FU Berlin
Stochastik I (Mono-Bachelor)
WiSe 2016–2017

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Exercise Sheet 2

These exercises concern the lectures of Week 2, and possibly previous lectures and exercises. Please submit your solutions to these exercises at the end of the second lecture of Week 3, i.e. 10:00 on 3 November 2016. Solutions in German and in English are equally valid. The numbers in the margin indicate approximately how many points are available for each part.

DO ANY <u>FOUR</u> OUT OF FIVE QUESTIONS

Exercise 2.1 (Sufficient conditions to be a σ-algebra)**.** Let Ω be a non-empty set.

(a) Show that $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is a σ-algebra on Ω if **[1]**

$$\varnothing \in \mathcal{F}, \tag{2.1}$$
$$E \in \mathcal{F} \iff \Omega \setminus E \in \mathcal{F}, \tag{2.2}$$
$$E_n \in \mathcal{F} \text{ for each } n \in \mathbb{N} \implies \bigcup_{n\in\mathbb{N}} E_n \in \mathcal{F}. \tag{2.3}$$

(b) Show that $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is a σ-algebra on Ω if it satisfies (2.1), (2.2), and **[2]**

$$E_1, \dots, E_n \in \mathcal{F} \implies \bigcap_{i=1}^{n} E_i \in \mathcal{F}, \tag{2.4}$$
$$E_n \in \mathcal{F} \text{ for each } n \in \mathbb{N}, \text{ pairwise disjoint} \implies \biguplus_{n\in\mathbb{N}} E_n \in \mathcal{F}. \tag{2.5}$$

Exercise 2.2 (Generated σ-algebras)**.** Let Ω be a non-empty set, and let $\mathcal{F}_i$ be a σ-algebra on Ω for each i in some non-empty index set I.

(a) Show that $\bigcap_{i\in I} \mathcal{F}_i$ is a σ-algebra on Ω. **[1]**

(b) Hence show that, given any collection $\mathcal{C}$ of subsets of Ω, there is a unique smallest σ-algebra $\sigma(\mathcal{C})$ on Ω that contains $\mathcal{C}$, i.e. $\sigma(\mathcal{C})$ is a σ-algebra on Ω, and if $\mathcal{F}$ is any σ-algebra on Ω with $\mathcal{F} \supseteq \mathcal{C}$, then $\mathcal{F} \supseteq \sigma(\mathcal{C})$. **[1]**

(c) Is $\bigcup_{i\in I} \mathcal{F}_i$ a σ-algebra on Ω? **[1]**

Exercise 2.3 (Defining a probability measure on basic events)**.** Motivation: it is often difficult to give a closed-form formula for the probability of an arbitrary measurable event, and it would be simpler to specify only the probabilities of some 'basic events' and know that this determines a probability measure on the σ-algebra generated by those basic events. This exercise shows one way to do this rigorously — other methods will come up later.

Let $\mathcal{B} \subseteq \mathcal{F}$ be a countable **system of basic events** for a measurable space $(\Omega, \mathcal{F})$: that is, $\mathcal{B} \ni \varnothing$, $\mathcal{B}$ is a countable partition of Ω into pairwise disjoint $\mathcal{F}$-measurable sets, and every event in $\mathcal{F}$ can be written uniquely — up to trivially including extra copies of $\varnothing$ — as a countable disjoint union of events in $\mathcal{B}$ (so, in particular, $\mathcal{F} = \sigma(\mathcal{B})$).

Suppose that $\mathbb{P}_0 \colon \mathcal{B} \to [0,1]$ satisfies $\mathbb{P}_0(\varnothing) = 0$ and $\sum_{B\in\mathcal{B}} \mathbb{P}_0(B) = 1$. Define $\mathbb{P} \colon \mathcal{F} \to [0,1]$ by

$$\mathbb{P}(F) \coloneqq \sum_{n\in\mathbb{N}} \mathbb{P}_0(B_n) \text{ if } F = \biguplus_{n\in\mathbb{N}} B_n \text{ with each } B_n \in \mathcal{B}.$$

(a) Show that $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$ and $\mathbb{P}|_{\mathcal{B}} = \mathbb{P}_0$, i.e. $\mathbb{P}(B) = \mathbb{P}_0(B)$ for all $B \in \mathcal{B}$. **[2]**

(b) Show that $\mathbb{P}$ is the unique extension of $\mathbb{P}_0$ to a probability measure on $(\Omega, \mathcal{F})$, i.e. $\mathbb{Q} \colon \mathcal{F} \to [0,1]$ is a probability measure with $\mathbb{Q}|_{\mathcal{B}} = \mathbb{P}_0$, then $\mathbb{Q} = \mathbb{P}$. **[1]**

Exercise 2.4 (Practice with generated σ-algebras and basic events). Suppose that Ω is a non-empty set, and that non-empty pairwise disjoint sets $A_1, \dots, A_5$ partition Ω. Let $\mathcal{F} \coloneqq \sigma(E_1, E_2)$ be the σ-algebra generated by E_1 and E_2, where

$$E_1 \coloneqq A_1 \uplus A_2 \uplus A_3, \qquad\qquad E_2 \coloneqq A_2 \uplus A_3 \uplus A_4 \uplus A_5.$$

(a) Show that $\mathcal{F}$ has exactly eight members, and list them all. **[1]**
(b) Supposing that each A_k has probability $k/15$, list the probabilities of all of the events in $\mathcal{F}$. **[1]**
(c) Find a system of basic events for $\mathcal{F}$. That is, find pairwise disjoint sets $B_1, \dots, B_k \subseteq \Omega$ whose union is Ω and such that $\mathcal{F} = \sigma(B_1, \dots, B_k)$. **[1]**

Exercise 2.5. Prove the union bound in the case of a countable union: for any sequence of events $(E_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}$ in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

$$\mathbb{P}\left(\bigcup_{n \in \mathbb{N}} E_n \right) \leq \sum_{n \in \mathbb{N}} \mathbb{P}(E_n).$$

Hint: re-write this union as the union of a increasing sequence $(F_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}$, and consider $F_n \setminus F_{n-1}$. **[3]**