{ij}a_{ji} \).

(i) Clear.
(ii) Follows from 10.1, 9.3.
(iii) Follows from 10.1.
(iv) Clear.

We incorporate this information into a graph. The Dynkin\(^\dagger\) diagram is a graph \( \Delta \) with \( l \) vertices, one for each fundamental root. If \( i \neq j \) then vertices \( i, j \) are joined by \( n_{ij} = a_{ij}a_{ji} \) edges, \( 0 \leq n_{ij} \leq 3 \). The Dynkin diagram may be disconnected, as in

\[\begin{array}{c}
\bullet \\
\bullet \\
\end{array}\]

\(\dagger\) After E.B. Dynkin. Also due to H.S.M. Coxeter.
It splits into connected components and the Cartan matrix splits into corresponding blocks; off-diagonal blocks are zero:

$$A = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$$

We define a corresponding quadratic form $Q$:

$$Q(x_1, \ldots, x_i) = \sum_{i=1}^{l} 2x_i^2 - \sum_{1 \leq i < j \leq l} \sqrt{n_{ij}} x_i x_j$$

Recall the correspondence between quadratic forms on $\mathbb{R}$ and real symmetric matrices:

$$M = (m_{ij}) \text{ symmetric }$$

$$xMx^T = \sum_{i,j} m_{ij} x_i x_j$$

The matrix of $Q(x_1, \ldots, x_i)$ is

$$
\begin{pmatrix}
2 & -\sqrt{n_{12}} & -\sqrt{n_{13}} \\
-\sqrt{n_{12}} & 2 & -\sqrt{n_{23}} \\
-\sqrt{n_{13}} & -\sqrt{n_{23}} & \ddots \\
\vdots & \vdots & \ddots & \ddots \\
\end{pmatrix}
$$

**Proposition 10.4.** The quadratic form $Q(x_1, \ldots, x_i)$ is positive definite, i.e. $Q(x_1, \ldots, x_i) \geq 0$ and $Q(x_1, \ldots, x_i) = 0 \iff x_1 = \ldots = x_i = 0$.

**Proof.**

$$4 \cos^2 \theta_{ij} = a_i a_j = n_{ij}$$

$$2 \cos \theta_{ij} = -\sqrt{n_{ij}}$$

$$\langle h_{\alpha_i}, h_{\alpha_j} \rangle = \frac{\langle h_{\alpha_i}, h_{\alpha_j} \rangle}{\frac{1}{h_{\alpha_i} h_{\alpha_j}}} \cos \theta_{ij}$$

$$Q(x_1, \ldots, x_i) = \sum_{i,j=1}^{l} 2 \left\langle \frac{h_{\alpha_i}, h_{\alpha_j}}{h_{\alpha_i} h_{\alpha_j}}, x_i x_j \right\rangle = 2 \left( \sum_i x_i h_{\alpha_i}, \sum_j x_j h_{\alpha_j} \right) = 2 \langle y, y \rangle$$

where $y = \sum_i x_i h_{\alpha_i} / \| h_{\alpha_i} \|$. So $Q(x_1, \ldots, x_i) \geq 0$. If $Q(x_1, \ldots, x_i) = 0$ then $\langle y, y \rangle = 0$, so $y = 0$, so all $x_i = 0$. The converse is clear.
Recall. Any quadratic form can be diagonalized; there exists a non-singular real \( l \times l \) matrix \( P \) such that \( PMP^T = D \), a diagonal matrix. Let \( y = xP^{-1} \); then \( xMx^T = yDy^T \).

**Proposition 10.5.** Let \( M = (m_{ij}) \) be an \( l \times l \) real symmetric matrix. Then the associated quadratic form \( \sum_{i,j} m_{ij}x_ix_j \) is positive definite if and only if all leading minors of \( M \) have positive determinant. (The leading minors are

\[
(m_{11})(m_{12} \ldots m_{1l}) (m_{22} \ldots M).
\]

**Proof.** We use induction on \( l \). Assume the quadratic form is positive definite. If \( l = 1 \), \( M = (m_{11}) \). \( m_{11}x^2 > 0 \Leftrightarrow m_{11} > 0 \). Suppose \( l > 1 \). \( \sum_{i,j=1}^l m_{ij}x_ix_j \) is still positive definite as it is the original with \( x_i = 0 \). By induction, the first \( l-1 \) leading minors of \( M \) have positive determinant; we require that \( \det(M) > 0 \). \( xMx^T = yDy^T \), \( D \) diagonal with entries \( d_1, \ldots, d_l > 0 \). Now if \( PMP^T = D \),

\[
\det(P)^2 \det(M) = \det(D) > 0.
\]

Conversely, suppose that all leading minors of \( M \) have positive determinant. The same is true of the smaller \( (l-1) \times (l-1) \) leading minor. By induction, \( \sum_{i,j=1}^{l-1} m_{ij}x_ix_j \) is positive definite. So we have a diagonal form in new coordinates \( y_1, \ldots, y_l \):

\[
\sum_{i,j=1}^l m_{ij}x_ix_j = \sum_{k=1}^{l-1} d_kx_k^2 \quad \text{with} \quad d_k > 0.
\]

\[
\sum_{i,j=1}^l m_{ij}x_ix_j = \sum_{k=1}^{l-1} d_kx_k^2 + 2e_1y_1x_1 + \ldots + 2e_{l-1}y_{l-1}x_l + e_l^2
\]

This may be diagonalized by a further transformation of coordinates:

\[
z_i = y_i + \frac{e_i}{d_i}x_i
\]

We get \( d_1z_1^2 + \ldots d_{l-1}z_{l-1}^2 + f\xi_l^2 \). So there is a non-singular \( P \) such that

\[
PMP^T = \begin{pmatrix}
d_1 & 0 \\
\ddots & 0 \\
0 & d_{l-1}
\end{pmatrix}
\]

\[
\det(P)^2 \det(M) = f \prod_{i=1}^{l-1} d_i
\]
We assume \( \det(M) > 0 \), so \( f \prod_{i=1}^{l-1} d_i > 0 \), so \( f > 0 \). Thus, the form is positive definite.

We consider graphs with the following properties:
(i) the graph is connected;
(ii) any two distinct vertices are joined by 0, 1, 2 or 3 edges;
(iii) the associated quadratic form is positive definite.

The Dynkin diagram of a semisimple Lie algebra has connected components satisfying (i)-(iii). It is possible to determine all graphs satisfying (i)-(iii).

**Theorem 10.6.** The only graphs satisfying (i)-(iii) are

\[
\begin{align*}
A_1 & \quad A_2 & \quad A_3 & \quad \cdots & \quad A_l \\
B_2 & \quad B_3 & \quad B_4 & \quad \cdots & \quad B_l \\
D_4 & \quad D_5 & \quad \cdots & \quad \cdots & \quad D_l \\
E_6 & \quad E_7 & \quad \cdots & \quad \cdots & \quad E_8 \\
F_4 & \quad & \quad & \quad & \quad G_2
\end{align*}
\]

**Proof.** The given graphs clearly satisfy (i) and (ii). We show that they satisfy (iii). We show \( Q(x_1, \ldots, x_l) \) is positive definite by induction on \( l \). If \( l = 1 \) we have \( Q(x_1) = 2x_1^2 \), which is positive definite. Suppose \( l > 1 \). There is a vertex \( l \) such that when it is removed we have another graph on the list. By induction, \( Q(x_1, \ldots, x_{l-1}) \) is positive definite, so all leading minors of the matrix of \( Q(x_1, \ldots, x_{l-1}) \) have positive determinant. To complete the induction we show that the matrix of \( Q(x_1, \ldots, x_l) \) has positive determinant.

Let \( Y_l \) be a graph of \( l \) vertices and \( y_l \) the determinant of the matrix of the associated quadratic form. In the case \( l = 1 \), \( a_l = |2| = 2 \). In the case \( l = 2 \) we have
Suppose \( l \geq 3 \). Remove a vertex \( l \) joined to just one other vertex \( l - 1 \) by a single edge. If \( Y \) is the given graph, let \( Y_{l-1} \) be the graph with vertex \( l \) removed in this way, and let \( Y_{l-2} \) be the graph with vertices \( l \) and \( l - 1 \) removed in this way.

\[
y_i = \det(Y_i) = \begin{vmatrix} 0 & \vdots & 0 & 2 & -1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & -1 & 2 \\ 2 & -1 & 0 & \cdots & 0 \\ \end{vmatrix} = 2y_{l-1} - (-1)(-1)y_{l-2} = 2y_{l-1} - y_{l-2}
\]

Hence:

Type \( A_i \)  \( a_i = 2a_{i-1} - a_{i-2} \Rightarrow a_i = l + 1 \)

Type \( B_i \)  \( b_i = 2b_{i-1} - b_{i-2} \Rightarrow b_i = 2 \)

Type \( D_i \)  \( d_4 = 2a_3 - a_1^2 = 4 \)

\( d_5 = 2d_4 - a_3 = 4 \)

\( \Rightarrow d_i = 4 \) by induction

Type \( E_6 \)  \( e_6 = 2d_5 - a_4 = 3 \)

Type \( E_7 \)  \( e_7 = 2e_6 - d_5 = 2 \)

Type \( E_8 \)  \( e_8 = 2e_7 - e_6 = 1 \)

Type \( F_4 \)  \( f_4 = 2b_3 - a_2 = 1 \)

Type \( G_2 \)  \( g_2 = 1 \)

Hence, \( Q(x_1, \ldots, x_l) \) is positive definite in each case.

In order to show the converse, i.e. that the graphs on our list are the only possible ones, we shall first require some additional results.

**Proposition 10.7.** For each of the following graphs the corresponding quadratic form \( Q(x_1, \ldots, x_l) \) has determinant zero.
Proof. In most cases we can calculate the determinant as before, but not in types $\tilde{A}_i$, $\tilde{C}_i$.

Type $\tilde{A}_i$

\[
\begin{vmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 & 0 \\
-1 & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & -1 \\
-1 & \ddots & \ddots & 2
\end{vmatrix} = 0
\]

since the row sum is $(0,\ldots,0)$

Type $\tilde{C}_i$

\[
\begin{vmatrix}
0 & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots \\
2 & -\sqrt{2} & \ddots & \ddots \\
0 & \ldots & 0 & -\sqrt{2} & 2
\end{vmatrix} = 2b_i - \left(-\sqrt{2}\right)\sqrt{2}b_{i-1} = 0
\]

Type $\tilde{B}_i$

$\tilde{b}_3 = 2b_3 - a_i^2 = 2.2 - 2^2 = 0$

Type $\tilde{D}_i$

$\tilde{d}_4 = 2d_4 - a_i^3 = 2.4 - 2^3 = 0$

Type $\tilde{E}_6$

$\tilde{e}_6 = 2e_6 - a_5 = 2.3 - 6 = 0$
Lemma 10.8. Let \( Y \) be a graph in which any two vertices are joined by at most three edges. Suppose the corresponding quadratic form is positive definite. Suppose \( Y' \) is a graph obtained from \( Y \) by omitting some of the vertices, or by reducing the number of edges, or both. Then the quadratic form for \( Y' \) is also positive definite.

We call \( Y' \) a subgraph of \( Y \).

Example.

\[
Y' = \bullet - \bullet \text{ is a subgraph of } Y = \bullet - \bullet - \bullet - \bullet - \bullet
\]

Proof. The quadratic form for \( Y \) is

\[
Q(x_1, \ldots, x_i) = \sum_{i=1}^{l} 2x_i^2 - \sum_{1 \leq i, j \leq l} x_i x_j.
\]

The quadratic form for \( Y' \) is

\[
Q'(x_1, \ldots, x_m) = \sum_{i=1}^{m} 2x_i^2 - \sum_{1 \leq i, j \leq m} x_i x_j,
\]

with \( m \leq l \) and \( n'_y \leq n_y \). Suppose \( Q' \) is not positive definite. Then there exists \((y_1, \ldots, y_m) \neq 0 \) with \( Q'(y_1, \ldots, y_m) \leq 0 \). Consider \( Q([y_1, \ldots, y_m], 0, \ldots, 0) \). This is

\[
\sum_{i=1}^{m} 2y_i^2 - \sum_{1 \leq i, j \leq m} n'_y y_i y_j \leq \sum_{i=1}^{m} 2y_i^2 - \sum_{1 \leq i, j \leq m} n_y y_i y_j \\
\leq \sum_{i=1}^{m} 2y_i^2 - \sum_{1 \leq i, j \leq m} n_y y_i y_j \\
\leq Q(y_1, \ldots, y_m)
\]

So \( Q([y_1, \ldots, y_m], 0, \ldots, 0) \leq Q'(y_1, \ldots, y_m) \leq 0 \). So \( Q(x_1, \ldots, x_i) \) is not positive definite, a contradiction.

We now return to the proof of 10.6.

Suppose \( Y \) is some graph satisfying conditions. (i)-(iii). By 10.7 and 10.8 we know that no graph of the form \( \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}, \tilde{F} \) or \( \tilde{G} \) can be obtained as a subgraph of \( Y \).
(a) \( Y \) contains no cycles, for otherwise \( Y \) would have a subgraph of the form \( \tilde{A}_i \).

(b) If \( Y \) has a triple edge then \( Y = G_2 \), for otherwise \( Y \) would have \( \tilde{G}_2 \) as a subgraph.

(c) Suppose \( Y \) has no triple edge. Then \( Y \) can have no more than one double edge, for otherwise \( Y \) has a subgraph of type \( \tilde{C}_i \).

(d) Suppose \( Y \) has one double edge. Then \( Y \) has no branch point, for otherwise \( Y \) has \( \tilde{B}_i \) as a subgraph.

(e) If the double edge is not at one end then \( Y = F_4 \), for otherwise \( Y \) has a subgraph \( \tilde{F}_4 \). If the double edge is at one end, \( Y = B_4 \).

(f) Now suppose \( Y \) has only single edges. Then \( Y \) cannot have a branch point with four or more branches, for otherwise \( Y \) has \( \tilde{D}_4 \) as a subgraph.

(g) \( Y \) can have no more than one branch point, for otherwise \( Y \) has a subgraph \( \tilde{D}_l \), \( l \geq 5 \).

(h) If \( Y \) has no branch points, \( Y = A_i \). So suppose \( Y \) has just one branch point with three branches of lengths \( l_1 \leq l_2 \leq l_3 \), \( l_1 + l_2 + l_3 + 4 = l \). Then \( l_1 = 1 \), for otherwise \( Y \) would have \( \tilde{E}_6 \) as a subgraph.

(i) If \( l_1 = l_2 = 1 \), \( Y = D_4 \). Also, \( l_2 \leq 2 \), for otherwise \( Y \) has \( \tilde{E}_6 \) as a subgraph.

(j) So assume \( l_1 = 1 \), \( l_2 = 2 \). Then \( l_3 \leq 4 \), for otherwise \( Y \) has \( \tilde{E}_8 \) as a subgraph.

\[
\begin{align*}
l_1 = 2 & \Rightarrow Y = E_6 \\
l_2 = 3 & \Rightarrow Y = E_7 \\
l_3 = 4 & \Rightarrow Y = E_8
\end{align*}
\]

**Corollary 10.9.** Every Dynkin diagram of a semisimple Lie algebra has connected components of type \( A_i, B_i, D_i, E_6, E_7, E_8, F_4 \) and \( G_2 \).

The Cartan matrix \( A = (a_{ij}) \) determines the Dynkin diagram since \( n_{ij} = a_{ij} a_{ji} \). However, the Dynkin diagram does not always determine the Cartan matrix. Recall that the \( a_{ij} \) satisfy
If \( n_j = 1 \) then \( a_j = a_j = 1 \). If \( n_j = 2 \) then \((a_j, a_j) = (-2, -1)\) or \((-1, -2)\). If \( n_j = 3 \) then \((a_j, a_j) = (-3, -1)\) or \((-1, -3)\). In this last case the Dynkin diagram is \( G_2 \). We have

\[
A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}
\]

and one is obtained from the other by re-labeling the vertices.

Suppose \( n_j = 2 \). If \( l = 2 \) the possibilities for the Cartan matrix are

\[
A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}
\]

again obtainable from one another by re-labeling. If \( l \geq 3 \) there are two possible Cartan matrices:

\[
B_i = \begin{pmatrix} 2 & -1 & \cdots & \cdots & \cdots \\ -1 & 2 & \cdots & \cdots & \cdots \\ \cdots & \cdots & 2 & -1 & \cdots \\ \cdots & \cdots & \cdots & 2 & -2 \\ -2 & -1 & \cdots & \cdots & 2 \end{pmatrix}, \quad C_i = \begin{pmatrix} 2 & -1 & \cdots & \cdots & \cdots \\ -1 & 2 & \cdots & \cdots & \cdots \\ \cdots & \cdots & 2 & -2 & \cdots \\ \cdots & \cdots & \cdots & -2 & -1 \\ -2 & -1 & \cdots & \cdots & 2 \end{pmatrix}
\]

\[
|h_{a_1}| = |h_{a_2}| = \sqrt{2}|h_{a_3}|
\]

We place an arrow on the Dynkin diagram when we have a double or triple edge; the arrow points from the longer root to the shorter one. For example, with \( G_2 \):

\[
\begin{array}{c}
\text{long} \\
\text{short}
\end{array}
\]

**Theorem 10.10.** The possible Cartan matrices with connected Dynkin diagrams are (up to permutation of the numbering of the vertices):
\[
\begin{align*}
A_t &= \begin{pmatrix}
2 & -1 \\
-1 & 2 \\
& \ddots \\
& & 2 & -1 \\
& & -1 & 2
\end{pmatrix} \\
B_t &= \begin{pmatrix}
2 & -1 \\
-1 & 2 \\
& \ddots \\
& & 2 & -1 \\
& & -2 & 2
\end{pmatrix} \\
C_t &= \begin{pmatrix}
2 & -1 \\
-1 & 2 \\
& \ddots \\
& & 2 & -2 \\
& & -1 & 2
\end{pmatrix} \\
D_t &= \begin{pmatrix}
2 & -1 \\
-1 & 2 \\
& \ddots \\
& & 2 & -1 & -1 \\
& & -1 & 2 & 0 \\
& & 0 & -1 & 2
\end{pmatrix} \\
E_6 &= \begin{pmatrix}
2 & -1 \\
-1 & 2 & -1 \\
& -1 & 2 & -1 \\
& & 0 & -1 & 2 \\
& & -1 & 2
\end{pmatrix} \\
E_7 &= \begin{pmatrix}
2 & -1 \\
-1 & 2 & -1 \\
& -1 & 2 & -1 \\
& & -1 & 2 & -1 & 0 & -1 \\
& & 0 & -1 & 2 \\
& & -1 & 2 
\end{pmatrix} \\
E_8 &= \begin{pmatrix}
2 & -1 \\
-1 & 2 & -1 \\
& -1 & 2 & -1 \\
& & -1 & 2 & -1 & 0 & -1 \\
& & 0 & -1 & 2 \\
& & -1 & 2 
\end{pmatrix} \\
F_4 &= \begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -2 & 2 & -1 \\
0 & 0 & -1 & 2
\end{pmatrix} \\
G_2 &= \begin{pmatrix}
2 & -1 \\
-3 & 2
\end{pmatrix}
\end{align*}
\]
11. The Indecomposable Root Systems

A root system is called *indecomposable* if it has a connected Dynkin diagram.

*Case* \( l = 1 \). We have only one possibility, \( A_1 \):

\[
\begin{align*}
\Phi &= \{ \pm \alpha_1 \} \\
W &= \langle s_{\alpha_1} \rangle; \ |W| = 2 
\end{align*}
\]

*Case* \( l = 2 \). Here we have three possibilities

- **Type** \( A_2 \).

\[
\Phi = \{ \pm \alpha_1, \pm \alpha_2, \pm (\alpha_1 + \alpha_2) \}
\]

- **Type** \( B_2 \).
\[ \Phi = \{ \pm \alpha_1, \pm \alpha_2, \pm (\alpha_1 + \alpha_2), \pm (2\alpha_1 + \alpha_2) \} \]

(In the above diagram, the dashed lines indicating the reflection axes are shown slightly offset for clarity.)

Type \( G_2 \).

\[ \Phi = \{ \pm \alpha_1, \pm \alpha_2, \pm (\alpha_1 + \alpha_2), \pm (2\alpha_1 + \alpha_2), \pm (3\alpha_1 + \alpha_2), \pm (3\alpha_1 + 2\alpha_2) \} \]
Case $l \geq 3$. Type $A_l$. It is convenient to describe the root system of type $A_l$ in a Euclidean space of dimension $l + 1$.

Let $V$ be an $(l + 1)$-dimensional Euclidean space. Let $\{e_0, \ldots, e_n\}$ be an orthogonal basis of vectors of the same length, so $\langle e_i, e_j \rangle = K \delta_{ij}$ for some $K > 0$.

Define $h_{\alpha_i} = e_0 - e_1$, $h_{\alpha_2} = e_1 - e_2$, $\ldots$, $h_{\alpha_{l-1}} = e_{l-1} - e_l$. The $h_{\alpha_i}$ are linearly independent.

Thus for suitable $K$ the $h_{\alpha_i}$ form a fundamental system of roots of type $A_l$. Let $V_0$ be the subspace spanned by these vectors; $\dim(V_0) = l$. Consider the map $V \rightarrow V$ given by $e_0 \leftrightarrow e_1$, $e_i \leftrightarrow e_i$ for $i \geq 2$.

$$
\begin{align*}
\langle h_{\alpha_i}, h_{\alpha_j} \rangle &= 2K \\
\langle h_{\alpha_i}, h_{\alpha_j} \rangle &= 0 \text{ if } j \neq i - 1, i, i + 1 \\
\langle h_{\alpha_i}, h_{\alpha_{i+1}} \rangle &= -K \\
a_{i,i+1} &= \frac{2\langle h_{\alpha_i}, h_{\alpha_{i+1}} \rangle}{\langle h_{\alpha_i}, h_{\alpha_i} \rangle} = \frac{-2K}{2K} = -1
\end{align*}
$$

This is $s_{\alpha_i}$. Similarly, the linear map $V \rightarrow V$ such that $e_{i-1} \leftrightarrow e_i$, all others fixed, is $s_{\alpha_i}$.

$W$ is generated by $s_{\alpha_1}, \ldots, s_{\alpha_l}$. The group generated by all transpositions $(e_i, e_j)$ is isomorphic to $S_{l+1}$. So we have a homomorphism $S_{l+1} \rightarrow V$. This map is surjective; it is also injective, since any permutation of $\{e_0, \ldots, e_n\}$ that fixes each $h_{\alpha_i}$ is the identity. Hence, $W \cong S_{l+1}$.

Each $h_\alpha$ has the form $h_\alpha = w(h_{\alpha_i})$ for some $w \in W$ and some $i$. Hence

$$\Phi = \{e_i - e_j \mid 0 \leq i \neq j \leq l\}.$$
So \(|\Phi|=l(l+1)|

Type \(B_l\):

\[
\begin{array}{cccccc}
1 & 2 & \cdots & l-1 & l \\
\bullet & \bullet & \cdots & \bullet & \bullet \end{array}
\]

Let \(V\) be a Euclidean space of dimension \(l\) with basis \(\{\epsilon_1, \ldots, \epsilon_l\}\) such that \(\langle \epsilon_i, \epsilon_j \rangle = K \delta_{ij}\). Define

\[
\begin{align*}
h_{\epsilon_1} &= \epsilon_1 - \epsilon_2 \\
h_{\epsilon_2} &= \epsilon_2 - \epsilon_3 \\
&\vdots \\
h_{\epsilon_{l-1}} &= \epsilon_{l-1} - \epsilon_l \\
h_{\epsilon_l} &= \epsilon_l
\end{align*}
\]

These form a fundamental system of vectors of type \(B_l\).

\[
\begin{align*}
|h_{\epsilon_1}| &= \ldots = |h_{\epsilon_{l-1}}| = \sqrt{2}|h_{\epsilon_l}| \\
\langle h_{\epsilon_i}, h_{\epsilon_j} \rangle &= 0 \text{ for } 1 \leq i \leq l-2 \\
\langle h_{\epsilon_{l-1}}, h_{\epsilon_l} \rangle &= -K \\
2\frac{\langle h_{\epsilon_{l-1}}, h_{\epsilon_{l-2}} \rangle}{2K} &= -1
\end{align*}
\]

\(s_{\epsilon_1} : \epsilon_1 \leftrightarrow \epsilon_2\) and leaves others fixed,

\(s_{\epsilon_2} : \epsilon_2 \leftrightarrow \epsilon_3\) and leaves others fixed,

\(\vdots\)

\(s_{\epsilon_{l-1}} : \epsilon_{l-1} \leftrightarrow \epsilon_l\) and leaves others fixed,

\(s_{\epsilon_l} : \epsilon_l \rightarrow -\epsilon_l\) and leaves others fixed.

\(W = \langle s_{\epsilon_1}, \ldots, s_{\epsilon_l} \rangle\); for \(w \in W\), \(w(\epsilon_i) = \pm \epsilon_j\). \(|W| = 2^l l!\). Each \(h_\alpha\) has the form \(h_\alpha = w(h_{\alpha_i})\) for some \(w \in W\) and some \(i\). Hence,

\[
\Phi = \{ \pm \epsilon_i \pm \epsilon_j \mid 1 \leq i \neq j \leq l \} \cup \{ \pm \epsilon_i \mid 1 \leq i \leq l \}
\]
\[ |\Phi| = 2^2 \binom{l}{2} + 2l = 2l(l - 1) + 2l = 2l^2 \]

**Type C**<sub>i</sub>.

\[ \begin{array}{cccccc}
1 & 2 & \cdots & l-1 & l \\
\end{array} \]

Let \( V \) be a Euclidean space of dimension \( l \) with basis \( \{ \varepsilon_1, \ldots, \varepsilon_i \} \) such that \( \langle \varepsilon_i, \varepsilon_j \rangle = K \delta_{ij} \). Define

\[
\begin{align*}
    h_{\alpha_1} &= \varepsilon_1 - \varepsilon_2 \\
    &\vdots \\
    h_{\alpha_i} &= \varepsilon_i - \varepsilon_{i-1} \\
    h_{\alpha_{i+1}} &= 2\varepsilon_i
\end{align*}
\]

\( W \) is the same as for \( B_i \). For \( w \in W \), \( w(\varepsilon_i) = \pm \varepsilon_j \). \( |W| = 2^l l! \). The \( h_\alpha \) are the vectors \( \pm \varepsilon_i \pm \varepsilon_j \) (for \( i \neq j \)) and \( \pm 2\varepsilon_i \). Hence,

\[ |\Phi| = 2^2 \binom{l}{2} + 2l = 2l(l - 1) + 2l = 2l^2 \]

**Type D**<sub>i</sub>.

\[ \begin{array}{cccc}
1 & 2 & \cdots & l-1 \\
\end{array} \]

Let \( V \) be a Euclidean space of dimension \( l \) with basis \( \{ \varepsilon_1, \ldots, \varepsilon_i \} \) such that \( \langle \varepsilon_i, \varepsilon_j \rangle = K \delta_{ij} \). Define

\[
\begin{align*}
    h_{\alpha_1} &= \varepsilon_1 - \varepsilon_2 \\
    &\vdots \\
    h_{\alpha_{i-1}} &= \varepsilon_{i-1} - \varepsilon_i \\
    h_{\alpha_i} &= \varepsilon_{i-1} + \varepsilon_i
\end{align*}
\]

This is a fundamental system of roots of type \( D_i \).
$s_{a_1} : e_1 \leftrightarrow e_2$ and leaves others fixed,

$\vdots$

$s_{a_{i-1}} : e_{i-1} \leftrightarrow e_i$ and leaves others fixed,

$s_{a_i} : \pm e_{i-1} \mapsto \mp e_i$ and leaves others fixed.

For $w \in W$, $w(e_i) = \pm e_j$. There will be an even number of sign changes, so $|W| = 2^{l-1}l!$.

The $h_\alpha$ have the form $\pm e_i \pm e_j$ for $i \neq j$. So

$$|\Phi| = 2^3 \binom{l}{2} = 4 \frac{l(l-1)}{2} = 2l(l-1)$$

**Type $F_4$.**

Let $V$ be a 4-dimensional Euclidean space with basis $\{e_1, e_2, e_3, e_4\}$, $\{e_i, e_j\} = K\delta_{ij}$.

$$h_{a_1} = e_1 - e_2$$

$$h_{a_2} = e_2 - e_3$$

$$h_{a_3} = e_3$$

$$h_{a_4} = \frac{1}{2}(-e_1 - e_2 - e_3 + e_4)$$

This is a fundamental system of vectors of type $F_4$. $s_{a_1}, s_{a_2}, s_{a_3}$ permute $e_1, e_2, e_3$ and change signs arbitrarily.

$$s_{a_1} : \begin{cases} e_1 \mapsto \frac{1}{2}(e_1 - e_2 - e_3 + e_4) = e_1 + h_{a_4} \\ e_2 \mapsto \frac{1}{2}(-e_1 + e_2 - e_3 + e_4) = e_2 + h_{a_3} \end{cases}$$

$$s_{a_2} : \begin{cases} e_3 \mapsto \frac{1}{2}(-e_1 - e_2 + e_3 + e_4) = e_3 + h_{a_4} \\ e_4 \mapsto \frac{1}{2}(e_1 + e_2 + e_3 + e_4) \end{cases}$$

$$s_{a_3} : \begin{cases} e_1 + e_2 \mapsto -e_3 + e_4 \end{cases}$$

Let $S = \{h_\alpha \mid \alpha \in \Phi \}$.

$$\{ \pm e_i \pm e_j \mid 1 \leq i \neq j \leq 3 \} \subseteq S$$

$$\{ \pm e_i \mid 1 \leq i \leq 3 \} \subseteq S$$
So $S$ contains

$$\pm \varepsilon_i \pm \varepsilon_j \text{ for } 1 \leq i \neq j \leq 4$$

$$\pm \varepsilon_i \text{ for } 1 \leq i \leq 4$$

$$\frac{1}{2} (\pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$$

This collection of vectors is closed under the actions of $s_{\alpha_1}, \ldots, s_{\alpha_4}$.

$$|\Phi| = 2^2 \binom{4}{2} + 2.4 + 2^4 = 48$$

(We have 24 short roots and 24 long ones.)

*Type $E_8$.*

Let $V$ be a Euclidean space of dimension 8 with basis $\{\varepsilon_1, \ldots, \varepsilon_8\}$ such that $\langle \varepsilon_i, \varepsilon_j \rangle = K \delta_{ij}$. Define

$$h_{\alpha_i} = \varepsilon_i - \varepsilon_2$$

$$\vdots$$

$$h_{\alpha_7} = \varepsilon_6 - \varepsilon_7$$

$$h_{\alpha_8} = \varepsilon_6 + \varepsilon_7$$

$$h_{\alpha_8} = -\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + \varepsilon_8)$$

$$\|h_{\alpha_8}\| = |h_{\alpha_8}|$$

$$\langle h_{\alpha_8}, h_{\alpha_8} \rangle = -K$$

$$2 \langle h_{\alpha_8}, h_{\alpha_8} \rangle = -2K$$

$$\frac{2K}{-2K} = -1$$
These vectors form a fundamental system of type $E_8$. $s_{\alpha_1}, \ldots, s_{\alpha_8}$ permute $\epsilon_i, \ldots, \epsilon_7$ and change an even number of signs.

$$s_{\alpha_k} : \epsilon_i \mapsto \frac{1}{4}(-\epsilon_i + \ldots + 3\epsilon_i + \ldots - \epsilon_8) = \epsilon_i + \frac{1}{2}h_{\alpha_k}$$

Let $S = \{h_\alpha \mid \alpha \in \Phi\}$. $S$ contains $\pm \epsilon_i \pm \epsilon_j$ for $1 \leq i \neq j \leq 7$. $S$ contains $\frac{1}{2}\sum_{i=1}^{8} \epsilon_i$ so $S$ contains $\frac{1}{2}(-\epsilon_1 - \ldots - \epsilon_6 + \epsilon_7 + \epsilon_8) = s_{\alpha_8}(\epsilon_7 + \epsilon_8)$. So $\epsilon_7 + \epsilon_8 \in S$. So $S$ contains $\pm \epsilon_i \pm \epsilon_8$ for $1 \leq i \leq 7$. $S$ also contains $\frac{1}{2}(\pm \epsilon_1 \pm \ldots \pm \epsilon_8)$ with an even number of negative signs. So $S$ contains

$$\pm \epsilon_i \pm \epsilon_j \text{ for } 1 \leq i \neq j \leq 8$$
$$\frac{1}{2}(\pm \epsilon_1 \pm \ldots \pm \epsilon_8) \text{ with an even number of negative signs}$$

This is the whole of $S$, for it is invariant under $s_{\alpha_1}, \ldots, s_{\alpha_8}$: this is clear for $s_{\alpha_1}, \ldots, s_{\alpha_8}$ but requires a little work to check for $s_{\alpha_8}$. So the roots of $E_8$ are

$$\pm \epsilon_i \pm \epsilon_j \text{ for } 1 \leq i \neq j \leq 8$$
$$\frac{1}{2}\sum_{i=1}^{8} \epsilon_i \text{ where } \Pi(\pm) = 1$$

$$|\Phi| = 2^2 \binom{8}{2} + 2^7 = 240$$

**Type $E_7$.**

Take $V$ as before – $\dim(V) = 8$. Take $V_0$ to be the subspace of $V$ perpendicular to $\epsilon_1 - \epsilon_8$. $\dim(V_0) = 7$ and $h_{\alpha_2}, \ldots, h_{\alpha_8}$ form a basis of $V_0$. This is a fundamental system of type $E_7$. Consider $S = \{h_\alpha \mid \alpha \in \Phi(E_7)\}$. This set lies in $\{h_\alpha \mid \alpha \in \Phi(E_8)\} \cap V_0$. This intersection is

$$\pm \epsilon_i \pm \epsilon_j \text{ for } 2 \leq i \neq j \leq 7$$
$$\pm(\epsilon_i + \epsilon_8)$$
$$\frac{1}{2}(\epsilon_i \pm \epsilon_2 \pm \ldots \pm \epsilon_7 + \epsilon_8) \text{ where } \Pi(\pm) = 1$$
$$\frac{1}{2}(-\epsilon_1 \pm \epsilon_2 \pm \ldots \pm \epsilon_7 - \epsilon_8) \text{ where } \Pi(\pm) = 1$$
All of these can be obtained from \( h_{a_1}, \ldots, h_{a_8} \) by means of \( s_{a_2}, \ldots, s_{a_8} \). This is obvious except for \( \pm (\varepsilon_1 + \varepsilon_8) \).

\[
\begin{align*}
\varepsilon_8 : \varepsilon_1 + \varepsilon_2 & \leftrightarrow \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \ldots - \varepsilon_7 + \varepsilon_8) \\
\end{align*}
\]

So \( \pm (\varepsilon_1 + \varepsilon_8) \in S \). So \( S \) is

\[
\begin{align*}
\pm \varepsilon_i \pm \varepsilon_j \text{ for } 2 \leq i \neq j \leq 7 \\
\pm (\varepsilon_1 + \varepsilon_8) \\
\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \ldots \pm \varepsilon_7 + \varepsilon_8) \text{ where } \Pi(\pm) = 1 \\
\frac{1}{2}(-\varepsilon_1 \pm \varepsilon_2 \pm \ldots \pm \varepsilon_7 - \varepsilon_8) \text{ where } \Pi(\pm) = 1
\end{align*}
\]

\[
|\Phi(E_7)| = 2^3 \left( \begin{array}{c}
6 \\
2
\end{array} \right) + 2 + 2^5 + 2^5
\]

Type \( E_6 \).

We proceed as before. \( h_{a_1}, \ldots, h_{a_8} \) form a fundamental system of vectors of type \( E_6 \). Let \( V_0 \) be the subspace of \( V \) for \( E_8 \) that is orthogonal to \( \varepsilon_1 - \varepsilon_8 \) and \( \varepsilon_2 - \varepsilon_8 \). \( \dim(V_0) = 6 \) and \( h_{a_3}, \ldots, h_{a_8} \) form a basis of this space.

\[
\{h_\alpha \mid \alpha \in \Phi(E_6)\} \subseteq \{h_\alpha \mid \alpha \in \Phi(E_8)\} \cap V_0
\]

The \( h_\alpha \) in \( V_0 \) are

\[
\begin{align*}
\pm \varepsilon_i \pm \varepsilon_j \text{ for } 3 \leq i \neq j \leq 7 \\
\frac{1}{2}(\varepsilon_1 + \varepsilon_2 \pm \varepsilon_3 \pm \ldots \pm \varepsilon_7 + \varepsilon_8) \text{ where } \Pi(\pm) = 1 \\
\frac{1}{2}(-\varepsilon_1 - \varepsilon_2 \pm \varepsilon_3 \pm \ldots \pm \varepsilon_7 - \varepsilon_8) \text{ where } \Pi(\pm) = 1
\end{align*}
\]

All of these are obtainable from \( h_{a_3}, \ldots, h_{a_8} \) by \( s_{a_1}, \ldots, s_{a_8} \).

\[
|\Phi(E_6)| = 2^2 \left( \begin{array}{c}
5 \\
2
\end{array} \right) + 2^4 + 2^4 = 72
\]

**Theorem 11.1.** The number of roots in each of the indecomposable root systems is
<table>
<thead>
<tr>
<th></th>
<th>(A_l)</th>
<th>(B_l)</th>
<th>(C_l)</th>
<th>(D_l)</th>
<th>(E_6)</th>
<th>(E_7)</th>
<th>(E_8)</th>
<th>(F_4)</th>
<th>(G_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(l(l+2))</td>
<td>(l(2l+1))</td>
<td>(l(2l+1))</td>
<td>(l(2l-1))</td>
<td>78</td>
<td>133</td>
<td>248</td>
<td>52</td>
<td>14</td>
</tr>
</tbody>
</table>
12. THE SEMISIMPLE LIE ALGEBRAS

**Theorem 12.1.** (a) If a semisimple Lie algebra $L$ has connected Dynkin diagram $\Delta$ then $L$ is simple.
(b) If $L$ is a semisimple Lie algebra whose Dynkin diagram $\Delta$ has connected components $\Delta_1, \ldots, \Delta_r$ then $L = L_1 \oplus \cdots \oplus L_r$ where $L_i$ is a simple Lie algebra with Dynkin diagram $\Delta_i$.

**Proof.** (a) Let $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$ be a Cartan decomposition with connected Dynkin diagram $\Delta$. Let $0 \neq I \subset L$. We first show that $I \cap H \neq 0$. Suppose not, i.e. that $I \cap H = 0$. Let $0 \neq x \in I$ with $x = h + \sum \mu_a e_\alpha$ and the number of non-zero $\mu_a$ as small as possible. Let $\mu_\beta \neq 0$.

$$[x h_\beta] = \sum \mu_a [e_\alpha h_\beta] = \sum \mu_a \alpha(h_\beta) e_\alpha$$

By 8.7 we can choose $e_-\beta$ with $[e_-\beta e_\beta] = h_\beta$.

$$[[x h_\beta] e_-\beta] = -\mu_\beta \beta(h_\beta) h_\beta + \sum_{\alpha \in \Phi, \alpha-\beta \in \Phi} \mu_a \alpha(h_\beta) N_{\alpha-\beta} e_\alpha$$

$[[x h_\beta] e_-\beta] \in I$ is non-zero since

$$-\mu_\beta \beta(h_\beta) h_\beta = -\mu_\beta \beta(h_\beta, h_\beta) h_\beta \neq 0$$

The number of non-zero $\mu_a$ with $\alpha - \beta \in \Phi$ is less than before, a contradiction. Hence, $I \cap H \neq 0$.

We next show that $I \supseteq H$. Suppose not. Then $0 \subset I \cap H \subset H$. $I \cap H$ is not orthogonal to all $h_{\alpha_i}, \alpha_i \in \Pi$. For suppose $I \cap H$ is not orthogonal to $h_{\alpha_i}$. Let $x \in I \cap H$ be such that $\langle x, h_{\alpha_i} \rangle \neq 0$. Then

$$[e_\alpha x] = \alpha(x) e_\alpha = \langle h_{\alpha_i}, x \rangle e_\alpha \in I$$

So $e_\alpha \in I$. So $[e_{-\alpha}, e_\alpha] = h_{\alpha_i} \in I$. So for each $\alpha_i \in \Pi$ either $\langle I \cap H, h_{\alpha_i} \rangle = 0$ or $h_{\alpha_i} \in I$.

Both classes are non-empty. Choose $h_{\alpha_i} \notin I$; then $\langle I \cap H, h_{\alpha_i} \rangle = 0$. This means $\Delta$ is disconnected, a contradiction. Hence, $H \subseteq I$.

Now let $\alpha \in \Phi$. 

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\[ [e_a h_a] = \alpha(h_a) e_a = \langle h_a, h_a \rangle e_a \]

So \( e_a \in I \). Hence \( I \) contains \( H \) and all \( e_a \). So \( I = L \); \( L \) is simple.

(b) Suppose \( \Delta \) is the disjoint union of connected components \( \Delta_1, \ldots, \Delta_r \). Then \( \Pi \) is the union of orthogonal components \( \Pi_1, \ldots, \Pi_r \). Let \( H_i \) be the subspace spanned by \( \{ h_\alpha \mid \alpha \in \Pi_i \} \). Then \( H = H_1 \oplus \ldots \oplus H_r \) and the \( H_i \) are mutually orthogonal. Now consider \( s_\alpha \) for some \( \alpha \in \Pi_i \). Then \( \alpha \) transforms \( H_i \) into itself and fixes each vector in \( H_j \) for \( j \neq i \); \( s_\alpha(H_j) = H_j \). Since the \( s_\alpha \) for \( \alpha \in \Pi \) generate \( W \), \( w(H_j) = H_j \) for each \( w \in W \).

For each \( \alpha \in \Phi \), \( h_\alpha = w(h_\alpha) \) for some \( w \in W \) and some \( i \). So \( h_\alpha \in H_i \) for some \( i \). Let \( \Phi_i = \{ \alpha \in \Phi \mid h_\alpha \in H_i \} \). Then \( \Phi = \Phi_1 \cup \ldots \cup \Phi_r \). Let \( L_i \) be the subspace of \( L \) spanned by \( H_i \) and the \( L_\alpha \) with \( \alpha \in \Phi_i \). We see that \( L = L_1 \oplus \ldots \oplus L_r \) as a direct sum of vector spaces.

To see that \( L_i \) is a subalgebra of \( L \) it is sufficient to show that \( \alpha, \beta \in \Phi_i \implies [e_\alpha e_\beta] \in L_i \).

If \( \alpha + \beta \notin \Phi \) then \( [e_\alpha e_\beta] = 0 \). If \( \beta = -\alpha \) then \( [e_\alpha e_\alpha] = -h_\alpha \in H_i \). If \( \alpha + \beta \in \Phi \) then \( \alpha + \beta \in \Phi_i \) and \( h_\alpha + h_\beta = h_\alpha + h_\beta \in H_i \). So \( L_i \) is a subalgebra.

We next check that \( i \neq j \implies [L_i L_j] = 0 \). Let \( \alpha \in \Phi_i \), \( \beta \in \Phi_j \).

\[ [h_\alpha h_\beta] = 0 \]
\[ [h_\alpha e_\beta] = 0 \text{ since } \langle h_\alpha, h_\beta \rangle = 0 \]
\[ [e_\alpha h_\beta] = 0 \text{ since } \langle h_\alpha, h_\beta \rangle = 0 \]
\[ [e_\alpha e_\beta] = 0 \text{ since } \alpha + \beta \notin \Phi \]

\( h_\alpha + h_\beta \) does not lie in any \( H_k \), \( h_\alpha \in H_i \), \( h_\beta \in H_j \). So \( [L_i L_j] = 0 \) for \( i \neq j \):

\[ [x_1 + \ldots + x_r, y_1 + \ldots + y_r] = [x_1 y_1] + \ldots + [x_r y_r] \]

So \( L = L_1 \oplus \ldots \oplus L_r \) as a direct sum of Lie algebras. We now see that each \( L_i \) is semisimple. Let \( I \triangleleft L_i \) be soluble. \( [IL_i] = 0 \) if \( i \neq j \), so \( I \triangleleft L \). \( I \) is a soluble ideal of \( L \), but \( L \) is semisimple, so \( I = 0 \), so \( L_i \) is semisimple.
We now show that $H_i$ is a Cartan subalgebra of $L_i$. $H$ is a Cartan subalgebra of $L$, so there is a regular element $x \in L$ such that $H$ is the 0-(generalized) eigenspace of $\text{ad} \, x$. Since $x \in H$ and $H = H_i \oplus \cdots \oplus H_i$, we can write $x = x_i + \cdots + x_i$ with $x_i \in H_i$.

$$\text{ad} \, x_i : L_j \to \begin{cases} L_j & i = j \\ 0 & i \neq j \end{cases}$$

So the 0-eigenspace of $\text{ad} \, x$ on $L$ is the direct sum of the 0-eigenspaces of the $\text{ad} \, x_i$ on the $L_i$. $x$ is regular in $L$ if and only if each $x_i$ is regular in $L_i$. So each $x_i$ is regular in $L_i$ and the 0-eigenspace of $\text{ad} \, x_i$ in $L_i$ is $H_i$. So $H_i$ is a Cartan subalgebra of $L_i$.

$$L_i = H_i \oplus \left( \oplus_{\alpha \in \Phi_i} L_{\alpha} \right)$$

is a Cartan decomposition of $L_i$. So $\Phi_i$ is the root system of $L_i$; $\Pi_i$ is a fundamental root system of $L_i$; the Dynkin diagram of $L_i$ is $\Delta_i$. But $\Delta_i$ is connected, so $L_i$ is simple by (a).

We next consider simple Lie algebras with a given indecomposable Cartan matrix $A$.

Existence Problem: Is there a simple Lie algebra with given Cartan matrix $A$?

Isomorphism Problem: Are any two such Lie algebras isomorphic?

Let $L$ be a simple Lie algebra and $H$ a Cartan subalgebra of $L$:

$$L = H \oplus \left( \oplus_{\alpha \in \Phi} L_{\alpha} \right)$$

$$\Phi = \Phi^+ \cup \Phi^-$$

For each $\alpha \in \Phi^+$ choose $0 \neq e_{\alpha} \in L_{\alpha}$; $L_{\alpha} = \mathbb{C} e_{\alpha}$. Choose $e_{-\alpha} \in L_{-\alpha}$ such that $[e_{-\alpha} e_\alpha] = h_\alpha$. If $\Pi = \{ \alpha_1, \ldots, \alpha_l \}$ then the $h_{\alpha_i}$ and $e_{\alpha}$ form a basis of $L$. $[L_{\alpha} L_{\beta}] \subseteq L_{\alpha + \beta}$ so $[e_{\alpha} e_{\beta}] = N_{\alpha, \beta} e_{\alpha + \beta}$ if $\alpha + \beta \in \Phi$, $\alpha \neq -\beta$.

$$[h_\alpha, h_\alpha] = 0$$

$$[e_\alpha h_\alpha] = \langle h_\alpha, h_\alpha \rangle e_\alpha$$

$$[e_{-\alpha} e_\alpha] = h_\alpha$$

$$[e_\alpha e_{\beta}] = \begin{cases} N_{\alpha, \beta} e_{\alpha + \beta} & \alpha + \beta \in \Phi \\ 0 & \text{otherwise} \end{cases}$$

The $N_{\alpha, \beta}$ are called the structure constants.
Proposition 12.2. The structure constants $N_{\alpha,\beta}$ satisfy

(i) $N_{\alpha,\beta} = -N_{\beta,\alpha}$;

(ii) if $\alpha, \beta, \gamma \in \Phi$ have $\alpha + \beta + \gamma = 0$ then $N_{\alpha,\beta} = N_{\beta,\gamma} = N_{\gamma,\alpha}$;

(iii) if $\alpha, \beta, \gamma, \delta \in \Phi$ have $\alpha + \beta + \gamma + \delta = 0$ and no pair have sum zero then

$$N_{\alpha,\beta}N_{\gamma,\delta} + N_{\beta,\gamma}N_{\alpha,\delta} + N_{\gamma,\alpha}N_{\beta,\delta} = 0$$

(if $\xi, \eta \in \Phi$, $\eta \neq -\xi$, $\eta + \xi \in \Phi$ take $N_{\xi,\eta} = 0$);

(iv) if $\alpha, \beta \in \Phi$ have $\alpha + \beta \in \Phi$ then

$$N_{\alpha,\beta}N_{-\alpha,-\beta} = -\frac{(p+1)q}{2} \langle h_\alpha, h_\beta \rangle$$

where the $\alpha$-chain of roots through $\beta$ is

$$-p\alpha + \beta, \ldots, \beta, \ldots, q\alpha + \beta.$$

In particular, $N_{\alpha,\beta} \neq 0$, so $[L_\alpha L_\beta] = L_{\alpha + \beta}$.

Proof. (i) $[e_\alpha e_\beta] = -[e_\beta e_\alpha]$ so $N_{\alpha,\beta} = -N_{\beta,\alpha}$.

(ii) Suppose $\alpha + \beta + \gamma = 0$.

$$[[e_\alpha e_\beta]e_\gamma] + [[e_\beta e_\gamma]e_\alpha] + [[e_\gamma e_\alpha]e_\beta] = 0$$

$$\Rightarrow N_{\alpha,\beta}[e_{\alpha + \beta}e_\gamma] + N_{\beta,\gamma}[e_{\beta + \gamma}e_\alpha] + N_{\gamma,\alpha}[e_{\gamma + \alpha}e_\beta] = 0$$

$$\Rightarrow N_{\alpha,\beta}h_\gamma + N_{\beta,\gamma}h_\alpha + N_{\gamma,\alpha}h_\beta = 0$$

$$\Rightarrow (-N_{\alpha,\beta} + N_{\beta,\gamma})h_\alpha + (-N_{\alpha,\beta} + N_{\gamma,\alpha})h_\beta = 0$$

Since $h_\alpha$ and $h_\beta$ are linearly independent, $N_{\alpha,\beta} = N_{\beta,\gamma} = N_{\gamma,\alpha}$.

(iii) Take $\alpha, \beta, \gamma, \delta \in \Phi$ with zero sum and no opposite pairs.

$$[[e_\alpha e_\beta]e_\gamma] + [[e_\beta e_\gamma]e_\alpha] + [[e_\gamma e_\alpha]e_\beta] = 0$$

$$\Rightarrow N_{\alpha,\beta}[e_{\alpha + \beta}e_\gamma] + N_{\beta,\gamma}[e_{\beta + \gamma}e_\alpha] + N_{\gamma,\alpha}[e_{\gamma + \alpha}e_\beta] = 0$$

$$\Rightarrow \left(N_{\alpha,\beta}N_{\alpha + \beta,\gamma} + N_{\beta,\gamma}N_{\beta + \gamma,\alpha} + N_{\gamma,\alpha}N_{\gamma + \alpha,\beta}\right)e_{\alpha + \beta + \gamma} = 0$$

$$\Rightarrow N_{\alpha,\beta}N_{\gamma,\delta} + N_{\beta,\gamma}N_{\alpha,\delta} + N_{\gamma,\alpha}N_{\beta,\delta} = 0$$

(iv) Let $\alpha, \beta \in \Phi$ with $\alpha + \beta \in \Phi$. 

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\[
\begin{align*}
[[e_\alpha e_{-\alpha}] e_\beta] + [[e_{-\alpha} e_\beta] e_\alpha] + [[e_\beta e_\alpha] e_{-\alpha}] &= 0 \\
\Rightarrow -[h_\alpha e_\beta] + N_{-\alpha,\beta}[e_{-\alpha} e_\beta e_\alpha] + N_{\beta,\alpha}[e_\alpha e_\beta e_{-\alpha}] &= 0 \\
\Rightarrow (\beta(h_\alpha) + N_{-\alpha,\beta}N_{-\alpha,\beta,\alpha} + N_{\beta,\alpha}N_{\alpha,\beta,-\alpha}) e_\beta &= 0 \\
\Rightarrow N_{-\alpha,\beta}N_{-\alpha,\beta,\alpha} + N_{\beta,\alpha}N_{\alpha,\beta,-\alpha} &= -\langle h_\alpha, h_\beta \rangle \\
\Rightarrow N_{-\alpha,\beta}N_{-\alpha,-\beta} - N_{-\alpha,\beta}N_{-\alpha,-\beta,\alpha} &= \langle h_\alpha, h_\beta \rangle
\end{align*}
\]
Take \( M_{\alpha,\beta} = N_{\alpha,\beta}N_{-\alpha,-\beta} \), so

\[
M_{\alpha,\beta} - M_{-\alpha,\alpha+\beta} = \langle h_\alpha, h_\beta \rangle
\]

Let the \( \alpha \)-chain of roots through \( \beta \) be

\[-p\alpha + \beta, \ldots, \beta, \ldots, q\alpha + \beta\]

So

\[
\begin{align*}
M_{\alpha,\beta} - M_{-\alpha,\alpha+\beta} &= \langle h_\alpha, h_\beta \rangle \\
M_{-\alpha,\alpha+\beta} - M_{-\alpha,-2\alpha+\beta} &= \langle h_\alpha, h_-\alpha+\beta \rangle \\
M_{-\alpha,-2\alpha+\beta} - M_{-\alpha,-3\alpha+\beta} &= \langle h_\alpha, h_-2\alpha+\beta \rangle \\
\vdots \\
M_{-\alpha,-p\alpha+\beta} &= \langle h_\alpha, h_-p\alpha+\beta \rangle \\
\end{align*}
\]

So

\[
M_{\alpha,\beta} = (p+1)\langle h_\alpha, h_\beta \rangle - \langle h_\alpha, h_-\alpha \rangle (1 + 2 + \ldots + p) = (p+1)\langle h_\alpha, h_\beta \rangle - \frac{p(p+1)}{2} \langle h_\alpha, h_-\alpha \rangle
\]

By, by 8.11, \( 2\langle h_\alpha, h_\beta \rangle / \langle h_\alpha, h_-\alpha \rangle = p - q \), so

\[
M_{\alpha,\beta} = \langle h_\alpha, h_\beta \rangle \left( \frac{(p+1)(p-q)}{2} - \frac{p(p+1)}{2} \right) = -\frac{(p+1)}{2} \langle h_\alpha, h_\beta \rangle
\]

So \( N_{\alpha,\beta} \neq 0 \); \( [L_\alpha L_\beta] = L_{\alpha+\beta} \).
This result has certain consequences. Let \( \alpha, \beta \in \Phi \), \( \alpha + \beta \in \Phi \), \( [e_{\alpha}e_{\beta}] = N_{\alpha,\beta}e_{\alpha + \beta} \). Let \( \gamma = -\alpha - \beta \). Then \( \alpha + \beta + \gamma = 0 \). We have the following ordered pairs of roots whose sum is a root:

\[
\begin{align*}
(\alpha, \beta) & \quad (\beta, \gamma) & \quad (\gamma, \alpha) & \quad (\beta, \alpha) & \quad (\gamma, \beta) & \quad (\alpha, \gamma) \\
(-\alpha, -\beta) & \quad (-\beta, -\gamma) & \quad (-\gamma, -\alpha) & \quad (-\beta, -\alpha) & \quad (-\gamma, -\beta) & \quad (-\alpha, -\gamma)
\end{align*}
\]

We have a total order \( \alpha \prec \beta \). An ordered pair \((\alpha, \beta)\) such that \(0 \prec \alpha \prec \beta\) is called a special pair.

Either one or two of \(\alpha, \beta, \gamma\) are positive; if one is positive two of \(-\alpha, -\beta, -\gamma\) are positive. Of the twelve pairs above just one is special.

\(N_{\alpha,\beta}\), for any ordered pair \((\alpha, \beta)\), can be expressed in terms of \(N_{\xi,\eta}\) for \((\xi, \eta)\) a special pair by using 12.2(i), (ii), (iv). So consider \(N_{\alpha,\beta}\) when \((\alpha, \beta)\) is special; \(\alpha + \beta \in \Phi^+ \setminus \Pi\). This root may be expressible as \(\alpha + \beta = \alpha' + \beta'\) where \((\alpha', \beta')\) is special and distinct from \((\alpha, \beta)\).

A special pair \((\alpha, \beta)\) is called extra special if for any special pair \((\alpha', \beta')\) with \(\alpha + \beta = \alpha' + \beta'\) we have \(\alpha \preceq \alpha'\).

The number of extra special pairs is \(|\Phi^+ \setminus \Pi|\).

Now let \((\alpha', \beta')\) be special but not extra special. Then \(\alpha + \beta = \alpha' + \beta'\) where \((\alpha, \beta)\) is extra special – such an extra special pair exists because the set of special and extra special pairs is finite.

\[
\begin{align*}
\alpha' + \beta' + (-\alpha) + (-\beta) &= 0 \\
N_{\alpha',\beta'}N_{-\alpha,-\beta} + N_{\beta,-\alpha}N_{\alpha,-\beta} + N_{-\alpha,\alpha}N_{\beta,-\beta} &= 0 \\
0 &< \alpha < \alpha' < \beta' < \beta \\
N_{\alpha',\beta'}N_{-\alpha,-\beta} + N_{\beta,-\alpha}N_{\alpha,-(\beta'-\alpha)} + N_{\beta,-\beta'}N_{-(\alpha'-\alpha),-\alpha} &= 0
\end{align*}
\]

We show that \(N_{\alpha',\beta'}\) is determined by \(N_{\xi,\eta}\)'s for extra special pairs \((\xi, \eta)\). We use induction on \(\alpha' + \beta'\):

\(N_{\alpha',\beta'}\) is determined by \(N_{\alpha,\beta}, N_{\beta,-\alpha,\alpha'}, N_{\alpha,\beta'-\alpha}, N_{\beta,-\beta',\alpha'}, N_{\alpha'-\alpha,\alpha} \cdot (\alpha, \beta)\) is extra special.

Either \((\beta - \alpha', \alpha')\) or \((\alpha', \beta - \alpha')\) is special.
\[(\beta - \alpha') + \alpha' = \beta < \alpha + \beta = \alpha' + \beta'\]

So \(N_{\beta-\alpha',\alpha'}\) can be expressed in terms of \(N_{\xi,\eta}\) for extra special pairs \((\xi, \eta)\).

Either \((\alpha, \beta' - \alpha)\) or \((\beta' - \alpha', \alpha)\) is special:
\[\alpha + (\beta' - \alpha) = \beta' < \alpha' + \beta'\]

Either \((\beta - \beta', \beta')\) or \((\beta', \beta - \beta')\) is special:
\[(\beta - \beta') + \beta' = \beta < \alpha' + \beta'\]

Either \((\alpha' - \alpha, \alpha)\) or \((\alpha, \alpha' - \alpha)\) is special:
\[(\alpha' - \alpha) + \alpha = \alpha' < \alpha' + \beta'\]

Hence, \(N_{\alpha', \beta'}\) can be expressed in terms of \(N_{\xi, \eta}\)'s for extra special pairs \((\xi, \eta)\). So relations 12.2(i)-(iv) expresses all \(N_{\alpha', \beta'}\)'s in terms of \(N_{\xi, \eta}\)'s for extra special pairs \((\xi, \eta)\).

**Theorem 12.3.** There is a unique simple Lie algebra, up to isomorphism, with a given indecomposable Cartan matrix.

\[L = H \oplus (\bigoplus_{\alpha \in \Phi} L_\alpha)\]
\[\dim(L) = l + |\Phi|\]

Thus, the simple Lie algebras and their dimensions are given by

\[
\begin{align*}
\dim(A_l) &= l(l + 2) \\
\dim(B_l) &= l(l + 1) \\
\dim(C_l) &= l(2l + 1) \\
\dim(D_l) &= l(2l - 1) \\
\dim(E_6) &= 78 \\
\dim(E_7) &= 133 \\
\dim(E_8) &= 248 \\
\dim(F_4) &= 52 \\
\dim(G_2) &= 14
\end{align*}
\]

**Note.** The following are isomorphic:

\[
\begin{align*}
B_2 &\cong C_2 \\
A_1 &\cong D_3 \\
D_2 &\cong A_1 \oplus A_1
\end{align*}
\]

**Proof.** Uniqueness. Let \(L, L'\) be simple Lie algebras with indecomposable Cartan matrix \(A = (a_{ij})\). \(L\) has a Cartan decomposition \(L = H \oplus (\bigoplus_{\alpha \in \Phi} L_\alpha)\). If \(\Pi = \{\alpha_1, \ldots, \alpha_l\}\) then \(H\)
has basis \{h_{i_1}, \ldots, h_{i_n}\}; L \text{ has basis } \{h_{i_1}, \ldots, h_{i_n}\} \cup \{e_\alpha \mid \alpha \in \Phi\}. \text{ Multiplication of basis elements:}

\[
\begin{align*}
[h_{i_j}, h_{i_k}] &= 0 \\
[e_\alpha h_{i_j}] &= \langle h_{i_j}, h_{i_j} \rangle e_\alpha \\
[e_\alpha e_\alpha] &= h_\alpha \\
[e_\alpha e_\beta] &= \begin{cases} 
N_{\alpha, \beta} e_{\alpha + \beta} & \alpha + \beta \in \Phi \\
0 & 0 \neq \alpha + \beta \notin \Phi
\end{cases}
\end{align*}
\]

All scalar products \langle h_\alpha, h_\beta \rangle are determined by \( A \). Also, all of the \( h_\alpha \) (as linear combinations of the \( h_{i_j} \)) are determined by \( A \).

\[
s_{i_j}(h_{i_k}) = h_{i_k} - a_\alpha h_{i_j}
\]

So the \( s_{i_j} \) are determined by \( A \). The Weyl group \( W \) is generated by \( s_{i_1}, \ldots, s_{i_n} \). So \( W \) is determined by \( A \). \( h_\alpha = w(h_{i_k}) \) for some \( i \) and some \( w \in W \). So the \( h_\alpha \) are determined by \( A \).

\[
s_{i_j}(h_\beta) = h_\beta - 2 \frac{\langle h_{i_j}, h_\beta \rangle}{\langle h_{i_j}, h_{i_j} \rangle} h_{i_j}
\]

So \( 2\langle h_\alpha, h_\beta \rangle / \langle h_{i_j}, h_{i_j} \rangle \) is determined by \( A \). But

\[
\frac{1}{\langle h_{i_j}, h_{i_j} \rangle} = \sum_{\beta \in \Phi} \left( \frac{\langle h_{i_j}, h_\beta \rangle}{\langle h_{i_j}, h_{i_j} \rangle} \right)^2
\]

by 8.13. So \( \langle h_{i_j}, h_{i_j} \rangle \) is determined by \( A \). So \( \langle h_\alpha, h_\beta \rangle \) is determined by \( A \).

Suppose a basis \( \{h'_{i_1}, \ldots, h'_{i_n}\} \cup \{e'_\alpha \mid \alpha \in \Phi\} \) of \( L' \) is given. We describe how to choose a basis of \( L \). The \( h_{i_j} \) are uniquely determined. Choose \( e_\alpha \neq 0 \) in \( L_\alpha \) for each \( \alpha \in \Pi \). For each \( \alpha \in \Phi^+ \setminus \Pi \) there is a unique extra special pair \( (\beta, \gamma) \) such that \( \alpha = \beta + \gamma \), \( \beta, \gamma < \alpha \).

Assume by induction that \( e_\beta, e_\gamma \) are already chosen. Choose \( e_\alpha \) by \( e_\alpha = N_{\beta, \gamma} [e_\beta e_\gamma] \) where \( N_{\beta, \gamma} = N'_{\beta, \gamma} \), the structure constant for \( L' \). Having thus chosen \( e_\alpha \) for \( \alpha \in \Phi^+ \), we choose \( e_{-\alpha} \) by \( [e_{-\alpha} e_\alpha] = h_\alpha \).
The $N_{\alpha,\beta}$ for arbitrary $\alpha, \beta$ are determined by the $N_{\xi,\eta}$, where $(\xi, \eta)$ is extra special, by 12.2. Since $N_{\xi,\eta} = N_{\xi,\eta}'$ for all extra special $(\xi, \eta)$ it follows that $N_{\alpha,\beta} = N_{\alpha,\beta}'$ for all $\alpha, \beta \in \Phi$ with $\alpha + \beta \in \Phi$.

This shows that $L$ and $L'$ are isomorphic.

Existence. (Sketch proof.) Begin with Cartan matrix $A = (a_{ij})$. Let $H$ be an $l$-dimensional vector space over $\mathbb{C}$ with basis $h_{\alpha_1}, \ldots, h_{\alpha_l}$. We define $s_{\alpha_i} : H \to H$ by $s_{\alpha_i}(h_{\alpha_i}) = h_{\alpha_i} - a_{\alpha_i}h_{\alpha_i}$, a self-inverse map. Let $W$ be the group of all non-singular linear maps $H \to H$ generated by $s_{\alpha_1}, \ldots, s_{\alpha_l}$. $W$ is finite. Correspondingly,

$$\{h_{\alpha_i} = w(h_{\alpha_i}) \mid w \in W, 1 \leq i \leq l\}$$

is also finite. (The $h_{\alpha_i}$ were determined in Chapter 11.) We now define a bilinear map

$$H \times H \to \mathbb{C}$$

$$(x, y) \mapsto \langle x, y \rangle$$

This form is uniquely determined by $A$. Define $\alpha \in H^*$ by $\alpha(x) = \langle h_{\alpha}, x \rangle$; let $\Phi$ be the set of all such $\alpha$.

Let $L$ be a vector space over $\mathbb{C}$ with $\dim(L) = \dim(H) + |\Phi|$ with basis

$$\{h_{\alpha_1}, \ldots, h_{\alpha_l}, \} \cup \{e_\alpha \mid \alpha \in \Phi\}$$

Define a bilinear map

$$L \times L \to L$$

$$(x, y) \mapsto [xy]$$

We define $[ \cdot ]$ on the basis elements by

$$[h_{\alpha_i}, h_{\alpha_j}] = 0$$

$$[e_\alpha h_{\alpha_i}] = -[h_{\alpha_i} e_\alpha] = \langle h_{\alpha_i}, h_{\alpha_i} \rangle e_\alpha$$

$$[e_{-\alpha} e_\alpha] = h_{\alpha}$$

$$[e_\alpha, e_\beta] = \begin{cases} N_{\alpha,\beta} e_{\alpha + \beta} & \alpha + \beta \in \Phi \\ 0 & 0 \neq \alpha + \beta \in \Phi \end{cases}$$
The \( N_{\alpha,\beta} \) can be chosen arbitrarily if \( (\alpha, \beta) \) is extra special, e.g. \( N_{\alpha,\beta} = 1 \). \( N_{\alpha,\beta} \) is determined for all other pairs by 12.2. So multiplication of basis elements is determined by \( A \). We make various checks:

Check \([xy] = 0\) for all \( x \in L \). (Easy.) Check \([[[xy]z] + [[[yz]x] + [[[zx]y] + 0.\) (Most are easy, but \( x = e_\alpha, \; y = e_\beta, \; z = e_\gamma \) is difficult.) Then \( L \) is a Lie algebra, \( L = H \oplus (\bigoplus_{\alpha \in \Phi} L_\alpha) \), \( L_\alpha = \mathbb{C}e_\alpha \). Check that \( H \) is a Cartan subalgebra of \( L \). (Difficult.) Then \( L = H \oplus (\bigoplus_{\alpha \in \Phi} L_\alpha) \) is a Cartan decomposition of \( L \) with respect to \( H \). (Easy.) Then \( \Phi \) is the set of roots of \( L \) with respect to \( H \). \( \Pi = \{\alpha_1, \ldots, \alpha_i\} \) is a fundamental system of roots inside \( \Phi \). We have

\[
2 \frac{\langle h_\alpha, h_\beta \rangle}{\langle h_\alpha, h_\alpha \rangle} = a_{ij}
\]

so \( A \) is the Cartan matrix. Finally, the argument of 12.1(a) proves that \( L \) is simple.

\[\hookrightarrow\]

**Review.**

If we choose a different Cartan subalgebra and a different fundamental system do we get a different \( A \)?

**Theorem 12.4.** (i) Let \( L \) be a Lie algebra and \( H_1, H_2 \) Cartan subalgebras. Then there exists an automorphism \( \theta : L \rightarrow L \) such that \( \theta(H_1) = H_2 \).
(ii) A subalgebra \( H \) of \( L \) is a Cartan subalgebra if and only if \( H \) is nilpotent and \( H = \mathcal{N}(H) \).

**Theorem 12.5.** Let \( \Phi \) be the root system of a semisimple Lie algebra and let \( \Pi_1, \Pi_2 \) be two fundamental systems in \( \Phi \). Then there is a \( w \in W \) such that \( w(\Pi_1) = \Pi_2 \).

12.4 and 12.5 imply that the Cartan matrix is uniquely determined by \( L \). So the simple Lie algebras on our list are pairwise non-isomorphic.

We have four infinite families of simple Lie algebras and five exceptional ones:
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<table>
<thead>
<tr>
<th>Classical</th>
<th>Exceptional</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_l$</td>
<td>$B_l$</td>
</tr>
<tr>
<td>$l(l+2)$</td>
<td>$l(2l+1)$</td>
</tr>
<tr>
<td>$l(l+1)$</td>
<td>$l(2l+1)$</td>
</tr>
<tr>
<td>$l(l+1)$</td>
<td>$l(2l-1)$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$E_7$</td>
</tr>
<tr>
<td>78</td>
<td>133</td>
</tr>
<tr>
<td>$E_8$</td>
<td>$F_4$</td>
</tr>
<tr>
<td>248</td>
<td>52</td>
</tr>
<tr>
<td>$G_2$</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td></td>
</tr>
</tbody>
</table>

Type $A_l$. We can write $\dim(A_l) = l(l+2) = (l+1)^2 - 1$. The set $\mathfrak{sl}_{l+1}(\mathbb{C})$ of all $(l+1) \times (l+1)$ matrices of trace zero forms a Lie algebra of type $A_l$. The diagonal subalgebra is a Cartan subalgebra.

Type $B_l$. The set $\mathfrak{so}_{2l+1}(\mathbb{C})$ of all $(2l+1) \times (2l+1)$ matrices $X$ satisfying

$$X^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_l & 0 \\ 0 & 0 & I_l \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & 0 & I_l \end{pmatrix} X$$

forms a simple Lie algebra of type $B_l$. The diagonal subalgebra is a Cartan subalgebra. $\mathfrak{so}_{2l+1}(\mathbb{C})$ is isomorphic to the Lie algebra of all $(2l+1) \times (2l+1)$ skew-symmetric matrices. Elements of $\mathfrak{so}_{2l+1}(\mathbb{C})$ have the block form

$$X = \begin{pmatrix} 0 & X_{01} & X_{02} \\ -X_{02}^T & X_{11} & X_{12} \\ -X_{01}^T & X_{22} & -X_{11}^T \end{pmatrix}$$

where $X_{11}$ is an arbitrary $l \times l$ matrix, $X_{12}$ and $X_{21}$ are $l \times l$ symmetric matrices and $X_{01}$ and $X_{02}$ are arbitrary $1 \times l$ matrices (row vectors).

Type $C_l$. The set $\mathfrak{sp}_{2l}(\mathbb{C})$ of all $2l \times 2l$ matrices $X$ satisfying

$$X^T \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix} X$$

forms a simple Lie algebra of type $C_l$. The diagonal subalgebra is a Cartan subalgebra. Elements of $\mathfrak{sp}_{2l}(\mathbb{C})$ have the block form

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & -X_{11}^T \end{pmatrix}$$

where $X_{11}$ is an arbitrary $l \times l$ matrix and $X_{12}$ and $X_{21}$ are $l \times l$ symmetric matrices.
**Type** $D_l$. The set $\mathfrak{so}_{2l}(\mathbb{C})$ of all $2l \times 2l$ matrices $X$ such that

$$X^T \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix} = -\begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix} X$$

forms a simple Lie algebra of type $D_l$. The diagonal subalgebra is a Cartan subalgebra. $\mathfrak{so}_{2l}(\mathbb{C})$ is isomorphic to the Lie algebra of all $2l \times 2l$ skew-symmetric matrices. Elements of $\mathfrak{so}_{2l}(\mathbb{C})$ have the block form

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & -X_{11}^T \end{pmatrix}$$

where $X_{11}$ is an arbitrary $l \times l$ matrix and $X_{12}$ and $X_{21}$ are $l \times l$ skew-symmetric matrices.

$\mathfrak{sl}_m(\mathbb{C})$ is the Lie algebra of $\text{SL}_m(\mathbb{C}) = \{X \in \text{GL}_m(\mathbb{C}) \mid \det(X) = 1\}$; $\mathfrak{so}_m(\mathbb{C})$ is the Lie algebra of $\text{SO}_m(\mathbb{C}) = \{X \in \text{GL}_m(\mathbb{C}) \mid X^T X = I_m \text{ and } \det(X) = 1\}$.

**Type** $G_2$. $\dim(G_2) = 14$. Consider the algebra of octonions (a.k.a. Cayley numbers), $\mathcal{O}$. $\dim(\mathcal{O}) = 8$. $\mathcal{O}$ has basis $1, e_1, e_2, \ldots, e_7$:

1 is the multiplicative identity;

$$e_i^2 = -1 \quad \text{for } 1 \leq i \leq 7;$$

$$e_i e_j = \pm e_k \quad \text{for } 1 \leq i \neq j \leq 7.$$

The projective plane over the 2-field.

$$e_i e_j = e_k \text{ if } i \to j; \quad e_i e_j = -e_k \text{ if } i \leftarrow j.$$
\( \mathcal{O} \) is a non-associative algebra. The set of all derivations of \( \mathcal{O} \), i.e. linear maps \( D: \mathcal{O} \to \mathcal{O} \) such that \( D(xy) = D(x)y + xD(y) \), forms a Lie algebra of type \( G_2 \).

**Type \( F_4 \).** Define the octonion conjugate:

\[
x = a_0 1 + \sum_{i=1}^{7} a_i e_i \\
\bar{x} = a_0 1 - \sum_{i=1}^{7} a_i e_i \\
x = \bar{x} \iff x = a_0 1
\]

A matrix \( M \) over \( \mathcal{O} \) is called **Hermitian** if \( M^T = \bar{M} \). Let \( \mathcal{J} \) be the \( \mathbb{C} \)-vector space of all \( 3 \times 3 \) Hermitian matrices over \( \mathcal{O} \). Such matrices have the form

\[
M = \begin{pmatrix}
a_1 & x & y \\
\bar{x} & b_1 & z \\
\bar{y} & \bar{z} & c_1
\end{pmatrix}
\]

where \( a, b, c \in \mathbb{C} \) and \( x, y, z \in \mathcal{O} \). \( \dim(\mathcal{J}) = 27 \). We define multiplication on \( \mathcal{J} \) by

\[
M_1 \times M_2 = \frac{1}{2}(M_1 M_2 + M_2 M_1)
\]

\( M_1 \times M_2 \in \mathcal{J} \) for \( M_1, M_2 \in \mathcal{J} \)

\( \mathcal{J} \) is a commutative non-associative algebra; it is an example of a **Jordan algebra**, the axioms for which are that

\[
X \times Y = Y \times X \\
(X^2 \times Y) \times X = X^2 \times (Y \times X)
\]

The derivations of \( \mathcal{J} \) form a simple Lie algebra of type \( F_4 \).

\( E_6, E_7 \) and \( E_8 \) can all be described in terms of \( \mathcal{O} \) and \( \mathcal{J} \).

There is an alternative approach to the existence theorem, which proceeds (in outline) as follows:

Let \( L \) be a simple Lie algebra with Cartan matrix \( A = (a_{ij}) \), \( L = H \oplus (\oplus_{a \in \Phi} L_a) \). \( H \) has basis \( h_{\alpha_1}, \ldots, h_{\alpha_l} \). Let

\[
h_i = \frac{2h_{\alpha_i}}{\langle h_{\alpha_i}, h_{\alpha_i} \rangle}
\]
\[ e_i h_j = \alpha_i(h_j) e_j \]
\[ = \langle h_{\alpha_j}, h_j \rangle e_j \]
\[ = \frac{2 \langle h_{\alpha_j}, h_{\alpha_j} \rangle}{\langle h_{\alpha_j}, h_{\alpha_j} \rangle} \]
\[ = a_{ij} e_j \]
\[ [f_i h_j] = -a_{ij} f_j \]

Choose \( f_i \) with \([f_i e_j] = h_j\); \( e_i, \ldots, e_l \) generate \( \oplus_{a \in \Phi} L_a \); \( f_1, \ldots, f_l \) generate \( \oplus_{a \in \Phi} L_a \); \( h_1, \ldots, h_l \) generate \( H \). So \( G = \{ e_i, f_i, h_j \mid 1 \leq i \leq l \} \) generates \( L \). We have relations \( R \):

\[ [h_i h_j] = 0 \]
\[ [e_i h_j] = a_{ij} e_j \]
\[ [f_i h_j] = -a_{ij} f_j \]
\[ [f_i e_j] = h_j \]
\[ [f_i e_j] = 0 \text{ if } i \neq j \]
\[ [e_i \ldots e_j [e_i e_j]] = 0 \text{ if } i \neq j (1 - a_{ij} e_i \text{'s}) \]
\[ [f_i \ldots f_j [f_i f_j]] = 0 \text{ if } i \neq j (1 - a_{ij} f_i \text{'s}) \]

(The requirements for \( 1 - a_{ij} e_i \text{'s} \) and \( f_i \text{'s} \) arise from consideration of the \( \alpha_j \)-chain of roots through \( \alpha_j \).) The Lie algebra generated by \( G \) with relations \( R \) is a finite-
dimensional Lie algebra with Cartan matrix \( A \).

\( L \) is constructed as follows: let \( R \) be the polynomial ring \( \mathbb{C}[e_i, \ldots, e_l, f_1, \ldots, f_l, h_1, \ldots, h_l] \) with non-commutative variables. \([R] \) is the Lie algebra obtained from \( R \). Let \( M \) be the subalgebra generated by \( e_i, \ldots, e_l, f_1, \ldots, f_l, h_1, \ldots, h_l \). Let \( I \) be the ideal of \( M \) generated by

\[ [h_i h_j], [e_i h_j] - a_{ij} e_j, [f_i h_j] + a_{ij} f_j, [f_i e_j] - \delta_{ij} h_j, [e_i \ldots e_j e_j], [f_i \ldots f_j [f_i f_j]]. \]

Then \( L = M/I \). We can show that \( L \) is finite-dimensional and has Cartan matrix \( A \).