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Exercise Sheet 6

These exercises relate to the material covered in the lecture of Week 6, and possibly previous weeks' lectures and exercises. Please submit your solutions (in German or English) to these exercises at the beginning of the lecture of Week 7, i.e. by 12:15 on 26 November 2015.

Exercise 6.1. Recall that the evolution of the simple harmonic oscillator satisfies the equation

$$\ddot{q}(t) = -\omega^2 q(t) \tag{6.1}$$

where $\omega > 0$ is the natural frequency. Let $q(t)$ and $p(t) = \dot{q}(t)$ denote the position and the velocity respectively of the harmonic oscillator at time t, let $\Delta t > 0$ be a time step, and let q_k and p_k denote the position and velocity of the harmonic oscillator at time $k\Delta t$. Define the state vector $x_k = (q_k, p_k) \in \mathbb{R}^2$. Assume that the observation of the state x_k at time k is simply a noise-corrupted position measurement, given by

$$y_k \coloneqq q_k + \eta_k, \quad \eta_k \sim \mathcal{N}(0, \tfrac{1}{2}). \tag{6.2}$$

(a) Express (6.1) as a first-order ODE, and then, for your choice of numerical integrator**, write down the corresponding evolution equations in discrete time. In particular, write down the matrices denoted F_k, H_k, Q_k, and R_k in lectures. **[4]**
(b) Estimate the state x_k using the linear Kalman filter, using the parameter set given at the end of the exercise sheet. Include your code. **[12]**
(c) For the position q_k, plot the following over the time interval $[0, T]$: **[4]**
  (i) the true evolution and the observed data,
  (ii) the filtered mean estimate of the position, plus/minus one standard deviation.
(d) Repeat part (c) for the velocity p_k. **[4]**

**If you wish, you may use the semi-implicit Euler scheme for (6.1), given by

$$\begin{aligned} q_k &\coloneqq q_{k-1} + p_k \Delta t \\ p_k &\coloneqq p_{k-1} - \omega^2 q_{k-1} \Delta t. \end{aligned}$$

Exercise 6.2. Recall that the evolution of the Van der Pol oscillator satisfies the equation

$$\ddot{q}(t) - \mu(1 - q(t)^2)\dot{q}(t) + \omega^2 q(t) = 0 \tag{6.3}$$

where $\omega > 0$ is the natural frequency and $\mu \geq 0$ is the damping. Assume that the observation of the state x_k at time k is a noise-corrupted position measurement given by (6.2).

(a) Write down the expressions for the matrices F_k, H_k, Q_k, R_k, $\tilde{u}_k$, and $\tilde{y}_k$. Write down the linearised system that describes the evolution of the state and observation processes in discrete time. **[8]**
(b) Estimate the state x_k using the extended Kalman filter (i.e. by applying the standard linear Kalman filter to the linearised system). Include your code. **[10]**
(c) Repeat Exercise 6.1(c) using your results from part (b) above. **[4]**
(d) Repeat Exercise 6.1(d) using your results from part (b) above. **[4]**

If you wish, you may use the semi-implicit Euler scheme.

Parameter Set.

$$x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \hat{x}_{0|0} = 0 \in \mathbb{R}^2, \quad C_{0|0} = 10^6 I_2, \quad \Delta t = \frac{1}{10}, \quad \omega = 1, \quad T = 6.0.$$

Note: given T and Δt, you should simulate each system for 600 steps.