FU Berlin
Stochastik I (Mono-Bachelor)
WiSe 2016–2017

Lectures: T. J. Sullivan [sullivan@zib.de]
TA 1: N. Gießing [noah.giessing@hushmail.com]
TA 2: I. Klebanov [klebanov@zib.de]
TA 3: H. C. Lie [hlie@zedat.fu-berlin.de]

Exercise Sheet 9

These exercises concern the lectures of Week 9, and possibly previous lectures and exercises. Please submit your solutions to these exercises at the end of the second lecture of Week 10, i.e. 10:00 on 5 January 2017. Solutions in German and in English are equally valid. The numbers in the margin indicate approximately how many points are available for each part.

Exercise 9.1 (Conditional variance)**.** Let X and Y be discrete random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We define the **conditional variance** of X given Y to be the random variable

$$\mathbb{V}[X|Y] := \mathbb{E}[X^2|Y] - (\mathbb{E}[X|Y])^2.$$

Assuming for simplicity that all the necessary series converge absolutely, prove that **[1]**

$$\mathbb{V}[X] = \mathbb{E}\big[\mathbb{V}[X|Y]\big] + \mathbb{V}\big[\mathbb{E}[X|Y]\big].$$

Exercise 9.2 (Conditional expectation for a coloured die)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a classical (uniform) probability space with 6 outcomes. Consider a fair 6-sided die with the usual arrangement of spots on the sides, the side showing one spot is red (r), the sides showing two and three spots are green (g) and other three sides are blue (b). Assume that the die is rolled fairly, and let X (resp. Y) denote the number of spots (resp. colour) on the uppermost side.

(a) Write down a table showing, for each $\omega \in \Omega$, the values $X(\omega)$, $Y(\omega)$, and $\mathbb{E}[X|Y](\omega)$. Verify the law of total expectation in this special case. **[2]**

(b) Write out all the sets in the σ-algebra on Ω generated by Y, i.e.

$$\sigma(Y) := \sigma\big(\{[Y = c] \mid c \in \{r, g, b\}\}\big).$$

Show that a discrete random variable $Z \colon \Omega \to \mathbb{R}$ is $\sigma(Y)$-measurable if and only if Z is constant on each event of the form $[Y = c]$. (Informally, such a Z can be 'written as a well-defined function of Y'.) Hence show that $\mathbb{E}[X|Y]$ calculated in part (a) is $\sigma(Y)$-measurable. **[2]**

(c) Recall from Exercise 7.2 that $m = \mathbb{E}[X]$ has the property that it minimises $\mathbb{E}[(X - m)^2]$, so m can be seen as the best constant approximation to X. By Exercise 6.1, this is the same as saying that the constant random variable $Z \equiv m$ is the best $\{\varnothing, \Omega\}$-measurable approximation to X.
Show that $\mathbb{E}[X|Y]$ minimises $\mathbb{E}[(X - Z)^2]$ among all $\sigma(Y)$-measurable Z, so $\mathbb{E}[X|Y]$ can be seen as the best $\sigma(Y)$-measurable approximation to X. **[1]**

Exercise 9.3 (Bonus question: Christmas is ruined!)**.** It looks like everything is all set for a beautiful Christmas, and the Weihnachtsmann has left $2n$ presents in $2n$ boxes for the $2n$ dwarf children ($n > 1$), each present and each box labelled with the dwarf child's unique name. However, when the dwarf children come to get their presents, they discover this note:

> Dear Dwarflings,
> I have randomly permuted the labels on the boxes, so that that each box still contains exactly one present, but perhaps not the one belonging to the dwarf named on the outside of the box. I have also placed a curse, so that each of you can only open n boxes and see which present is inside. If every one of you can find the present that is labelled with your name, then you can all have your presents; otherwise, the presents will all disappear in a puff of smoke and none of you will get a present! Also, after each of you has had her/his turn at opening n out of $2n$ boxes, my curse will re-close all the boxes and prevent you from saying anything about which boxes you opened or what you found inside.
> Wishing you a terrible Christmas,
> The Evil Witch

The dwarf children are terribly disappointed, thinking that this means that they have only a tiny 2^{-n} probability of getting their presents. However, their parents say that there is a way to get the presents with probability strictly greater than 1/4. What could this clever method be? **[+3]**
