FU Berlin
Stochastik I (Mono-Bachelor)
WiSe 2016–2017

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Exercise Sheet 6

These exercises concern the lectures of Week 6, and possibly previous lectures and exercises. Please submit your solutions to these exercises at the end of the second lecture of Week 7, i.e. 10:00 on 1 December 2016. Solutions in German and in English are equally valid. The numbers in the margin indicate approximately how many points are available for each part.

Exercise 6.1 (Measurability with respect to small and large σ-algebras)**.** Let $(\Omega, \mathcal{F})$ be a measurable space, and let $\mathcal{G}$ be a σ-algebra on $\mathbb{R}$. Recall that as part of the definition of a (discrete) random variable, $X : \Omega \to \mathbb{R}$ was called $(\mathcal{F}, \mathcal{G})$**-measurable** if, for every $G \in \mathcal{G}$, $[X \in G] \equiv X^{-1}(G) \equiv \{\omega \in \Omega \mid X(\omega) \in G\} \in \mathcal{F}$. Assume that $\mathcal{G}$ is sufficiently fine[6.1] that, whenever $x, y \in \mathbb{R}$ are distinct, there exists $G \in \mathcal{G}$ such that $x \in G$ and $y \in \mathbb{R} \setminus G$.

(a) Show that, if $\mathcal{F} = \mathcal{P}(\Omega)$, then every function $X : \Omega \to \mathbb{R}$ is $(\mathcal{F}, \mathcal{G})$-measurable.
(b) Show that, if $\mathcal{F} = \{\varnothing, \Omega\}$, then $X : \Omega \to \mathbb{R}$ is $(\mathcal{F}, \mathcal{G})$-measurable if and only if it is constant. **[1]**

Bonus. Explain why the assumption on $\mathcal{G}$ is necessary. **[+0.5]**

Exercise 6.2 (The Borel σ-algebra)**.** Thus far, the lectures have not discussed the choice of a 'good' σ-algebra on $\mathbb{R}$, and have simply asserted that $\mathcal{P}(\mathbb{R})$ is 'too big'. We now begin to fill in this gap.

(a) Show that if $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{P}(\Omega)$ and every set in $\mathcal{C}_1$ can be obtained by taking countably many unions/intersections/complements of sets in $\mathcal{C}_2$, then $\sigma(\mathcal{C}_2) \supseteq \sigma(\mathcal{C}_1)$. **[1]**
(b) Hence show that the following collections $\mathcal{C}_i \subseteq \mathcal{P}(\mathbb{R})$ all generate the same σ-algebra:

$$
\begin{aligned}
\mathcal{C}_1 &:= \{(a, b) \mid -\infty < a < b < +\infty\}, \\
\mathcal{C}_2 &:= \{[a, b] \mid -\infty < a \leq b < +\infty\}, \\
\mathcal{C}_3 &:= \{(a, b] \mid -\infty < a \leq b < +\infty\}, \\
\mathcal{C}_4 &:= \{(-\infty, b] \mid -\infty < b < +\infty\}, \\
\mathcal{C}_5 &:= \{[a, \infty) \mid -\infty < a < +\infty\}.
\end{aligned}
$$

We call $\sigma(\mathcal{C}_i)$ the **Borel σ-algebra**, $\mathcal{B}(\mathbb{R})$, and it is the 'usual' choice of σ-algebra on $\mathbb{R}$. **[1]**
(c) Find a countable collection $\mathcal{C}_6$ that also generates $\mathcal{B}(\mathbb{R})$. **[1]**
(d) Let $\mathbb{P}$ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and define its **cumulative distribution function** $F : \mathbb{R} \to \mathbb{R}$ by $F(x) := \mathbb{P}((-\infty, x])$. Show that F is a distribution function (i.e. increasing, tending to 0 at $-\infty$, to 1 at $+\infty$, continuous from the right and with well-defined left limits). What goes wrong with this result if $\mathbb{P}$ is defined on a σ-algebra $\mathcal{F}$ that has 'too few' measurable sets, e.g. $\mathcal{F} := \sigma(\{(a, b) \mid a, b \in \mathbb{Z}\})$? **[2]**

Exercise 6.3 (Equality in distribution)**.** Let $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ be probability spaces, and let $X_1 : \Omega_1 \to \mathbb{R}$ and $X_2 : \Omega_2 \to \mathbb{R}$ be discrete random variables with PMFs p_1 and p_2 and CDFs F_1 and F_2 respectively. Show that following are equivalent:

(a) $X_1 \stackrel{\mathrm{d}}{=} X_2$, i.e. X_1 and X_2 are equal in distribution, $\mathbb{P}_1 \circ X_1^{-1} = \mathbb{P}_2 \circ X_2^{-1} : \mathcal{P}(\mathbb{R}) \to [0, 1]$;
(b) $p_1 = p_2 : \mathbb{R} \to [0, 1]$, i.e. their probability mass functions coincide;
(c) $F_1 = F_2 : \mathbb{R} \to [0, 1]$, i.e. their cumulative distribution functions coincide.

Show also that, if one (and hence all) of these conditions hold, then $\mathbb{E}[X_1] = \mathbb{E}[X_2]$. **[2]**

Exercise 6.4 (Practice with expected values)**.** A fair coin is tossed twice (independently), yielding N heads. Then, the coin is tossed a further N times (these tosses being independent), yielding X heads.

(a) Construct a probability model for this experiment: decide on a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$, define N and X as discrete random variables on that space, and explicitly write out ω, $\mathbb{P}(\{\omega\})$, $N(\omega)$, and $X(\omega)$ for each $\omega \in \Omega$. **[1]**

(b) Calculate $\mathbb{E}[N]$, $\mathbb{E}[X]$, and the expected total number of heads obtained. **[1]**

Exercise 6.5 (Eight is Enough). The dwarf queen and dwarf king have no children at present, but decide that they will continue to have children until either they have either one girl or eight children, whichever comes first. Suppose that girl-dwarves and boy-dwarves are equally probable, and that the sexes of the various dwarf children are mutually independent. Construct a probability model and discrete random variables for this problem, and use your model to find the expected number of girl-dwarves in the family, the expected number of boy-dwarves in the family, and the expected total number of children. **[2]**

[6.1] If $\mathcal{G}$ were a topology rather than a σ-algebra, then this condition would be analogous to the Hausdorff separation axiom.