FU Berlin
Stochastik I (Mono-Bachelor)
WiSe 2016–2017

Lectures: T. J. Sullivan [sullivan@zib.de]
TA 1: N. Gießing [noah.giessing@hushmail.com]
TA 2: I. Klebanov [klebanov@zib.de]
TA 3: H. C. Lie [hlie@zedat.fu-berlin.de]

Exercise Sheet 7

These exercises concern the lectures of Week 7, and possibly previous lectures and exercises. Please submit your solutions to these exercises at the end of the second lecture of Week 8, i.e. 10:00 on 8 December 2016. Solutions in German and in English are equally valid. The numbers in the margin indicate approximately how many points are available for each part.

Exercise 7.1 (Covariance). Let X, Y, and Z be discrete real-valued random variables defined over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, all with finite expected value and variance, and let $\alpha, \beta \in \mathbb{R}$. Show that

(a) $\operatorname{Cov}[X, Y] = \operatorname{Cov}[Y, X]$;
(b) if X or Y is a.s. constant, then $\operatorname{Cov}[X, Y] = 0$; if $\operatorname{Cov}[X, X] = 0$, then X is a.s. constant.
(c) $\operatorname{Cov}[\alpha X + \beta Y, Z] = \alpha \operatorname{Cov}[X, Z] + \beta \operatorname{Cov}[Y, Z]$ and $\operatorname{Cov}[X, \alpha Y + \beta Z] = \alpha \operatorname{Cov}[X, Y] + \beta \operatorname{Cov}[X, Z]$;
(d) $\mathbb{V}[\alpha X] = \alpha^2 \mathbb{V}[X]$;
(e) $|\operatorname{Cov}[X, Y]| \leq \sqrt{\mathbb{V}[X]\mathbb{V}[Y]}$;
(f) and, if X and Y are uncorrelated, then $\mathbb{V}[X + Y] = \mathbb{V}[X] + \mathbb{V}[Y]$. **[4]**

(These facts amount to showing that covariance is an indefinite inner product on the vector space $\mathcal{L}^2$ of finite-variance random variables, and a *bona fide* inner product on the vector space of finite-variance random variables modulo $\mathbb{P}$-a.s. constancy.)

Exercise 7.2 (Minimal mean squared deviation). Let X be a discrete random variable defined over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with finite mean $\mathbb{E}[X] =: \mu$ and finite variance $\mathbb{V}[X] =: \sigma^2$. Show that

$$\sigma^2 = \mathbb{E}[X(X-1)] + \mu - \mu^2$$

and that, for any $c \in \mathbb{R}$,

$$\mathbb{E}[(X - c)^2] = \sigma^2 + (\mu - c)^2,$$

so that the minimal mean squared deviation about c occurs when $c = \mu$. **[2]**

Exercise 7.3 (Means, products, and independence). Let $X, Y : \Omega \to \mathbb{R}$ be independent discrete random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with finite mean and variance, with absolute convergence of the respective series. Show that

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$$

and give an example of non-independent random variables for which this equality still holds. **[2]**

Exercise 7.4. A dwarf and an evil witch have three 6-sided dice, A, B, and C. The dice are fair in the sense that each of the 6 sides is equiprobable, but instead of the usual assignment of 21 spots onto the sides, the spots are assigned to sides as follows:

A	1	1	4	5	5	5
B	2	2	2	3	6	6
C	1	3	3	4	4	6

The dwarf and the witch play the following games; because she is evil and seeks an advantage over the dwarf, the witch always goes first. In each case, use probability to explain who should expect to win. **[4]**

(a) One of the three dice is selected fairly, they both roll the selected die, and the higher score wins; in case of a tie, they re-roll their dice until a winner emerges.
(b) The witch selects a die, but then keeps that die, so that the dwarf selects fairly from the two remaining dice; ties are handled as before. Analyse the case that the witch selects die A (respectively B, C), and also comment on what happens if the witch selects randomly.
(c) The witch selects a die according to any probability distribution on the three dice and keeps the die that she selects; ties are handled as before. Surely she has the advantage now?! The dwarf follows a strange strategy of opposing the witch's A with B, her B with C, and her C with A.