MA475 Riemann Surfaces
Condensed Notes

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1 Preliminaries

Notation. $U \mathring{\subseteq} X$ will indicate that U is an open subset of X. p_k denotes the map $z \mapsto z^k$ for $k \in \mathbb{N}$. Γ will denote a lattice in C of the form $\Gamma := \mathbb{Z}w_1 + \mathbb{Z}w_2$ with $w_1, w_2 \in \mathbb{C}$ linearly independent over $\mathbb{R}$. All diagrams of maps are commutative unless otherwise indicated. $f\colon X, x \to Y, y$ denotes a function $X \to Y$ such that $f(x) = y$. $I := [0, 1] \subseteq \mathbb{R}$ is the unit interval.

Definitions 1.1. Let X be a (real) 2-dimensional manifold. A *complex chart* on X is a homeomorphism $\phi\colon U \to V$ with $U \mathring{\subseteq} X$, $V \mathring{\subseteq} \mathbb{C}$. Complex charts $\phi_j\colon U_j \to V_j$, $j = 1, 2$ are *compatible* if $U \cap V = \emptyset$ or

$$\phi_2 \circ \phi_1^{-1}\colon \phi_1(U \cap V) \to \phi_2(U \cap V)$$

is holomorphic with holomorphic inverse, i.e. *biholomorphic*. A *complex atlas* $\mathcal{A}$ on X is a collection of pairwise-compatible complex charts whose domains cover X. Two complex atlases $\mathcal{A}, \mathcal{A}'$ are *analyticially equivalent* if every chart in $\mathcal{A}$ is compatible with every chart in $\mathcal{A}'$. A *complex structure* Σ on X is a choice of analytic equivalence class of complex atlases on X. A *Riemann surface* is a 2-dimensional manifold X with a complex structure $\Sigma = [\mathcal{A}]$ on X.

Examples 1.2. (i) $\mathbb{C}$, or any $U \mathring{\subseteq} \mathbb{C}$ is a Riemann surface with the identity map as a global chart.

(ii) $\mathbb{P}^1$ denotes the *Riemann sphere*, S^2 identified with $\mathbb{C} \cup \{\infty\}$, with two charts given by stereographic projection from the North and South poles, with coordinates z and $w := 1/z$ respectively.

(iii) A torus $\mathbb{C}/\Gamma$.

2 Maps of Riemann Surfaces

Definitions 2.1. Let X be a Riemann surface and $U \overset{\circ}{\subseteq} X$: $f \colon U \to \mathbb{C}$ is *holomorphic* if for all charts $\phi_1 \colon U_1 \to V_1$ on X, $f \circ \phi_1^{-1} \colon \phi_1(U \cap U_1) \to \mathbb{C}$ is holomorphic. $\mathcal{O}(U) := \{f \colon U \to \mathbb{C} \mid f \text{ holomorphic}\}$ forms a commutative algebra over $\mathbb{C}$. Let X, Y be Riemann surfaces: $f \colon X \to Y$ is *holomorphic* if for all charts $\phi_1 \colon U_1 \to V_1$ on X and $\phi_2 \colon U_2 \to V_2$ on Y with $f(U_1) \subseteq U_2$, $\phi_2 \circ f \circ \phi_1^{-1} \colon V_1 \to V_2$ is holomorphic.

$$\begin{tikzcd} U_1 \arrow[r, "f"] \arrow[d, "\phi_1"'] & U_2 \arrow[d, "\phi_2"] \\ V_1 \arrow[r, "\phi_2 \circ f \circ \phi_1^{-1}"'] & V_2 \end{tikzcd}$$

Proposition 2.2. *A continuous map $f \colon X \to Y$ is holomorphic $\iff$ for all $V \overset{\circ}{\subseteq} Y$ and $\phi \in \mathcal{O}(V)$, $\phi \circ f \in \mathcal{O}(f^{-1}(V))$.*

Proposition 2.3. *If $\phi \colon U \to V$ is a chart on a Riemann surface X then ϕ is holomorphic.*

Definitions 2.4. Let X be a Riemann surface and $U \overset{\circ}{\subseteq} X$: a *meromorphic* function on U is a holomorphic function $f \colon U' \to \mathbb{C}$, where $U' \overset{\circ}{\subseteq} U$, $U \backslash U'$ is discrete, and for all $p \in U \backslash U'$, $\lim_{z \to p} |f(z)| = \infty$. $U \backslash U'$ is called the set of *poles* of f. $\mathcal{M}(U) := \{f \colon U \to \mathbb{C} \mid f \text{ meromorphic}\}$ forms a commutative division algebra over $\mathbb{C}$.

Proposition 2.5. *Let $f \in \mathcal{M}(X)$ with poles $X \backslash X'$. Define $\tilde{f} \colon X \to \mathbb{P}^1$ by*

$$\tilde{f}(z) := \begin{cases} f(z) & z \in X', \\ \infty & z \in X \backslash X'. \end{cases}$$

Then $\tilde{f}$ is a holomorphic map of Riemann surfaces.

Theorem 2.6 (The Identity Theorem for $\mathbb{C}$)**.** *Let $U \overset{\circ}{\subseteq} \mathbb{C}$ be connected, $f, g \in \mathcal{O}(U)$ and $f|_A = g|_A$ for some $A \subseteq U$ with a limit point. Then $f = g$ on U.*

Theorem 2.7 (The Identity Theorem for Riemann surfaces)**.** *Let $f, g\colon X \to Y$ be holomorphic maps of Riemann surfaces with $f|_A = g|_A$ for some $A \subseteq X$ with a limit point. Then $f = g$ on X.*

Proposition 2.8. *$f\colon X \to \mathbb{P}^1$ holomorphic $\implies$ $f(z) \equiv \infty$ or else $f^{-1}(\infty)$ is discrete and $f \in \mathcal{M}(X)$.*

Proposition 2.9 (Branching theorem)**.** *Let $f\colon X \to Y$ be a non-constant holomorphic map of Riemann surfaces, $a \in X$, $b := f(a) \in Y$. Then there exists a $k \in \mathbb{N}$ and charts $\phi_1\colon U_1 \to V_1$ and $\phi_2\colon U_2 \to V_2$ with $a \in U_1$, $b \in U_2$, $\phi_1(a) = \phi_2(b) = 0$ such that $\phi_2 \circ f \circ \phi_1^{-1}\colon V_1 \to V_2$ is $p_k\colon z \mapsto z^k$.*

$$\begin{tikzcd}
U_1, a \arrow[r, "f"] \arrow[d, "\phi_1"'] & U_2, b \arrow[d, "\phi_2"] \\
V_1, 0 \arrow[r, "p_k = \phi_2 \circ f \circ \phi_1^{-1}"'] & V_2, 0
\end{tikzcd}$$

Definitions 2.10. If $k > 1$ above, we call a a *branch point* of f. If f has no branch points we say that it is *unbranched*. If $f\colon Y \to X$ is locally given by $z \mapsto z^k$ at $y \in Y$, then we say that the *multiplicity* of f at y is $\nu(f, y) := k$.

Corollary 2.11 (Open mapping theorem)**.** *For $f\colon X \to Y$ a non-constant holomorphic map of Riemann surfaces, $U \overset{\circ}{\subseteq} X \implies f(U) \overset{\circ}{\subseteq} Y$.*

Definition 2.12. A map of topological spaces $f\colon X \to Y$ is *discrete* if $f^{-1}(p) \subseteq X$ is discrete for all $p \in Y$.

Corollary 2.13. *A non-constant holomorphic map of connected Riemann surfaces $f\colon X \to Y$ is both open and discrete.*

Corollary 2.14. *If $f\colon X \to Y$ injective and holomorphic, then $f\colon X \to f(X)$ is biholomorphic.*

Corollary 2.15 (Maximum principle)**.** *If $f \in \mathcal{O}(X)$ is non-constant, then $|f|\colon X \to \mathbb{R}$ does not attain a maximum.*

Corollary 2.16. *If X is a compact Riemann surface and $f \in \mathcal{O}(X)$, then f is constant.*

Theorem 2.17. *If $f\colon X \to Y$ is a non-constant holomorphic map with X compact and Y connected, then Y is compact and f is surjective.*

Proposition 2.18. *$\mathcal{M}(\mathbb{P}^1)$ is the space $\mathbb{C}(z)$ of rational functions in one complex variable.*

Definition 2.19. Let Γ be a 2-dimensional lattice in $\mathbb{C}$. $f \in \mathcal{M}(\mathbb{C})$ is *doubly periodic* if $f(z+\gamma) = f(z)$ for all $\gamma \in \Gamma$.

Corollary 2.20. *Let $f\colon \mathbb{C} \to \mathbb{P}^1$ be doubly periodic. Then f determines a function $F\colon \mathbb{C}/\Gamma \to \mathbb{P}^1$ such that $f = F \circ \pi$, where $\pi\colon \mathbb{C} \to \mathbb{C}/\Gamma$ is the quotient map, and f is holomorphic / meromorphic if and only is F has the same property. Moreover, f is either constant or surjective.*

Definition 2.21. A map of topological spaces $f\colon X \to Y$ is a *local homeomorphism* if every $x \in X$ has a neighbourhood $x \in U \overset{\circ}{\subseteq} X$ with $f|_U$ a homeomorphism.

Theorem 2.22. *Let X be a Riemann surface, Y a Hausdorff topological space and $f\colon Y \to X$ a local homeomorphism. Then there is a unique complex structure on Y making f holomorphic.*

3 Covering Maps and Spaces

Definitions 3.1. $f\colon Y \to X$ is a *covering map* if every $x \in X$ has a neighbourhood $x \in U \overset{\circ}{\subseteq} X$ such that $f^{-1}(U) = \bigcup_{j\in J} V_j$ where the V_j are disjoint open sets of Y with $f|_{V_j}\colon V_j \to U$ a homeomorphism for each j. Let X, Y, Z be topological spaces and $f\colon Z \to X$, $g\colon Y \to X$ continuous: a *lifting of f with respect to g* is a continuous $h\colon Z \to Y$ such that $f = g \circ h$, i.e.

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ Z \arrow[ur, dashed, "h"] \arrow[r, "f"'] & X \end{tikzcd}$$

Theorem 3.2. *If $p\colon Y \to X$ is a covering map and $\gamma\colon I \to X$ is a curve, then for all $y_0 \in Y$ with $p(y_0) = \gamma(0)$ there is a unique lifting $\check{\gamma}\colon I \to Y$ with $\check{\gamma}(0) = y_0$.*

Theorem 3.3. *If $p\colon Y \to X$ is a covering map, $\gamma_1, \gamma_2\colon I \to X$ curves from $a \in X$ to $b \in X$, with $\gamma_1 \simeq \gamma_2$ by a homotopy $H\colon I \times I \to X$, and $\check{\gamma}_1, \check{\gamma}_2\colon I \to Y$ liftings with $\check{\gamma}_1(0) = \check{\gamma}_2(0) = y_0 \in Y$, then there is is lifting $\check{H}\colon I \times I \to Y$ of H such that $\check{H}$ is a homotopy from $\check{\gamma}_1$ to $\check{\gamma}_2$. Conversely, if $\check{H}\colon I \times I \to Y$ is a homotopy from $\check{\gamma}_1$ to $\check{\gamma}_2$, then $H = p \circ \check{H}$ is a homotopy from γ_1 to γ_2.*

Definitions 3.4. If $p\colon Y \to X$ is a covering map, a *deck transformation* of Y over X is a homeomorphism $f\colon Y \to Y$ such that $p \circ f = p$:

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

The set of all deck transformations of Y over X is denoted $\mathrm{Deck}(Y/X)$. If $p\colon Y \to X$ is a covering map with X path connected and Y connected and simply connected, we call $p\colon Y \to X$ the *universal covering space* of X.

Theorem 3.5. *If $p\colon Y \to X$ is a universal covering and $q\colon Z \to X$ a connected covering, then for all $y_0 \in Y$ and $z_0 \in Z$ with $p(y_0) = q(z_0) =: x_0 \in X$, there exists a unique $f\colon Y \to Z$ such that $q \circ f = p$ and $f(y_0) = z_0$:*

$$\begin{tikzcd} Y, y_0 \arrow[rr, dashed, "f"] \arrow[dr, "p"'] & & Z, z_0 \arrow[dl, "q"] \\ & X, x_0 & \end{tikzcd}$$

Theorem 3.6. *If $p\colon Y \to X$ is a universal covering space, then* $\mathrm{Deck}(Y/X) \cong \pi_1(X, x_0)$ *as groups for any $x_0 \in X$.*

Definitions 3.7. A topological space X is *locally compact* if every $x \in X$ has a neighbourhood $x \in U \overset{\circ}{\subseteq} X$ with $U \subseteq K$ for some compact $K \subseteq X$. A map of locally compact spaces $f\colon Y \to X$ is *proper* if $K \subseteq X$ compact $\implies f^{-1}(K) \subseteq Y$ compact.

Lemma 3.8. *If $f\colon Y \to X$ is a discrete proper map of locally compact spaces, then for all $x \in X$, $f^{-1}(x)$ is finite, and for all open neighbourhoods $f^{-1}(x) \subseteq V \overset{\circ}{\subseteq} Y$ there is a neighbourhood $x \in U \overset{\circ}{\subseteq} X$ with $f^{-1}(U) \subseteq V$.*

Theorem 3.9. *If X, Y are locally compact Hausdorff spaces with $f\colon Y \to X$ a discrete proper local homeomorphism, then f is a covering map.*

Proposition 3.10. *If f is an unbranched proper non-constant map of Riemann surfaces then f is a local homeomorphism, and hence a covering map.*

Theorem 3.11. *If $f\colon Y \to X$ is a proper non-constant holomorphic map of connected Riemann surfaces, then there exists an $n \in \mathbb{N}$ such that for all $x \in X$, $\sum_{y \in f^{-1}(x)} \nu(f, y) = n$.*

Definition 3.12. The above n is called the number of *sheets* of f, denoted $\mathrm{sh}(f)$.

Corollary 3.13. *If X is a connected Riemann surface with $f \in \mathcal{M}(X)$ then f has the same number of poles as it has zeroes (both counted with multiplicity).*

Corollary 3.14. *A polynomial $p \in \mathbb{C}[z]$ has $\deg p$ roots counted with multiplicity.*

Corollary 3.15. *There is no $f \in \mathcal{M}(\mathbb{C}/\Gamma)$ with a single pole of multiplicity 1.*

Theorem 3.16 (Uniformization theorem)**.** *If Y is a simply connected Riemann surface, then Y is biholomorphic to either $\mathbb{P}^1$, $\mathbb{C}$ or the unit disc $D := \{z \in \mathbb{C} \mid |z| < 1\}$.*

Theorem 3.17. *If X is a Riemann surface and $f\colon X \to D^* := D \backslash \{0\}$ a holomorphic covering map, then either*

(i) $\mathrm{sh}(f) = \infty$ *and there is a biholomorphic* $\phi\colon X \to H := \{z \in \mathbb{C} \mid \operatorname{Re} z < 0\}$ *with* $\exp \circ \phi = f$*:*

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & H \arrow[dl, "\exp"] \\ & D^* & \end{tikzcd}$$

or

(ii) $\mathrm{sh}(f) = k \in \mathbb{N}$ *and there is a biholomorphic* $\phi\colon X \to D^*$ *with* $p_k \circ \phi = f$*:*

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & D^* \arrow[dl, "p_k"] \\ & D^* & \end{tikzcd}$$

Theorem 3.18. *If X is a Riemann surface and $f\colon X \to D$ is a proper non-constant biholomorphic map unbranched over D^*, then there is a $k \in \mathbb{N}$ and a biholomorphic $\phi\colon X \to D$ such that $f = p_k \circ \phi$:*

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & D \arrow[dl, "p_k"] \\ & D & \end{tikzcd}$$

4 Analytic Continuation

Definitions 4.1. Let X be a topological space. A *presheaf* $\mathcal{F}$ of Abelian groups[1] is a collection of data:

(i) for each $U \overset{\circ}{\subseteq} X$, an Abelian group $\mathcal{F}(U)$;

(ii) for each inclusion $U \subseteq V$, a *restriction map* $\rho_{UV} \colon \mathcal{F}(V) \to \mathcal{F}(U)$, a homomorphism of Abelian groups such that

  (b) $\rho_{UU} = \mathrm{id}$;

  (b) $U \subseteq V \subseteq W \implies \rho_{UV} \circ \rho_{VW} = \rho_{UW}$.

For $s \in \mathcal{F}(V)$, write $s|_U := \rho_{UV}(s)$. A presheaf $\mathcal{F}$ is a *sheaf* if, for $\{U_i\}_{i\in I}$ a collection of open subsets of X, with $U := \bigcup_{i\in I} U_i$,

(i) if $s \in \mathcal{F}(U)$ and $s|_{U_i} = 0$ for all $i \in I$ then $s = 0$. (Hence $\mathcal{F}(\emptyset) = 0$.)

(ii) given $s_i \in \mathcal{F}(U_i)$ with $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all $i, j \in I$, there is an $s \in \mathcal{F}(U)$ with $s|_{U_i} = s_i$.

Let $\mathcal{F}$ be a sheaf on X and $a \in X$. A *germ* of $\mathcal{F}$ at $a \in X$ is a pair (U, s), where $a \in U \overset{\circ}{\subseteq} X$ and $s \in \mathcal{F}(U)$. Two germs (U, s) and (U', s') are *equivalent* if there is an open neighbourhood $a \in V \subseteq U \cap U'$ with $s|_V = s'|_V$. The *stalk* of $\mathcal{F}$ at a is

$$\mathcal{F}_a := \frac{\{\text{germs } (U, s) \text{ of } \mathcal{F} \text{ at } a\}}{\sim}.$$

$\mathcal{F}_a$ is an Abelian group via

$$(U, s) + (U', s') := (U \cap U', s|_{U\cap U'} + s'|_{U\cap U'}) .$$

Define $\rho_a \colon \mathcal{F}(U) \to \mathcal{F}_a$, where $a \in U$ by $\rho_a(s) := (U, s)$. The *éspace étalé* of $\mathcal{F}$ is $|\mathcal{F}| := \coprod_{a\in X} \mathcal{F}_a$, with $p\colon |\mathcal{F}| \to X$, $p(\phi) := a \in X$ for $\phi \in \mathcal{F}_a$. Define *open sets in* $|\mathcal{F}|$ by $[U, s] := \{\rho_a(s) | a \in U\}$, and $[U, s] \overset{\circ}{\subseteq} |\mathcal{F}|$ if $U \overset{\circ}{\subseteq} X$, $s \in \mathcal{F}(U)$.

Theorem 4.2. *$\mathcal{B} := \left\{[U, f] \,\middle|\, U \overset{\circ}{\subseteq} X, f \in \mathcal{F}(U)\right\}$ is a basis for a topology on $|\mathcal{F}|$ with respect to which $p : |\mathcal{F}| \to X$ is a local homeomorphism.*

Corollary 4.3. *Let X be a Riemann surface and $\mathcal{O}$ the sheaf of holomorphic functions, with $p\colon |\mathcal{O}| \to X$. If $|\mathcal{O}|$ is Hausdorff and Y a connected component of $|\mathcal{O}|$ then $p\colon Y \to X$ is a local homeomorphism and there is a unique complex stucture on Y making p holomorphic.*

[1] We can also have presheaves of sets, rings, fields, vector spaces, etc., with the appropriate morphisms.

Definition 4.4. A presheaf $\mathcal{F}$ *satisfies the identity theorem* if whenever $U \overset{\circ}{\subseteq} X$ is connected and $f, g \in \mathcal{F}(U)$ are such that $\rho_a(f) = \rho_a(g)$ for some $a \in U$, it follows that $f = g$.

Corollary 4.5. *$\mathcal{O}$ satisfies the identity theorem for any Riemann surface X.*

Theorem 4.6. *If X is a locally connected Hausdorff space and $\mathcal{F}$ is a presheaf on X satisfying the identity theorem, then $|\mathcal{F}|$ is Hausdorff.*

Definition 4.7. Let X be a Riemann surface, $\gamma\colon I \to X$ a curve from a to b: $\psi \in \mathcal{O}_b$ is an *analytic continuation* of $\phi \in \mathcal{O}_a$ along γ if the following holds: there exists a family $\phi_t \in \mathcal{O}_{\gamma(t)}$, $t \in I$, with $\phi_0 = \phi$, $\phi_1 = \psi$ such that for all $\tau \in I$ there is an open neighbourhood $\tau \in T \overset{\circ}{\subseteq} I$and a $U \overset{\circ}{\subseteq} X$ with $\gamma(T) \subseteq U$ and $f \in \mathcal{O}(U)$ such that $\phi_t = \rho_{\gamma(t)}(f)$ for all $t \in T$. *Equivalently:* there exists a partition $0 = t_0 < t_1 < \cdots < t_r = 1$ of I and $U_i \overset{\circ}{\subseteq} X$ with $\gamma([t_{i-1}, t_i]) \subseteq U_i$ and $f_i \in \mathcal{O}(U_i)$ such that

(i) ϕ is a germ of f_1 at a; ψ is a germ of f_r at b;

(ii) $f_i|_{V_i} = f_{i+1}|_{V_i}$ where V_i is the connected component of $U_i \cap U_{i+1}$ containing $\gamma(t_i)$.

Lemma 4.8. *Let X be a Riemann surface, $\gamma\colon I \to X$ a curve from a to b. Then $\psi \in \mathcal{O}_b$ is an analytic continuation of $\phi \in \mathcal{O}_a$ if and only if there is a lift $\check{\gamma}\colon I \to |\mathcal{O}|$ of γ such that $\check{\gamma}(0) = \phi$, $\check{\gamma}(1) = \psi$.*

Theorem 4.9 (Monodromy theorem)**.** *Let X be a Riemann surface, $\gamma_0, \gamma_1\colon I \to X$ two homotopic curves from a to b. Suppose that γ_s, $0 \leq s \leq 1$, is a homotopy of γ_0 into γ_1, and suppose that $\phi \in \mathcal{O}_a$ admits an analytic continuation to $\psi_s \in \mathcal{O}_b$ for every s. Then ψ_s is independent of s.*

Theorem 4.10. *Let X, Y be Hausdorff topological spaces, $p\colon Y \to X$ a local homeomorphism. Let $a, b \in X$, $\check{a} \in Y$ with $p(\check{a}) = a$. Let $\gamma_s\colon I \to X$ be a continuous family of curves connecting a to b. If each γ_s can be lifted to a curve $\check{\gamma}_s$ with $\check{\gamma}_s(0) \equiv \check{a}$, then $\check{\gamma}_0(1) = \check{\gamma}_1(1)$.*

Corollary 4.11. *If X is a simply connected Riemann surface, $a \in X$, and $\phi \in \mathcal{O}_a$ admits an analytic continuation along every curve in X starting at a, then there is an $f \in \mathcal{O}(X)$ such that $\rho_a(f) = \phi$.*

Definitions 4.12. Let $p\colon Y \to X$ be an unbranched holomorphic map of Riemann surfaces, a local homeomorphism, so locally biholomorphic. Define the *pull-back isomorphism* of rings (stalks) $p^*\colon \mathcal{O}_{X,p(y)} \to \mathcal{O}_{Y,y}$ by

$$p^*(U, f) := \left(p^{-1}(U), f \circ p\right).$$

Define the inverse, the *push-forward isomorphism*, $p_*\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,p(y)}$, by choosing a neighbourhood $y \in V \mathring{\subseteq} Y$ such that $p|_V\colon V \to p(V)$ is biholomorphic and setting

$$p_*(U,g) := \left(p(U \cap V), g \circ p|_V^{-1}\right).$$

Let X be a Riemann surface, $a \in X$, $\phi \in \mathcal{O}_a$. (Y,p,f,b) is an *analytic continuation* of ϕ if

(i) Y is a Riemann surface;

(ii) $p\colon Y \to X$ is an unbranched holomorphic map;

(iii) $f\colon Y \to \mathbb{C}$ is holomorphic;

(iv) $b \in Y$ satisifies $p(b) = a$ and $p_*(\rho_b(f)) = \phi$.

(Y,p,f,b) is *maximal* if it satisfies the following: if (Z,q,g,c) is any other analytic continuation of ϕ then there is a unique holomorphic $F\colon Z \to Y$ with $F(c) = b$, $q = p \circ F$, and $g = f \circ F$, i.e.

$$\begin{tikzcd} Z, c \arrow[rr, dashed, "F"] \arrow[dr, "q"'] & & Y, b \arrow[dl, "p"] \\ & X, a & \end{tikzcd}$$

Lemma 4.13. *If X is a Riemann surface, $a \in X$, $\phi \in \mathcal{O}_a$, (Y,p,f,b) an analytic continuation of ϕ and $\check{\gamma}\colon I \to Y$ is a curve with $\check{\gamma}(0) = b$, $\check{\gamma}(1) = y \in Y$, then $\psi := p_*(\rho_y(f)) \in \mathcal{O}_{X,p(y)}$ is an analytic continuation of ϕ along $\gamma := p \circ \check{\gamma}$.*

Theorem 4.14 (Existence of maximal analytic continuations)**.** *Let X be a Riemann surface, $a \in X$, $\phi \in \mathcal{O}_a$. Then there exists a maximal analytic continuation of ϕ.*

Theorem 4.15. *Let X be a Riemann surface, $A \subseteq X$ discrete, $X' := X \backslash A$, Y' another Riemann surface, and $\pi'\colon Y' \to X'$ a proper unbranched holomorphic map. Then π' extends to a branched proper map, i.e. there is a Riemann surface Y with $\pi\colon Y \to X$ proper, and ϕ biholomorphic:*

$$\begin{tikzcd} Y \backslash \pi^{-1}(A) \arrow[rr, dashed, "\phi"] \arrow[dr, "\pi"'] & & Y' \arrow[dl, "\pi'"] \\ & X' & \end{tikzcd}$$

Definition 4.16. We define the *Riemann surface of* $\sqrt{f(z)}$, where f is a polynomial with no multiple roots, $\deg f = 2g+2$, as follows:

(i) $\phi :=$ a germ of some branch of $\sqrt{f(z)}$ at some $a \in X' := \mathbb{C}\backslash\{z|f(z) = 0\}$.

(ii) $(Y', p', f, b) :=$ a maximal analytic continuation of ϕ.

(iii) Extend $p'\colon Y' \to X'$ to $p\colon Y \to X := \mathbb{P}^1$ by adding one branch point over each zero of f and two distinct points over ∞.

Y with $p\colon Y \to X$ is the *Riemann surface of* $\sqrt{f(z)}$.

5 Integration of Differential 1-Forms

Definitions 5.1. Writing $z = x + iy$, a *differential* 1*-form* on $V \mathring{\subseteq} \mathbb{C}$ is an expression $f_1\,\mathrm{d}x + f_2\,\mathrm{d}y$ for some differentiable $f_1, f_2\colon V \to \mathbb{C}$. If $p = p_1 + ip_2\colon U \to V$ is holomorphic and $\omega := f_1\,\mathrm{d}x + f_2\,\mathrm{d}y$ is a 1-form on V the *pull-back* is

$$p^*\omega := \left((f_1 \circ p)\frac{\partial p_1}{\partial x} + (f_2 \circ p)\frac{\partial p_2}{\partial y}\right)\,\mathrm{d}x + \left((f_1 \circ p)\frac{\partial p_1}{\partial y} + (f_2 \circ p)\frac{\partial p_2}{\partial x}\right)\,\mathrm{d}y$$

If $f\colon V \to \mathbb{C}$ is differentiable, define $\mathrm{d}f := \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y$, so that $p^*\omega = (f_1 \circ p)\,\mathrm{d}p_1 + (f_2 \circ p)\,\mathrm{d}p_2$. We define $\mathrm{d}z := \mathrm{d}x + i\,\mathrm{d}y$ and $\mathrm{d}\bar{z} := \mathrm{d}x - i\,\mathrm{d}y$.

Proposition 5.2. *If $f\colon V \to \mathbb{C}$ is holomorphic, then $\mathrm{d}f = f'(z)\,\mathrm{d}z$.*

Definition 5.3. A *differential* 1*-form* on a Riemann surface X is a choice, for each chart $\psi\colon U \to V \mathring{\subseteq} \mathbb{C}$, of a 1-form ω_ψ on V such that if $\psi_j\colon U_j \to V_j$, $j = 1, 2$, are compatible charts with $\psi_2 \circ \psi_1^{-1}\colon \psi_1(U_1 \cap U_2) \to \psi_2(U_1 \cap U_2)$ holomorphic, then

$$(\psi_2 \circ \psi_1^{-1})^* \left(\omega_{\psi_2}|_{\psi_2(U_1 \cap U_2)}\right) = \omega_{\psi_1}|_{\psi_1(U_1 \cap U_2)}$$

A *holomorphic* 1*-form* on X is a 1-form locally of the form $f(z)\,\mathrm{d}z$ for some holomorphic f; a *meromorphic* 1*-form* on X is a 1-form locally of the form $f(z)\,\mathrm{d}z$ for some mermorphic f. Denote by $\Omega^1(X)$ the $\mathbb{C}$-vector space of holomorphic 1-forms on X.

Proposition 5.4. *If X is a Riemann surface and $\mathcal{F}(U) := \{$differential 1-forms on $U\}$ for $U \mathring{\subseteq} X$, then $\mathcal{F}$ is a sheaf of vector spaces. Similarly for holomorphic and meromorphic* 1*-forms.*

Proposition 5.5. *If $p : U \to V$ and $f(z)\,\mathrm{d}z$ are holomorphic then so is $p^*(f(z)\,\mathrm{d}z) = f(p(z))\,\mathrm{d}p$.*

Definition 5.6. If $p\colon Y \to X$ is holomorphic and ω is a 1-form on X then define the *pull-back* $p^*\omega$ on Y by $(p^*\omega)_{\psi_1} := (\psi_1 \circ f \circ \psi_2^{-1})(\omega_{\psi_2})$ for $\psi_j\colon U_j \to V_j$, $j = 1, 2$, charts on Y and X respectively.

Theorem 5.7 (Chain rule)**.** *If $f\colon X \to Y$, $g\colon Y \to Z$ are maps of Riemann surfaces and ω is a 1-form on Z then $(g \circ f)^*(\omega) = f^*(g^*(\omega))$.*

Theorem 5.8. *If X is a compact Riemann surface, then $\dim_{\mathbb{C}} \Omega^1(X) < \infty$; in fact, $\dim_{\mathbb{C}} \Omega^1(X) =$ the genus of X.*

Definition 5.9. *Let ω be a 1-form on X, $\gamma\colon I \to X$ a piecewise differentiable curve, i.e. take a partition $0 = t_0 < t_1 < \cdots < t_r = 1$ of I and charts $z_k = x_k + iy_k\colon U_k \to V_k$ with $\gamma([t_{k-1}, t_k]) \subseteq U_k$ and $x_k \circ \gamma|_{[t_{k-1},t_k]}, y_k \circ \gamma|_{[t_{k-1},t_k]}$ of class C^1. Write $\omega = f_k\,\mathrm{d}x_k + g_k\,\mathrm{d}y_k$ on U_k. The* integral *of ω over γ is*

$$\int_\gamma \omega := \sum_{k=1}^{r} \int_{t_{k-1}}^{t_k} f_k(\gamma(t))\frac{\mathrm{d}(x_k(\gamma(t)))}{\mathrm{d}t} + g_k(\gamma(t))\frac{\mathrm{d}(y_k(\gamma(t)))}{\mathrm{d}t}\,\mathrm{d}t$$

Theorem 5.10 (Fundamental theorem of calculus)**.** *Let X be a Riemann surface, $\gamma\colon I \to X$ piecewise smooth, and $F\colon X \to \mathbb{C}$ differentiable. Then $\int_\gamma \mathrm{d}F = F(\gamma(1)) - F(\gamma(0))$.*

Definition 5.11. $\omega = f_1\,\mathrm{d}x + f_2\,\mathrm{d}y$ is *closed* if $\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}$. Holomorphic 1-forms are always closed.

Proposition 5.12. *If ω is a closed 1-form on X and $p \in X$ then there is a neighbourhood $p \in U \overset{\circ}{\subseteq} X$ such that $\omega|_U = \mathrm{d}F$ for some $F\colon U \to \mathbb{C}$.*

Theorem 5.13. *Let X be a Riemann surface and $\omega \in \Omega^1(X)$. Then there is a covering map $p\colon \check{X} \to X$ with $\check{X}$ connected and a function $F\colon \check{X} \to \mathbb{C}$ such that $p^*\omega = \mathrm{d}F$.*

Corollary 5.14. *If X is simply connected and $\omega \in \Omega^1(X)$ then there is a $F\colon X \to \mathbb{C}$ such that $\omega = \mathrm{d}F$.*

Corollary 5.15. *We can assume that $\check{X}$ is the universal cover $\tilde{X}$ of X.*

Theorem 5.16. *If X is a Riemann surface, $p\colon \tilde{X} \to X$ the universal cover, $\omega \in \Omega^1(X)$, $p^*\omega = \mathrm{d}F$, $\gamma\colon I \to X$ a curve that lifts to $\check{\gamma}\colon I \to \tilde{X}$, then $\int_\gamma \omega = F(\check{\gamma}(1)) - F(\check{\gamma}(0))$.*

Corollary 5.17. *If X is a Riemann surface, $\omega \in \Omega^1(X)$, $\gamma_1, \gamma_2 \colon I \to X$ homotopic curves from a to b, then $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$.*

Definitions 5.18. Let X be a Riemann surface and $\omega \in \Omega^1(X)$. We have the *period homomorphism* of ω, $\pi_1(X, x_0) \to \mathbb{C}$, defined by $[\gamma] \mapsto \int_\gamma \omega$. $\int_\gamma \omega$ is called the *period integral* of ω. The image of the period homomorphism is the *period lattice* $\Gamma \subset \mathbb{C}$.

Theorem 5.19. *If X is a Riemann surface and $\omega \in \Omega^1(X)$, then $\omega = \mathrm{d}F$ for some $F \in \mathcal{O}(X)$ if and only if the period homomorphism of ω is the zero homomorphism.*

Corollary 5.20. *If X is a compact Riemann surface and $\omega_1, \omega_2 \in \Omega^1(X)$ have the same period homomorphism, then $\omega_1 = \omega_2$.*