FU Berlin
Stochastik I (Mono-Bachelor)
WiSe 2016–2017

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Exercise Sheet 11

These exercises concern the lectures of Week 13, and possibly previous lectures and exercises. Please submit your solutions to these exercises at the end of the second lecture of Week 14, i.e. 10:00 on 19 January 2017. Solutions in German and in English are equally valid. The numbers in the margin indicate approximately how many points are available for each part.

Recall. In this exercise sheet, you may find it useful to recall the following following facts about d-dimensional Gaussian integrals:

$$\int_{\mathbb{R}^d} \exp\bigl(-\|x\|^2/2\bigr)\,\mathrm{d}x = (2\pi)^{d/2},$$

$$\int_{\mathbb{R}^d} x_i \exp\bigl(-\|x\|^2/2\bigr)\,\mathrm{d}x = 0, \qquad i \in \{1, \dots, d\},$$

$$\int_{\mathbb{R}^d} x_i x_j \exp\bigl(-\|x\|^2/2\bigr)\,\mathrm{d}x = (2\pi)^{d/2}\delta_{ij}, \qquad i, j \in \{1, \dots, d\},$$

where $\delta_{ij} \coloneqq 1$ if $i = j$ and $\delta_{ij} \coloneqq 0$ otherwise, and $\|x\| \coloneqq (x_1^2 + \dots + x_d^2)^{1/2}$ denotes the Euclidean norm.

Recall also that a square matrix $S \in \mathbb{R}^{d\times d}$ is **symmetric** if $S_{ij} = S_{ji}$ for all $i, j \in \{1, \dots, d\}$. Such a matrix is **positive definite** (resp. **positive semidefinite**) if $v \cdot Sv > 0$ (resp. $v \cdot Sv \geq 0$) for all $v \in \mathbb{R}^d \setminus \{0\}$; the abbreviations **SPD** and **SPSD** are sometimes used. An SPD matrix is always invertible, and an SP(S)D matrix always has an SP(S)D square root, i.e. a SP(S)D matrix $S^{1/2} \in \mathbb{R}^{d\times d}$ such that $S^{1/2}S^{1/2} = S$.

Exercise 11.1 (Uniform distribution)**.** Let $X \sim \mathrm{Uniform}([a, b])$ be a continuously uniformly distributed random variable on $[a, b] \subset \mathbb{R}$ with $-\infty < a < b < \infty$. Write down the PDF $\rho\colon \mathbb{R} \to [0, \infty]$ and CDF $F\colon \mathbb{R} \to [0, 1]$ of X and show that its mean and variance are **[2]**

$$\mathbb{E}[X] \coloneqq \int_{\mathbb{R}} x\rho(x)\,\mathrm{d}x = \frac{b+a}{2},$$

$$\mathbb{V}[X] \coloneqq \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{(b-a)^2}{12}$$

Exercise 11.2 (Normalisation, mean and (co)variance of the normal distribution)**.**

(a) Let $m \in \mathbb{R}$ and $\sigma > 0$. Use the above facts and a change of variables to find a normalising constant $c \in \mathbb{R}$ so that $\rho(x) \coloneqq c\exp(-|x-m|^2/2\sigma^2)$ is a PDF on $\mathbb{R}$ (i.e is non-negative and has $\int_{\mathbb{R}} \rho(x)\,\mathrm{d}x = 1$). A random variable X with this PDF is said to have a (univariate) **normal distribution** with mean m and variance σ^2, denoted $X \sim \mathcal{N}(m, \sigma^2)$. Show that the mean and variance of X are indeed what the terminology suggests, i.e. that **[2]**

$$\mathbb{E}[X] \coloneqq \int_{\mathbb{R}} x\rho(x)\,\mathrm{d}x = m,$$

$$\mathbb{V}[X] \coloneqq \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \sigma^2.$$

(b) Now let $d \in \mathbb{N}$, $m \in \mathbb{R}^d$, and let $C \in \mathbb{R}^{d\times d}$ be a symmetric and positive-definite matrix. Find a normalising constant c so that $\rho(x) \coloneqq c\exp(-\frac{1}{2}(x-m)\cdot C^{-1}(x-m))$ is a PDF on $\mathbb{R}^d$. A random vector X with this PDF is said to have a (multivariate) **normal distribution** with mean m and covariance C, denoted $X \sim \mathcal{N}(m, C)$. Show that, indeed **[2]**

$$\mathbb{E}[X_i] \coloneqq \int_{\mathbb{R}^d} x_i\rho(x)\,\mathrm{d}x = m_i, \qquad i \in \{1, \dots, d\},$$

$$\mathrm{Cov}[X_i, X_j] \coloneqq \mathbb{E}[X_iX_j] - \mathbb{E}[X_i]\mathbb{E}[X_j] = C_{ij}, \qquad i, j \in \{1, \dots, d\}.$$

Hint: prove the claim for a standard normal random vector and then use Exercise 11.3.

Exercise 11.3 (Covariances and images of random vectors). Suppose that $X = (X_1, \dots, X_d)$ is a random (column) vector taking values in $\mathbb{R}^d$ with mean $m \in \mathbb{R}^d$ and covariance $C \in \mathbb{R}^{d \times d}$, i.e.

$$\begin{aligned} m_i &= \mathbb{E}[X_i] && \text{for } i \in \{1, \dots, d\}, \\ C_{ij} &= \mathbb{E}[X_i X_j] - m_i m_j && \text{for } i, j \in \{1, \dots, d\}. \end{aligned}$$

(a) (Preparation / hint for later.) Convince yourself that the above definition of C can be rewritten as $C = \mathbb{E}\big[XX^{\mathsf{T}}\big] - mm^{\mathsf{T}} = \mathbb{E}\big[(X-m)(X-m)^{\mathsf{T}}\big]$. Here, for column vectors $a, b \in \mathbb{R}^d$, $ab^{\mathsf{T}} \in \mathbb{R}^{d \times d}$ is the matrix with $(i,j)^{\text{th}}$ entry $a_i b_j$. Note that this is *not* the same as $a^{\mathsf{T}}b$ or $b^{\mathsf{T}}a$, both of which are the scalar Euclidean dot product of the vectors a and b. **[0]**

(b) Show that C is symmetric and positive semidefinite. **[2]**

(c) Let $A \in \mathbb{R}^{d' \times d}$ and $v \in \mathbb{R}^{d'}$. Show that $Y := AX + v$ is a random vector in $\mathbb{R}^{d'}$ with mean $Am + v$ and covariance ACA^{T}. **[2]**