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# Exercise Sheet 1

These exercises concern the lectures of Week 1, and possibly previous lectures and exercises. Please submit your solutions to these exercises at the end of the second lecture of Week 2, i.e. 10:00 on 27 October 2016. Solutions in German and in English are equally valid. The numbers in the margin indicate approximately how many points are available for each part.

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**Exercise 1.1** (De Morgan's laws). For logical propositions $P$ and $Q$, we can write $P \vee Q$ for the disjuntion "$P$ or $Q$", i.e. the proposition that evaluates to true if either or both of $P$ and $Q$ is/are true, and false otherwise. Similarly, we write $P \wedge Q$ for the conjunction "$P$ and $Q$", i.e. the proposition that evaluates to true if both $P$ and $Q$ are true, and false otherwise. We denote the negation of $P$, i.e. the proposition that is false exactly when $P$ is true, by $\neg P$.

(a) Prove De Morgan's laws **[1]**

$$\neg(P \vee Q) \iff (\neg P) \wedge (\neg Q), \qquad \neg(P \wedge Q) \iff (\neg P) \vee (\neg Q).$$

(b) For a family of propositions $P_i$, indexed by an index $i \in I$, we define $\bigvee_{i\in I} P_i$ and $\bigwedge_{i\in I} P_i$ similarly to the above. Again, prove De Morgan's laws **[1]**

$$\neg\left(\bigvee_{i\in I} P_i\right) \iff \bigwedge_{i\in I}(\neg P_i), \qquad \neg\left(\bigwedge_{i\in I} P_i\right) \iff \bigvee_{i\in I}(\neg P_i).$$

(c) Let $\Omega$ be a set, and let $E_i \subseteq \Omega$ for each $i \in I$. Prove De Morgan's laws **[1]**

$$\Omega \setminus \bigcup_{i\in I} E_i = \bigcap_{i\in I}(\Omega \setminus E_i), \qquad \Omega \setminus \bigcap_{i\in I} E_i = \bigcup_{i\in I}(\Omega \setminus E_i).$$

**Exercise 1.2** (Algebras of sets). Let $\Omega$ be a set, and let $\mathcal{P}(\Omega)$ denote its power set, i.e. the set of all subsets of $\Omega$. An **algebra** on $\Omega$ is a collection $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ such that $\varnothing \in \mathcal{A}$ and $\mathcal{A}$ is closed under finite unions, intersections, and complementations of its members, i.e.

$$\begin{aligned} E \in \mathcal{A} &\implies E^{\complement} \equiv \Omega \setminus E \in \mathcal{A}, \\ E_1, \dots, E_n \in \mathcal{A} &\implies E_1 \cup \dots \cup E_n \in \mathcal{A} \text{ and } E_1 \cap \dots \cap E_n \in \mathcal{A}. \end{aligned}$$

(a) Show that if $\mathcal{A}$ contains $\varnothing$ and is closed under complementations and finite unions of its members, then $\mathcal{A}$ is an algebra. **[1]**
(b) Show that if $\mathcal{A}$ contains $\varnothing$ and is closed under complementations and finite intersections of its members, then $\mathcal{A}$ is an algebra. **[1]**
(c) Show that if $\mathcal{A}_i$ is an algebra on $\Omega$ for each $i \in I$, then so is $\mathcal{A} := \bigcap_{i\in I} \mathcal{A}_i$. **[1]**
(d) The previous part implies that, if $\mathcal{C}$ is any collection of subsets of $\Omega$, then we can define

$$a(\mathcal{C}) := \bigcap \{\mathcal{A} \mid \mathcal{A} \supseteq \mathcal{C} \text{ and } \mathcal{A} \text{ is an algebra on } \Omega\}.$$

to be the smallest algebra containing $\mathcal{C}$, also known as the algebra generated by $\mathcal{C}$. What is $a(\{\{1\}, \{2, 3\}, \{2, 3, 4\}\})$ on $\Omega = \{1, 2, 3, 4\}$? **[1]**

**Exercise 1.3** (Symmetry isn't everything). Writing in 1754, d'Alembert considered two tosses of a single fair coin. He argued that there are three possible outcomes (double heads, double tails, and one of each), so these three outcomes must be equiprobable with probability $\frac{1}{3}$. Experimentation contradicted d'Alembert's probability model. Explain this. What probability model would you suggest instead? **[2]**

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