FU Berlin
Stochastik I (Mono-Bachelor)
WiSe 2016–2017

Lectures: T. J. Sullivan [sullivan@zib.de]
TA 1: N. Gießing [noah.giessing@hushmail.com]
TA 2: I. Klebanov [klebanov@zib.de]
TA 3: H. C. Lie [hlie@zedat.fu-berlin.de]

Exercise Sheet 8

These exercises concern the lectures of Week 8, and possibly previous lectures and exercises. Please submit your solutions to these exercises at the end of the second lecture of Week 9, i.e. 10:00 on 15 December 2016. Solutions in German and in English are equally valid. The numbers in the margin indicate approximately how many points are available for each part.

Exercise 8.1 (Practice with Poisson random variables)**.** Let $X \sim \text{Poisson}(\lambda)$,

(a) Show that $\mathbb{E}[X] = \mathbb{V}[X] = \lambda$. **[1]**

(b) Let $Y \sim \text{Poisson}(\mu)$ be independent of X. Show that $X + Y \sim \text{Poisson}(\lambda + \mu)$. **[1]**

Exercise 8.2 (Binomial versus Poisson)**.** The evil witch has captured 400 dwarves, who make mutually independent escape attempts from her evil castle, each with a probability of a successful escape being 1-in-2000. Use both the binomial model and the Poisson approximation to give the probabilities that

(a) at least one dwarf escapes;

(b) exactly one dwarf escapes;

(c) exactly two dwarves escape.

Give both the exact values for these probabilities under the two models and decimal approximations to 4 significant digits. **[2]**

Exercise 8.3 (Exit times for a random walk)**.** At time $t = 0$, a dwarf starts at location $X_0 \coloneqq a \in \{0, 1, \dots, 9, 10\}$ and, each second, takes a step of size ± 1 with equal probability. Let X_t denote the location of the dwarf at time $t \in \mathbb{N} \cup \{0\}$, and let

$$T_a \coloneqq \min\{t \in \mathbb{N} \cup \{0\} \mid X_t = 0 \text{ or } X_t = 10, \text{ given that } X_0 = a\}$$

denote the number of seconds (i.e. steps) that pass before the dwarf leaves the interval $(0, 10)$. Let $m(a) \coloneqq \mathbb{E}[T_a]$ denote the expected exit time.

(a) For the initial condition $a = 5$, explicitly write out the PMFs of X_0, X_1, X_2, and X_3. **[1]**

(b) Show that $m(0) = m(10) = 0$ and that, for $0 < a < 10$, **[1]**

$$m(a) = 1 + \frac{m(a+1)}{2} + \frac{m(a-1)}{2}. \tag{8.1}$$

(c) Show that $m(a) = (10 - a)a$ is a solution to these equations. Is it unique? **[1]**

After subtracting $m(a)$ from both sides, equation (8.1) can be seen as a finite-differences approximation to the differential equation $m'' = -1$. This suggests that there are interesting connections between random walks and (partial) differential equations — and there are!

Exercise 8.4 (Memoryless geometric distributions)**.** A discrete random variable $T \colon \Omega \to \mathbb{N}$ is said to be **memoryless** if, for any $m, n \in \mathbb{N}$,

$$\mathbb{P}[T = m + n | T > m] = \mathbb{P}[T = n].$$

If T is seen as a waiting time until some condition is met, then memorylessness can be interpreted as saying that the *additional* waiting time given that you have already been waiting for m seconds does not depend on m. But take care: memorylessness does not mean that $\mathbb{P}[T = n | T > m] = \mathbb{P}[T = n]$! This holds only when $[T = n] \perp\!\!\!\perp [T > m]$.

(a) Show that, if $T \sim \text{Geometric}(p)$ (the $\mathbb{N}$-valued 'index of the first success' version of the geometric distribution), then it is memoryless. **[1]**

(b) Show that, if T is memoryless, then it must be geometrically distributed. **[2]**