FU Berlin
Stochastik I (Mono-Bachelor)
WiSe 2016–2017

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Exercise Sheet 4

These exercises concern the lectures of Week 4, and possibly previous lectures and exercises. Please submit your solutions to these exercises at the end of the second lecture of Week 5, i.e. 10:00 on 17 November 2016. Solutions in German and in English are equally valid. The numbers in the margin indicate approximately how many points are available for each part.

Exercise 4.1 (Conditioning done wrongly). It is very easy to make plausible-sounding but false statements about conditional probability. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For each of the following claims about arbitrary events $A, B \in \mathcal{F}$, state whether it is true or false, and give a proof or a (simple!) counterexample accordingly. (Assume, whenever necessary for expressions to be well-defined, that $\mathbb{P}(A), \mathbb{P}(B) > 0$.) **[3]**

(a) $\mathbb{P}(A|B) \geq \mathbb{P}(A)$.
(b) $\mathbb{P}(A|B) \leq \mathbb{P}(A)$.
(c) $\mathbb{P}(A|B) \approx \mathbb{P}(B|A)$.
(d) $\mathbb{P}(A^{\complement}|B) = 1 - \mathbb{P}(A|B)$.
(e) $\mathbb{P}(A|B^{\complement}) = 1 - \mathbb{P}(A|B)$.
(f) $\mathbb{P}(A) = 0 \implies \mathbb{P}(A|B) = 0$.
(g) $\mathbb{P}(A) = 1 \implies \mathbb{P}(A|B) = 1$.

Exercise 4.2 (Conditioning and selection bias). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $A, B \in \mathcal{F}$, with $\mathbb{P}(B) > 0$, and let $C := A \cup B$.

(a) Show that $\mathbb{P}(A|C) \geq \mathbb{P}(A)$, with equality if and only if $\mathbb{P}(A) = 0$ or $\mathbb{P}(C) = 1$. **[1]**
(b) Show that, if $A \perp\!\!\!\perp B$, then $\mathbb{P}(A|B \cap C) \leq \mathbb{P}(A|C)$, with the same condition for equality. **[1]**
(c) An evil witch is looking to recruit a big, bad wolf into her service. There is a population of candidate wolves, each of which is either big or small and — independently — bad or good, with both bigness and badness being relatively rare attributes in the population. In order to not waste time interviewing many unsuitable small, good wolves, the witch decides that she will only interview candidate wolves who are either big or bad (or both). However, when she does so, she finds that the bad wolves tend to be small, and the big wolves tend to be good. Formulate a probability model for this problem (i.e. choose an appropriate sample space, σ-algebra of events, and a probability measure) and explain the witch's observation. **[2]**

Exercise 4.3 (A fair price for a card game). A card trickster has a deck of three cards, which are stacked vertically so that only the top card can be seen. Before the game begins, you inspect the cards and see that one card is red on both sides, the second card has a red side and a black side, and the third card is black on both sides.

(a) Starting with the red side of the two-coloured card uppermost, the trickster fairly shuffles the three cards, preserving their up-down orientations. You observe that the topside of the top card is red. What is the probability that its underside is black? Would you pay 1 EUR for the opportunity to receive 2 EUR if its underside is black, and 0 EUR if the underside is red? **[1]**
(b) Suppose that as well as randomising the order of the three cards, the trickster's shuffle also fairly randomises the up-down orientations of all three cards. Answer part (a) for this new situation. **[1]**

Exercise 4.4 (The great escape). The evil witch has captured a dwarf, but offers him/her[4.1] a chance to escape in the following way. The dwarf is presented with three boxes, one of which is empty and two of which contain big, bad wolves; the dwarf is allowed to leave the evil witch's castle if he/she opens the empty box. The dwarf selects a box, but then the witch opens a different box — one which she knows to contain a big, bad wolf — and shows this to the dwarf. She now gives the dwarf a choice: stick with the original choice of box, or switch to the other unopened one? Formulate a probability model for this problem and use it to explain which choice the dwarf should make and why. **[3]**

[4.1]It is notoriously difficult to tell the sex of a dwarf, since both he-dwarves and she-dwarves have magnificent beards.