Exercises for MA4A2 Advanced Partial Differential Equations (Reading Course)

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Attempt as many of these exercises as you can. If you can comfortably solve all of these exercises and questions from past exams, then you will be in good shape for this year’s exam. You are encouraged to discuss your solutions with, and present them to, fellow MA4A2 students in the weekly meetings. Evans’ and Renardy and Rogers’ books are good sources of further exercises, should you feel the need for them.

Week 1 Review and some basics

1. Prove the fundamental lemma of the calculus of variations: for $u \in L^1(\Omega)$,

$$u(x) = 0 \text{ for almost all } x \in \Omega \iff \int_\Omega u(x) \phi(x) \, dx = 0 \text{ for all } \phi \in C^\infty_c(\Omega).$$

2. Let $\Omega$ be the open unit ball in $\mathbb{R}^2$ and let $u(x) = \sqrt{|x|}$. For which values of $p \in [1, +\infty]$ is $u \in L^p(\Omega)$? For which values of $p \in [1, +\infty]$ is $u \in W^{1,p}(\Omega)$?

Week 2 Sobolev spaces

1. Give (with proof) an example of a function that is not weakly differentiable.

2. Prove directly that any function $u \in W^{1,1}(\Omega)$ is continuous on $\Omega$.

3. Difference quotients: show that for $1 < p < \infty$,

$$u \in W^{1,p}(\mathbb{R}^n) \iff \sup_{h \in \mathbb{R}^n} \left( \int_{\mathbb{R}^n} \frac{|u(x + h) - u(x)|^p}{|h|^p} \, dx \right)^{1/p} < +\infty.$$

4. Show that for $k \in \mathbb{N}$, $1 \leq p < +\infty$, $\Omega$ open in $\mathbb{R}^n$, $W^{k,p}(\Omega)$ is a separable Banach space. What happens for $p = +\infty$?

5. Let $u \in L^p(\Omega)$, $1 \leq p < +\infty$; extend $u$ by zero outside $\Omega$. Show that

$$\limsup_{\varepsilon \downarrow 0} \sup_{|z| \leq \varepsilon} \int_{\mathbb{R}^n} |u(x + z) - u(x)|^p \, dx = 0.$$
6. Prove that the weak derivative satisfies the usual sum, product, quotient and chain rules. Show that if \( u \in W^{k,p}(\Omega) \) has \( D^\alpha u = 0 \) for all \( |\alpha| = m \leq k \), then \( u \) is a polynomial of order \( m - 1 \).

7. If you have some familiarity with the Schwarz space and tempered distributions, compare and contrast the weak and distributional derivative.

**Week 3 More Sobolev spaces**

1. Let \( \Omega \) be the open square \((-1, +1)^2\) in \( \mathbb{R}^2 \) and let

\[
  u(x_1, x_2) = \begin{cases} 
    1 - x_1^2, & x_1 > 0; \\
    1 + x_1^2, & x_1 \leq 0.
  \end{cases}
\]

Show that \( u \) is weakly differentiable; for which \( p \in [1, +\infty) \) is \( u \in W^{1,p}(\Omega) \)?

2. For which values of \( \alpha > 0 \), \( n \in \mathbb{N} \) and \( p \geq 1 \) does \( x \mapsto x^{-\alpha} \) belong to \( W^{1,p}(B_1(0)) \)?

3. Let \( u \in L^p(\Omega), 1 \leq p < +\infty \); extend \( u \) by zero outside \( \Omega \). Set

\[
  u_\varepsilon(x) = (u \ast \eta_\varepsilon)(x) = \int_{\mathbb{R}^n} u(y) \eta_\varepsilon(x - y) \, dy,
\]

where \( \eta_\varepsilon : \mathbb{R}^n \to \mathbb{R} \) is the standard mollifier. Show that \( u_\varepsilon \rightharpoonup u \) in \( L^p(\Omega) \). If \( u \in W^{k,p}_0(\Omega) \), show that \( u_\varepsilon \rightharpoonup u \) in \( W^{k,p}_0(\Omega) \) as well.

4. Show that Friedrichs’ inequality fails for \( W^{1,p}(\Omega) \).

5. Let \( T \) be a trace operator for \( W^{1,p}(\Omega) \); show that \( u \in W^{1,p}_0(\Omega) \iff Tu = 0 \) on \( \partial \Omega \).

6. Show that \( W^{k,p}(\mathbb{R}^n) = W^{k,p}_0(\mathbb{R}^n) \). Show that \( W^{k,p}_{\text{loc}}(\mathbb{R}^n) \neq W^{k,p}(\mathbb{R}^n) \).

7. Show that \( C^{\infty}(\overline{\Omega}) \) is \( W^{1,p} \)-dense in \( W^{1,p}(\Omega) \). (Hint: use the definition of \( W^{1,p}_0(\Omega) \) and the extension theorem.)

**Week 4 Embeddings and inequalities**

1. Let \( B_1(0) \) be the open ball in \( \mathbb{R}^n \). Show that \( u(x) = \log \log (1 + \frac{1}{|x|}) \) belongs to \( W^{1,n}(B_1(0)) \) for \( n \geq 2 \) but that \( u \notin L^\infty(B_1(0)) \). Conclude that the Gagliardo-Nirenberg inequality fails for \( p = n > 1 \).

2. Show that the operator \( S_\varepsilon : L^2(\Omega) \to L^2(\Omega), S_\varepsilon u = u_\varepsilon \), is a continuous, symmetric and compact operator. (Hint: use Rellich’s theorem.) Apply the spectral theorem and the earlier parts to show that there exists an orthonormal basis of \( L^2(\Omega) \) consisting of eigenfunctions of \( S_\varepsilon \).

3. A function \( u : (a, b) \to \mathbb{R} \) is said to be absolutely continuous (\( u \in AC((a, b)) \)) if \( u \) is continuous on \([a, b]\) and there exists \( v \in L^1([a, b]) \) such that

\[
  u(\beta) - u(\alpha) = \int_\alpha^\beta v(x) \, dx \quad \text{for all} \quad \alpha, \beta \in [a, b].
\]
Show that \( u \in W^{1,1}((a,b)) \iff u \sim u^* \in AC((a,b)) \), i.e., possibly after redefinition on a set of measure zero, \( u \in W^{1,1}((a,b)) \) is absolutely continuous and vice versa. (Hint: the weak differentiability of an absolutely continuous \( u \) can be shown by writing \( \int_a^b u(x)\phi'(x) \, dx \) as an iterated integral and reversing the order of integration.)

**Week 5  More embeddings and inequalities**

1. Hölder spaces: for \( 0 < \alpha \leq 1 \), \( K \subset \mathbb{R}^n \) compact, let \( C^{0,\alpha}(K) \) be the space of all functions \( u: K \to \mathbb{R} \) with \( \|u\|_{C^{0,\alpha}(K)} < +\infty \), where

\[
\|u\|_{C^{0,\alpha}(K)} = \sup_{x \in K} |u(x)| + \sup_{x \neq y} \left\{ \frac{|u(x) - u(y)|}{|x - y|^\alpha} : x, y \in K, x \neq y \right\}.
\]

(a) Show that \( \| \cdot \|_{C^{0,\alpha}(K)} \) is a norm.
(b) Show that \( C^{0,\alpha}(K) \) is a Banach space but not a Hilbert space. (Hint: use the polarization identity.)
(c) Prove the interpolation inequality that, for \( 0 < \alpha < \beta \leq 1 \),

\[
\|u\|_{C^{0,\beta}(K)} \leq \|u\|_{C^{0,\alpha}(K)}^{1-\beta} \|u\|_{C^{0,1}(K)}^{\beta-\alpha}.
\]

2. Failure of Morrey’s inequality for irregular domains: give an example of an open set \( \Omega \subset \mathbb{R}^n \) such that \( u \) is not \( \alpha \)-Hölder continuous for any exponent \( \alpha \in (0, 1) \). (Hint: try removing a slit from disc in \( \mathbb{R}^2 \).)

3. Bochner spaces: let \( \Omega = (0, 1) \subset \mathbb{R} \) and consider the function

\[
u(x, t) = \begin{cases} \arctan(x/t), & x \in \Omega, t \in (0, T]; \\ 1, & x \in \Omega, t = 0. \end{cases}\]

Show that \( u \in C([0, T]; L^2(\Omega)) \) and \( u \in L^2([0, T]; H^1(\Omega)) \) but \( u \notin C([0, T]; H^1(\Omega)) \).

4. A more “philosophical” question: how sensitive are Sobolev functions are to changes of their values. For example, does \( \|u - \tilde{u}\|_{H^1(\Omega)} = 0 \) if \( \tilde{u} \neq u \) at only one point? Along a line segment?

**Week 6  Elliptic PDEs**

1. Consider the operator \( Lu = \sum_{i,j=1}^2 \frac{\partial}{\partial x_i} (a_{ij} \frac{\partial u}{\partial x_j}) \). For which of these values of the coefficient matrix \( A = (a_{ij}) \) is \( L \) elliptic?

\[
\begin{align*}
A &= \begin{pmatrix} 17 & 0.1 \\ 0.1 & 1 \end{pmatrix}, & A &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & A &= \begin{pmatrix} 0.1 & 10 \\ 10 & 0.1 \end{pmatrix}.
\end{align*}
\]

2. Can the Lax-Milgram theorem be applied to the bilinear form \( B: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \), \( B[x, y] = x_1 y_2 + x_2 y_1 \)? Can it be applied to the bilinear form \( B: H^1_0((0, \pi)^2) \times H^1_0((0, \pi)^2) \to \mathbb{R} \),

\[
B[u, v] = \int_0^\pi \int_0^\pi \frac{\partial u}{\partial x_1} \frac{\partial v}{\partial x_1} - \frac{\partial u}{\partial x_2} \frac{\partial v}{\partial x_2} + 5000u(x)v(x) \, dx.
\]

3
Can it be applied to the bilinear form
\[
B[u, v] = \int_{0}^{\pi} \int_{0}^{\pi} \frac{\partial u}{\partial x_2} \frac{\partial v}{\partial x_2} - \frac{\partial u}{\partial x_1} \frac{\partial v}{\partial x_1} \, dx
\]

**Week 7  Some spectral theory of elliptic operators**

1. Explicitly construct a basis for \(L^2((0, 1))\) consisting of eigenfunctions of the Laplacian \(\Delta\) with Dirichlet boundary conditions (i.e. \(u(0) = u(1) = 0\)). (Hint: use Fourier series.)

2. Let \(\Omega = (0, 1)^2 \subseteq \mathbb{R}^2\). Show that for \(f \in L^2(\Omega)\), the Lax-Milgram theorem applies to the boundary value problem
\[
\begin{aligned}
-\Delta u(x) &= f(x), \quad x \in \Omega \\
u(x) &= 0, \quad x \in \partial \Omega.
\end{aligned}
\]

Show that the solution operator \(S : L^2(\Omega) \to L^2(\Omega)\) is symmetric and compact, and apply the spectral theorem to construct and eigenfunction basis of \(L^2(\Omega)\).

**Week 8  Parabolic PDEs**

1. Decay estimates for the heat equation: let \(u : [0, \infty) \to H^1_0(\Omega)\) be a weak solution of the heat equation
\[
\begin{aligned}
\dot{u}(x, t) - \Delta u(x, t) &= 0, \quad (x, t) \in \Omega \times (0, \infty); \\
u(x, t) &= 0, \quad (x, t) \in \partial \Omega \times (0, \infty); \\
u(x, t) &= g, \quad (x, t) \in \Omega \times \{t = 0\}.
\end{aligned}
\]

Prove the exponential decay estimate: for some \(\lambda > 0\) and all \(t \geq 0\),
\[
\|u(t)\|_{L^2(\Omega)} \leq e^{-\lambda t} \|g\|_{L^2(\Omega)}.
\]

(Hint: apply the spectral theorem for compact, symmetric operators to the solution operator for the Poisson equation and use Grönwall’s inequality.)

2. Fix \(T > 0\) and \(\Omega \subseteq \mathbb{R}^n\) and suppose that \(u_k \xrightarrow{k \to \infty} u\) in \(L^2([0, T]; H^1_0(\Omega))\) and \(\dot{u}_k \xrightarrow{k \to \infty} v\) in \(L^2([0, T]; H^{-1}(\Omega))\). Show that \(\dot{u} = v\). (Hint: for \(\phi \in C^1_c([0, T])\) and \(w \in H^1_0(\Omega)\),
\[
\int_0^T \langle \dot{u}_k(t), \phi(t)w \rangle_{L^2(\Omega)} \, dt = - \int_0^T \langle u_k(t), \phi(t)w \rangle_{L^2(\Omega)} \, dt.
\]

**Week 9  More parabolic PDEs**

1. Ritz-Galerkin approximation: let \(Lu = -\sum_{i,j} (au_{x_j})_{x_j} + \sum_{i=1}^d b_i u_{x_i} + cu\) be an elliptic partial differential operator and, for \(N \in \mathbb{N}\), let \(V_N \subset H^1_0(\Omega)\) be an \(N\)-dimensional subspace. Show that, for each \(N \in \mathbb{N}\), there exists a unique function \(u_N = u_N(f) \in V_N\) such that, for all \(\zeta \in V_N\),
\[
B[u_N, \zeta] = \langle f, \zeta \rangle_{L^2(\Omega)},
\]
where $B[\cdot,\cdot]$ is the bilinear form associated to $L$. For a suitable basis of $V_N$ determine the $N$-dimensional system of linear equations which determines $u_N$. Let $u \in H_0^1(\Omega)$ be the unique weak solution of the Dirichlet problem

\[
\begin{cases}
Lu(x) = f(x), & x \in \Omega, \\
u(x) = 0, & x \in \partial\Omega.
\end{cases}
\]

Prove Céa’s inequality:

\[
\|u - u_N\|_{H^1(\Omega)} \leq C \inf_{v \in V_N} \|u - v\|_{H^1(\Omega)},
\]

where $C$ is independent of the spaces $V_N$. Discuss whether it is possible to find vector spaces $V_1 \subset V_2 \subset \ldots \subset H_0^1(\Omega)$ so that

\[
\lim_{N \to \infty} \|u_N - u\|_{H^1(\Omega)} = 0.
\]

2. Assume the following result:

**Theorem 1 (Lax-Milgram-Lions).** Let $H$ be a real Hilbert space and $V$ a normed real vector space. Suppose that $E : H \times V \to \mathbb{R}$ is bilinear and that $E(\cdot, v) \in H^*$ for all $v \in V$. Then the following are equivalent:

\[\begin{align*}
(\ast) & \quad \inf_{\|v\|_V = 1} \sup_{\|h\|_H \leq 1} |E(h, v)| \geq \theta; \\
(\ast\ast) & \quad \text{for each } \ell \in V^*, \text{ there exists } h \in H \text{ such that, for all } v \in V, E(h, v) = \langle \ell | v \rangle.
\end{align*}\]

(a) Show that the solution $h$ in the Lax-Milgram-Lions theorem satisfies the estimate

\[\|h\|_H \leq \frac{1}{C} \|\ell\|_{V^*}.\]

(b) Show that if $V$ is continuously imbedded in $H$ and the bilinear form $E$ is $V$-coercive, that is, there is a $\theta > 0$ such that

\[E(v, v) \geq \theta \|v\|^2_V \text{ for all } v \in V,
\]

then $(\ast\ast)$ holds.

3. Compare and contrast the Lax-Milgram-Lions theorem with the following:

**Theorem 2 (Babuška-Lax-Milgram).** Let $H$ and $V$ be real Hilbert spaces and $E : H \times V \to \mathbb{R}$ a continuous, bilinear function. Suppose that, for some constant $c > 0$ and all $h \in H$,

\[\sup_{\|v\|_V = 1} |E(h, v)| \geq c \|h\|_H
\]

and, for $0 \neq v \in V$, $\sup_{0 \neq h \in H} |E(h, v)| > 0$. Then, for each $f \in V^*$, there exists a unique $h_f \in H$ such that $E(h_f, v) = \langle f | v \rangle$ for all $v \in V$; moreover, $\|h_f\|_H \leq \frac{1}{c} \|f\|_{V^*}$.
4. Apply one of the above refinements of the Lax-Milgram theorem to the following initial boundary value problem on a non-cylindrical domain: let $G \subset \mathbb{R}^2$ be a bounded domain such that $\partial G$ is a smooth curve with unit outward normal $(\nu_x, \nu_t)$. Solve the linear parabolic heat equation

$$\partial_t u(t, x) + \lambda u(t, x) - \partial_x^2 u(t, x) = F$$

for $(t, x) \in G,$ with $u \equiv 0$ on the non-horizontal sides of $G$ and $u = u_0$ on the horizontal bottom of $G$.

(Hint: try $H = \{u \in L^2(G) | \partial_x u \in L^2(G) \text{ and } (u \nu_x)|_{\partial G} = 0\};$

$V = \{\phi \in H \cap H^1(G) | \phi|_{\partial G} = 0 \text{ wherever } (\nu_x, \nu_t) = (0, 1)\};$

$E(u, \phi) = \int_G (\partial_x u(t, x) \partial_x \phi(t, x) + \lambda u(t, x) \phi(t, x) - u(t, x) \partial_t \phi(t, x)) \, dt \, dx.$

Week 10 Methods for non-linear problems

1. Let $p > 2$, $q = \frac{p}{p-1}$; let $\Omega$ be a bounded, open domain in $\mathbb{R}^n$ and let $f \in L^q(\Omega; \mathbb{R}).$

Show that the functional $I : W^{1,p}_0(\Omega; \mathbb{R}) \to \mathbb{R}$ given by

$$I[u] := \frac{1}{p} \int_{\Omega} |Du(x)|^p + f(x)u(x) \, dx$$

has a unique minimizer. (Hints: first try the case $p = q = 2$; use lower boundedness of $I$ and weak compactness for existence; use convexity for uniqueness.) Relate this result to the $p$-Laplacian equation.

2. Prove the following result, known as Kachurovskii’s theorem: a differentiable, real-valued function $f$ defined on a convex subset $K$ of a Banach space $V$ is a convex function if, and only if, its Fréchet derivative $df : K \to V^*$ is an (increasing) monotone operator, i.e., for all $x, y \in K$, $(df(x) - df(y))(x - y) \geq 0.$