FU Berlin
Stochastik I (Mono-Bachelor)
WiSe 2016–2017

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Exercise Sheet 3

These exercises concern the lectures of Week 3, and possibly previous lectures and exercises. Please submit your solutions to these exercises at the end of the second lecture of Week 4, i.e. 10:00 on 10 November 2016. Solutions in German and in English are equally valid. The numbers in the margin indicate approximately how many points are available for each part.

Exercise 3.1. Let $\mathbb{P}_0$ and $\mathbb{P}_1$ be probability measures defined on the same measurable space $(\Omega, \mathcal{F})$.

(a) Let $t \in [0, 1]$, and define $\mathbb{P}_t : \mathcal{F} \to [0, 1]$ by

$$\mathbb{P}_t(E) := (1 - t)\mathbb{P}_0(E) + t\mathbb{P}_1(E).$$

Show that $\mathbb{P}_t$ is a probability measure on $(\Omega, \mathcal{F})$. **[1]**

(b) Show that setting $\mathbb{P}_{\min}(E) := \min\{\mathbb{P}_0(E), \mathbb{P}_1(E)\}$ defines a probability measure on $(\Omega, \mathcal{F})$ if and only if $\mathbb{P}_0 = \mathbb{P}_1$. **[2]**

Exercise 3.2 (Independence and almost sure events)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $E, F \in \mathcal{F}$. Show that, if F occurs either almost surely or almost never, then E and F are independent. Show also that, if E is independent of itself, then E occurs almost surely or almost never. **[2]**

Exercise 3.3 (Product spaces and independence)**.** For $i = 1, 2$, let $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$ be a probability space with a countable system of basic events $\mathcal{B}_i$ (see Exercise 2.3), and let $\Omega := \Omega_1 \times \Omega_2$.
Bonus. Formulate and solve this exercise for the case of a k-fold product of probability spaces, $k \in \mathbb{N}$. If you do this bonus version, then you do not need to treat the special case $k = 2$ separately. **[+2]**

(a) Show that $\mathcal{B} := \{B_1 \times B_2 \mid B_i \in \mathcal{B}_i \text{ for } i = 1, 2\}$ is a countable partition of Ω. **[1]**

(b) It is a fact (which we will prove later using Dynkin's π-λ theorem) that $\sigma(\mathcal{B})$ consists exactly of countable disjoint unions of sets from $\mathcal{B}$. Define $\mathbb{P} : \mathcal{B} \to [0, 1]$ by

$$\mathbb{P}(B_1 \times B_2) := \mathbb{P}_1(B_1) \cdot \mathbb{P}_2(B_2) \text{ for each } B_1 \in \mathcal{B}_1,\ B_2 \in \mathcal{B}_2.$$

Show that this extends to a unique probability measure on $(\Omega, \sigma(\mathcal{B}))$, which we also denote by $\mathbb{P}$. **[1]**

(c) Show that, for $i = 1, 2$, the marginal distribution of $\mathbb{P}$ on $(\Omega_i, \mathcal{F}_i)$ is $\mathbb{P}_i$, i.e.

$$\mathbb{P}(E_1 \times \Omega_2) = \mathbb{P}_1(E_1) \text{ and } \mathbb{P}(\Omega_1 \times E_2) = \mathbb{P}_2(E_2) \text{ for all } E_1 \in \mathcal{F}_1,\ E_2 \in \mathcal{F}_2. \qquad (*)$$

Show also that, under $\mathbb{P}$, 'cylindrical events at right angles to each other' are independent, i.e. **[1]**

$$(E_1 \times \Omega_2) \perp\!\!\!\perp (\Omega_1 \times E_2) \text{ for all } E_1 \in \mathcal{F}_1,\ E_2 \in \mathcal{F}_2. \qquad (**)$$

(d) Suppose that $\mathbb{Q}$ is any probability measure on $(\Omega, \sigma(\mathcal{B}))$ satisfying $(*)$ and $(**)$. Show that $\mathbb{Q} = \mathbb{P}$. (Thus, taking products is the *only* way to couple probability spaces and get independent factors.) **[1]**

Exercise 3.4 (Fair use of an unfair coin/die)**.** (a) An evil witch has captured a dwarf, but her dungeons are already full. She decides that she will use a coin with two sides — heads (outcome h) and tails (outcome t) — to determine whether the dwarf will be released or executed. Since the coin is unfair, with $0 < \mathbb{P}(\{h\}) < 1$, the dwarf suggests that the coin should be tossed twice, independently, with the following results:

$$\begin{aligned} \text{ordered outcome } (h, t) &\implies \text{dwarf is released,} \\ \text{ordered outcome } (t, h) &\implies \text{dwarf is executed,} \\ \text{ordered outcome } (h, h) \text{ or } (t, t) &\implies \text{ignore result, re-toss twice again.} \end{aligned}$$

Show that this game is fair, i.e. probabilities of the dwarf being released and executed are equal. **[1]**

(b) Now suppose that the evil witch has captured $n \in \mathbb{N}$ dwarves, and wishes to use an unfair n-sided die to select one dwarf *uniformly* at random to be released. How can this be done? **[2]**