FU Berlin
Stochastik I (Mono-Bachelor)
WiSe 2016–2017

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Exercise Sheet 10

These exercises concern the lectures of Week 12, and possibly previous lectures and exercises. Please submit your solutions to these exercises at the end of the second lecture of Week 13, i.e. 10:00 on 12 January 2017. Solutions in German and in English are equally valid. The numbers in the margin indicate approximately how many points are available for each part.

Recall. A collection $\mathcal{C} \ni \varnothing$ of subsets of a non-empty set Ω is called a

- **π-system** if $\mathcal{C}$ is closed under binary intersections of its members;
- **λ-system** if $\mathcal{C}$ is closed under complementations and countable disjoint unions of its members;
- **σ-algebra** if $\mathcal{C}$ is closed under complementations and countable unions and intersections of its members.

Dynkin's π-λ theorem states that if $\mathcal{P}$ is a π-system on Ω and $\mathcal{L}$ a λ-system on Ω, and $\mathcal{P} \subseteq \mathcal{L}$, then $\sigma(\mathcal{P}) \subseteq \mathcal{L}$.

Exercise 10.1 (Intervals and π- and λ-systems). (a) Show that the collection $\mathcal{I}$ of all intervals in $\mathbb{R}$ is a π-system. **[1]**

(b) Show that the collection $\mathcal{I}_{\mathrm{L}}$ of all intervals in $\mathbb{R}$ that are open on the left (i.e. all intervals of the form $(a, b]$, for $-\infty < a \leq b < \infty$) is a π-system. **[1]**

(Recall in passing from Exercise 6.2 that $\sigma(\mathcal{I}) = \sigma(\mathcal{I}_{\mathrm{L}}) = \mathcal{B}(\mathbb{R})$, the Borel σ-algebra of the real line.)

Exercise 10.2 (Uniqueness of extension). Let Ω be a non-empty set, $\mathcal{P}$ a π-system on Ω, and $\mathcal{F} = \sigma(\mathcal{P})$. Suppose that μ and ν are measures on $(\Omega, \mathcal{F})$ with $\mu(\Omega) = \nu(\Omega) < \infty$ and $\mu(P) = \nu(P)$ for all $P \in \mathcal{P}$. Show that $\mu = \nu$. Hint: show that $\mathcal{L} := \{E \in \mathcal{F} | \mu(E) = \nu(E)\}$ is a λ-system and use Dynkin's π-λ theorem to show that $\mathcal{L} = \mathcal{F}$. **[2]**

Exercise 10.3 (Independence of σ-algebras and π-systems). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathcal{P}_1, \mathcal{P}_2 \subseteq \mathcal{F}$ be π-systems. Suppose that $\mathcal{P}_1 \perp\!\!\!\perp \mathcal{P}_2$, i.e.

$$\mathbb{P}(E_1 \cap E_2) = \mathbb{P}(E_1)\mathbb{P}(E_2) \tag{$*$}$$

holds for all $E_1 \in \mathcal{P}_1$, $E_2 \in \mathcal{P}_2$. This exercise aims to show that $\sigma(\mathcal{P}_1) \perp\!\!\!\perp \sigma(\mathcal{P}_2)$, i.e. $(*)$ holds for all $E_1 \in \sigma(\mathcal{P}_1)$, $E_2 \in \sigma(\mathcal{P}_2)$.

(a) Fix $E_1 \in \mathcal{P}_1$ and, for $E \in \mathcal{F}$, define $\mu(E) := \mathbb{P}(E_1 \cap E)$ and $\nu(E) := \mathbb{P}(E_1)\mathbb{P}(E)$. Use Exercise 10.2 to show that these measures agree on $\sigma(\mathcal{P}_2)$, and hence that $\mathcal{P}_1 \perp\!\!\!\perp \sigma(\mathcal{P}_2)$. **[1]**

(b) Use a similar strategy to part (a) to show that $\sigma(\mathcal{P}_1) \perp\!\!\!\perp \sigma(\mathcal{P}_2)$. **[1]**

Exercise 10.4 (Zero-one law). Let $X_1, X_2, \dots$ be a sequence of mutually independent real-valued random variables, taking values in $\mathbb{R}$ with its Borel σ-algebra $\mathcal{B}(\mathbb{R})$.

(a) For $n \in \mathbb{N}$, show that

$$\big\{[X_1 \leq x_1, X_2 \leq x_2, \dots, \text{and } X_n \leq x_n] \big| x_1, \dots, x_n \in \mathbb{R} \cup \{\pm\infty\}\big\}$$

is a π-system; let $\mathcal{F}_n$ denote the generated σ-algebra. Show also that

$$\big\{[X_{n+1} \leq x_{n+1}, X_{n+2} \leq x_{n+2}, \dots, \text{and } X_{n+k} \leq x_{n+k}] \big| x_{n+1}, \dots, x_{n+k} \in \mathbb{R} \cup \{\pm\infty\}, k \in \mathbb{N}\big\}$$

is a π-system; let $\mathcal{T}_n$ denote the generated σ-algebra. Show that $\mathcal{F}_n \perp\!\!\!\perp \mathcal{T}_n$. **[2]**

(b) Let $\mathcal{F}_\infty := \sigma\big(\bigcup_{n \in \mathbb{N}} \mathcal{F}_n\big)$ and $\mathcal{T} := \bigcap_{n \in \mathbb{N}} \mathcal{T}_n$. The σ-algebra $\mathcal{T}$ is called the **tail σ-algebra**; informally, it consists of events or statements about the sequence $(X_n)_{n \in \mathbb{N}}$ whose occurence does not depend on any finite sub-collection of the X_n. Show that $\mathcal{F}_n \perp\!\!\!\perp \mathcal{T}$, that $\mathcal{F}_\infty \perp\!\!\!\perp \mathcal{T}$, and that $\mathcal{T} \perp\!\!\!\perp \mathcal{T}$. Conclude that, for all $E \in \mathcal{T}$, $\mathbb{P}(E) \in \{0, 1\}$. **[2]**

(c) Show that every $\mathcal{T}$-measurable random variable is almost surely constant. **[1]**

(d) Show that a sequence or series of independent random variables either almost surely converges or almost surely fails to converge. (So that, for example, convergence with probability $\frac{1}{2}$ is impossible.) **[1]**