FU Berlin
Stochastik I (Mono-Bachelor)
WiSe 2016–2017

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Exercise Sheet 12

These exercises concern the lectures of Week 14, and possibly previous lectures and exercises. Please submit your solutions to these exercises at the end of the second lecture of Week 15, i.e. 10:00 on 26 January 2017. Solutions in German and in English are equally valid. The numbers in the margin indicate approximately how many points are available for each part.

Full Marks = 12. There are 14 points available on this exercise sheet, but full marks will be awarded for solutions that score 12 points. Therefore, scores above 12 will count as bonus marks. You are free to use results even from questions that you do not solve in your solutions to the other questions.

Exercise 12.1 (Discontinuities of a CDF). In lectures, it was mentioned that any real-valued random variable X can be expressed as a combination of a discrete part, an absolutely continuous part, and a singular continuous part. The values of the discrete part of X are the jump discontinuities of the CDF F_X. Since we insist that discrete random variables take at most countably many values, we should check that a distribution function cannot have uncountably many discontinuities.

(a) Let $F\colon \mathbb{R} \to [0, 1]$ be a distribution function, i.e. continuous from the right, has limits from the left, increasing, and tends to 0 and 1 at $-\infty$ and $+\infty$ respectively. Show that F has at most countably many discontinuities. Hint: consider the set $D \subseteq \mathbb{R}$ of discontinuities of F and find a way to label each one with a unique rational number. **[1.5]**

(b) Show that, for a random variable X, $\mathbb{P}[X < x] = \mathbb{P}[X \leq x]$ for Lebesgue almost every $x \in \mathbb{R}$. **[0.5]**

Exercise 12.2 (Moments and tail probabilities).

(a) Let X be a non-negative random variable. Show that

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}[X \geq x] \, \mathrm{d}x.$$

Hint: use the fact that, for all $y \in \mathbb{R}$,

$$y = \int_0^y \mathrm{d}x = \int_0^\infty \mathbb{I}_{y \geq x} \, \mathrm{d}x,$$

where I_S is 1 if the statement S is true and is 0 if S is false; apply this with X in place of y. **[1]**

(b) Let X be a real-valued random variable with finite mean. Show that **[1]**

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}[X \geq x] \, \mathrm{d}x - \int_{-\infty}^0 \mathbb{P}[X \leq x] \, \mathrm{d}x.$$

Exercise 12.3 (Medians). A **median** for a real-valued random variable X is a number $m \in \mathbb{R}$ such that

$$\mathbb{P}[X \leq m] \geq \frac{1}{2} \text{ and } \mathbb{P}[X \geq m] \geq \frac{1}{2}.$$

(a) Give an example of a random variable X with more than one median. Give an example of a random variable Y with no mean but a well-defined median. Give an example of a random variable Z with finite mean and a unique median m but for which $\mathbb{E}[Z] \neq m$. **[2]**

(b) Show that every random variable X has at least one median. Hint: consider the infimum of **[2]**

$$M := \{x \in \mathbb{R} \mid \mathbb{P}[X \leq x] \geq 1/2\}.$$

(c) Show that if $m_1 < m_2$ are both medians for X, then so is every $m \in (m_1, m_2)$, i.e. the set of medians forms an interval. **[1]**

(d) Let $I \subseteq \mathbb{R}$ be an interval and suppose that X has a continuous PDF $\rho_X : I \to [0, \infty]$ with $\rho_X(x) > 0$ for all $x \in I$. Show that the median of X is unique. **[2]**

(e) For $c \in \mathbb{R}$, let $J(c) := \mathbb{E}\big[|X - c|\big]$ be the **mean absolute deviation** of X about c, and assume for simplicity that X is such that $J(c)$ is finite for all c. Show that

$$J(c) = \int_c^\infty \mathbb{P}[X \geq s] \, \mathrm{d}s + \int_{-\infty}^c \mathbb{P}[X \leq s] \, \mathrm{d}s$$

and use this to show that medians of X minimise the function J. (For context, in Exercise 7.2 we showed that choosing $c = \mathbb{E}[X]$ minimises the mean *squared* deviation $\mathbb{E}\big[|X - c|^2\big]$.) **[2]**

(f) Suppose that X has finite variance σ^2. Show that

$$\big|\mathbb{E}[X] - m\big| \leq \sigma.$$

Hint: you may find it useful to use Jensen's inequality $f\big(\mathbb{E}[X]\big) \leq \mathbb{E}\big[f(X)\big]$ for any convex function f, which we already proved for discrete random variables X, but in fact holds in general. **[1]**